Solution Set for Chapter 21

(Xinkai)

1. Ex. 21.4 Landau Contour Deduced Using Laplace Transforms

(a) This part is nothing but a definition of Laplace transformation.

(b) The z-dependence is e^{ikz}, giving $\partial/\partial z \to ik$. Also integration by parts gives

$$\int_0^\infty dt e^{-pt}\partial F_{s1}/\partial t = -F_{s1}(v,0) + p\int_0^\infty dt e^{-pt} F_{s1}(v,t)$$

Noticing the above facts, we get by Laplace transforming the Vlasov equation:

$$0 = -F_{s1}(v,0) + p\,\tilde{F}_{s1}(v,p) + vik\,\tilde{F}_{s1}(v,p) + (q_s/m_s)F'_{s0}\,\tilde{E}(p) \quad \text{where } s = p, e$$

Laplace transforming $\nabla \cdot \vec{E} = \rho/\epsilon_0$ gives us a second equation:

$$ik\,\tilde{E}(p) = \sum_s (q_s/\epsilon_0)\int_{-\infty}^{\infty} dv[F_{s0}(v)/p + \tilde{F}_{s1}(v,p)] = \sum_s (q_s/\epsilon_0)\int_{-\infty}^{\infty} dv\,\tilde{F}_{s1}(v,p)$$

Where to get the last equality we've used the fact that the unperturbed charge density is zero, i.e. the contribution from $F_{s0}(v)$ vanishes.

Combining these two equations we easily get (21.41).

(c) Setting $ip = \omega$, and plugging (21.41) into (21.42), we immediately get (21.26) without that overall minus sign(I guess that sign is a typo)

Q.E.D.

2.Ex. 21.6 Dispersion Relations for a Non-Maxwellian Distribution Function

This problem is quite instructive in that it clarifies what approximations we are making when considering Langmuir waves and ion acoustic waves, respectively.

Now the distribution function is given by $F(v) = F_e(v) + (m_e/m_p)F_p(v)$, where $F_e(v) = nv_{0e}/[\pi(v_{0e}^2 + v^2)]$, and $F_p(v)$ is obtained by replacing v_{0e} with v_{0p}. Note that these distribution functions are normalized so that n is the electron(and also proton) density.

We then plug this $F(v)$ into the general expression (21.30) for $\epsilon(\omega,k)$. Let's first assume that $\mathrm{Im}(\omega/k) > 0$. Thus we take the Landau contour to be the real axis in the v-plane. Closing the contour in the lower half plane and evaluating the residues carefully we get:

$$\epsilon(\omega,k) = 1 + e^2 n\{\epsilon_0 k^2 m_e v_{0e}^2[1 - i\omega/(kv_{0e})]^2\}^{-1} + e^2 n\{\epsilon_0 k^2 m_p v_{0p}^2[1 - i\omega/(kv_{0p})]^2\}^{-1}$$

Obviously the above expression for $\epsilon(\omega,k)$ can be analytically continued to the whole ω-plane. Note that so far no approximation has been made.

(a) For Langmuir waves, $T_e \sim T_{p,}(i.e.\, m_e v_{0e}^2 \sim m_p v_{0p}^2)$, and $\omega/k >> v_{0e} >> v_{0p}$, thus we can ignore the third term in $\epsilon(\omega,k)$, namely the contribution from the protons. Then by letting $\epsilon(\omega,k) = 0$, we easily get $\omega = \omega_{pe} - ikv_{0e}$

(b) For ion acoustic waves, $T_e >> T_{p,}(i.e.\, m_e v_{0e}^2 >> m_p v_{0p}^2)$, and $v_{0e} >> \omega/k >> v_{0p}$. Thus we have to include both electron and proton contribution. We expand the electron contribution to the first power of $\omega/(kv_{0e})$, and the proton contribution to the leading order of $v_{0p}k/\omega$. When solving $\epsilon(\omega,k) = 0$ for $\omega(k)$, we will assume $\omega_i << \omega_r$, which we expect for the case of weak damping.

$\mathrm{Im}(\epsilon(\omega,k)) = 0$ gives $\omega_i = -(m_p/m_e)[\omega_r^4/(kv_{0e})^3]$

Plugging the above relation into the equation $\mathrm{Re}(\epsilon(\omega,k)) = 0$, we turn the equation $\mathrm{Re}(\epsilon(\omega,k)) = 0$ into:

$0 = 1 + [\omega_{pe}/(kv_{0e})]^2\{1 + (m_p/m_e)[2\omega_r/(kv_{0e})]^4 - (m_e/m_p)(kv_{oe}/\omega_r)^2\}$

The ratio between the second and the third term in $\{\ldots\}$ is ~$(m_p/m_e)^2[\omega_r/(kv_{0e})]^6$, which will be very small when T_e is very high(i.e. v_{0e} very large compared with ω_r/k). Thus we can neglect the second term. After making this approximation, we readily get ω_r, and in turn ω_i. The ω we get is precisely the same as given in (21.45), excpet that we use v_{0e} instead of v_0(which is just a difference in notation).

Q.E.D.

3. Ex 21.7 Penrose Criterion

We can set u=1, i.e. choose u to be the unit of velocity. Solving this problem involves some straightforward but tedious algebra, but using Mathematica saves us all the trouble:)

We find that $F'(v)$ vanishes at three values of v, namely $v_1 = 0, v_{2,3} = \pm\sqrt{-v_0^2 - 1 + 2v_0\sqrt{v_0^2+1}}$, where $v_{2,3}$ are real only when $v_0 > 1/\sqrt{3}$.

Then we go on to calculate $F''(v)$ and find that $F''(v_1)$ is positive if and only if $v_0 > 1/\sqrt{3}$, while $F''(v_{2,3})$ is always negative when $v_0 > 1/\sqrt{3}$ (i.e. when $v_{2,3}$ are real).

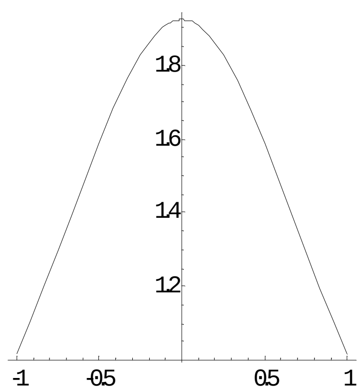
Thus we have shown that $F(v)$ possesses a minimum if and only if $v_0 > 1/\sqrt{3}$, and this minimum, when it exists, is located at $v_{\min} = 0$.

Carrying out the integration defined in (21.49) we find that $Z_r(v_{\min}) = 2\pi(-1 + v_0^2)/(1 + v_0^2)^2$, which is positive when $v_0 > 1$.

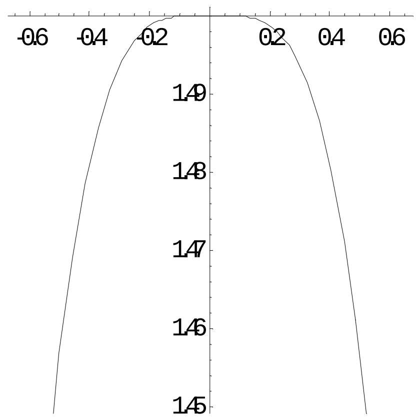
So by Penrose criterion we see that when $v_0 > u$, there will be instability in our plasma.

Here we give a few plots of $F(v)$ with different v_0 parameters.

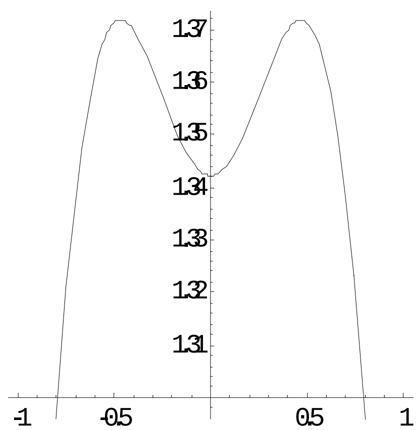
$v_0 = 0.2u$



$v_0 = (1/\sqrt{3})u = 0.577u$



$v_0 = 0.7u$



Q.E.D.

4. Ex.21.10 Correlations in a Tokamak Plasma

Plugging the relevant numbers for the Tokamak plasma as given in table 19.1 into the expression (21.74) for the two-point correlation function, we find:

(a) $\xi_{12}(\lambda_D) \sim 10^{-9}$

(b) $\xi_{12}(n^{-1/3}) \sim 10^{-6}$

Q.E.D.