Chapter 22 Nonlinear Dynamics of Plasmas

1.Ex22.2 (Particle Energy in a Wave) [original solution by Chris Hirata]

$d\varepsilon_k/dt = (d/dt)\int_{-\infty}^{\infty}(1/2)m_e v^2 F_0(v)dv$

$= \int_{-\infty}^{\infty}(1/2)m_e v^2 \partial F_0(v)/\partial t dv$

$= \int_{-\infty}^{\infty}(1/2)m_e v^2\{\partial[D(v)\partial F_0(v)/\partial v]/\partial v - v\partial F_0/\partial z\}dv$ by equation (22.16)

The second term in {..} is odd in v, making its contribution to the integral vanish.

After integrating by part, we get

$d\varepsilon_k/dt = -\int_{-\infty}^{\infty} D(v)[\partial F_0(v)/\partial v]m_e v dv$ (*)

Now look at the expression (22.17) for $D(v)$:

$D(v) = (e^2/\epsilon_0 m_e^2)\int_0^{\infty} dk\varepsilon_k \frac{\omega_i}{(\omega_r - kv)^2+\omega_i^2}$

For the nonresonant piece of $D(v)$, $kv << \omega_r$, so we can replace the denominator of the integrand with $\omega_r^2 \approx \omega_p^2 = ne^2/\epsilon_0 m_e$. Then we get,

$D(v) = (1/nm_e)\int_0^{\infty} dk\varepsilon_k\omega_i$, which is actually v-independent.

Plugging the above expression into equation (*) and integrate by part, we get

$d\varepsilon_k/dt = \int_{-\infty}^{\infty} Dm_e F_0(v)dv = Dm_e n = \int_0^{\infty} dk\varepsilon_k\omega_i$

Writing this as :

$d\varepsilon_k/dt = (1/2)\int_0^{\infty} dk 2\omega_i\varepsilon_k = (1/2)\int_0^{\infty} dk(d\varepsilon_k/dt)$

we see this implies half the wave energy is kinetic.

2.Ex 22.4(Cerenkov Power)[by Alexander Putilin]

The emission rate of plasmons is given by (22.37)

$W = (\pi e^2\omega_r/\epsilon_0 k^2\hbar)\delta(\omega_r - \vec{k}\cdot\vec{v})$

Each plasmon has energy $\hbar\omega_r$, so the radiated power per unit time is:

$P = (1/2\pi)^3\int d^3k W\hbar\omega_r = (e^2/8\pi^2\epsilon_0)\int d^3k(\omega_r^2/k^2)\delta(\omega_r - \vec{k}\cdot\vec{v})$

The integration is over the region $\left|\vec{k}\right| < k_{\max}$(outside this region waves are strongly Landau damped). A good estimate of $k_{\max}$ is inverse Debye length: $k_{\max}\sim 1/\lambda_D$. (see the discussion at the end of Chapter 21.3.5) Since $k\lambda_D < 1$, we can approximate $\omega_r(k)$ by a constant ω_{pe}.

Choosing $\vec{v}$ to point along z-axis, we have

$P = (e^2\omega_{pe}^2/8\pi^2\epsilon_0)\int_{\left|\vec{k}\right|<k_{\max}} d^3k(1/k^2)\delta(\omega_r - \vec{k}\cdot\vec{v})$

$= (e^2\omega_{pe}^2/8\pi^2\epsilon_0)\int_{\left|\vec{k}\right|<k_{\max}} d^2k_\perp dk_z[1/(k_\perp^2 + k_z^2)]\delta(\omega_r - \vec{k}\cdot\vec{v})$

$= (e^2\omega_{pe}^2/8\pi^2\epsilon_0 v)\int_{k_\perp^2<k_{\max}^2-(\omega_{pe}^2/v^2)} d^2k_\perp[1/(k_\perp^2 + \omega_{pe}^2/v^2)]$

$= (e^2\omega_{pe}^2/8\pi^2\epsilon_0 v)\int_0^{k_{\max}^2-(\omega_{pe}^2/v^2)} 2\pi k_\perp dk_\perp/(k_\perp^2 + \omega_{pe}^2/v^2)$

$= (e^2\omega_{pe}^2/4\pi\epsilon_0 v)\ln(k_{\max}v/\omega_{pe})$

Note that P depends on $k_{\max}$ logarithmically. So if v is sufficiently large it doesn't make much difference what particular definition we use for $k_{\max}$.

3. Non-linear Excitation of Particle Motion [original solution by Chris Hirata]

Power expand $\vec{x} = \overrightarrow{x_{(0)}} + \overrightarrow{x_{(1)}} + \overrightarrow{x_{(2)}}$ +higher order terms, where $\overrightarrow{x_{(n)}}$ is of order E^n.

The equation of motion is: $d^2\vec{x}/dt^2 = (-e/m)\{\overrightarrow{E_1}\cos(\overrightarrow{k_1}\cdot\vec{x} - \omega_1 t) + \overrightarrow{E_2}\cos(\overrightarrow{k_2}\cdot\vec{x} - \omega_2 t + \phi)\}$

At the 1st order, the equation of motion becomes:
$d^2\overrightarrow{x_{(1)}}/dt^2 = (-e/m)\{\overrightarrow{E_1}\cos(\overrightarrow{k_1}\cdot\overrightarrow{x_{(0)}} - \omega_1 t) + \overrightarrow{E_2}\cos(\overrightarrow{k_2}\cdot\overrightarrow{x_{(0)}} - \omega_2 t + \phi)\}$
which gives:
$\overrightarrow{x_{(1)}} = (e/m)\{(\overrightarrow{E_1}/\omega_1^2)\cos(\overrightarrow{k_1}\cdot\overrightarrow{x_{(0)}} - \omega_1 t) + (\overrightarrow{E_2}/\omega_2^2)\cos(\overrightarrow{k_2}\cdot\overrightarrow{x_{(0)}} - \omega_2 t + \phi)\}$
We see that $x_{(1)}$ is of order $eE/(m\omega^2)$

At the 2nd order, the equation of motion is:
$d^2\overrightarrow{x_{(2)}}/dt^2 = (e/m)\{\overrightarrow{E_1}(\overrightarrow{k_1}\cdot\overrightarrow{x_{(1)}})\sin(\overrightarrow{k_1}\cdot\overrightarrow{x_{(0)}} - \omega_1 t) + \overrightarrow{E_2}(\overrightarrow{k_2}\cdot\overrightarrow{x_{(1)}})\sin(\overrightarrow{k_2}\cdot\overrightarrow{x_{(0)}} - \omega_2 t + \phi)\}$
Plugging the expression of $\overrightarrow{x_{(1)}}$ into the above expression and noticing that for EM wave $\overrightarrow{k_1}\cdot\overrightarrow{E_1} = \overrightarrow{k_2}\cdot\overrightarrow{E_2} = 0,$ we find(using some elementary trignometry),

$d^2\overrightarrow{x_{(2)}}/dt^2 = (1/2)(e/m)^2\{(\overrightarrow{E_1}/\omega_2^2)(\overrightarrow{k_1}\cdot\overrightarrow{E_2})[\sin((\overrightarrow{k_1}+\overrightarrow{k_2})\cdot\overrightarrow{x_{(0)}} - (\omega_1+\omega_2)t + \phi) + \sin((\overrightarrow{k_1}-\overrightarrow{k_2})\cdot\overrightarrow{x_{(0)}}$

$+ (\overrightarrow{E_2}/\omega_1^2)(\overrightarrow{k_2}\cdot\overrightarrow{E_1})[\sin((\overrightarrow{k_1}+\overrightarrow{k_2})\cdot\overrightarrow{x_{(0)}} - (\omega_1+\omega_2)t + \phi) + \sin((\overrightarrow{k_1}-\overrightarrow{k_2})\cdot\overrightarrow{x_{(0)}} - (\omega_1-\omega_2)t + \phi)]\}$
So we see that at the 2nd order there are oscillations of the electron at $(\omega_1 \pm \omega_2, \overrightarrow{k_1} \pm \overrightarrow{k_2})$.
Easily seen, $x_{(2)}$ is of order: $e^2kE^2/m^2\omega^4$
And we get: $x_{(2)}/x_{(1)} \sim ekE/m\omega^2$
In laser fusion experiments where $E\sim 10^{10}V/m, k\sim 10^7 m^{-1}, \omega\sim 3\times 10^{15}s^{-1}, x_{(2)}/x_{(1)}\sim 10^{-3}$

4.Ex 22.10(Solar Wind Termination Shock) [by Roger Blandford]
Solar wind is supersonic. Momentum flux at Earth ~ $nm_pu^2 \sim 10^{-9}$ N m^{-2}. The density will decline$\propto r^{-2}$ as the outflow is speherical and the velocity will be constant as the flow is supersonic. The momentum flux will balance the interstellar pressure at $r \sim (10^{-9}/10^{-13})^{1/2} \sim 100$ times the radius of the Earth's orbit or 1.5×10^{13}m.

If there is flux freezing, we expect $B \propto r^{-1}$, so the field should be ~ 10 pT at the termination shock.

Assuming the same temperature, $\omega_p \sim 1000$ rad s^{-1}, $T_e/m_i \sim 30$ km s^{-1}, so that $M \sim 13$, and $r_{Li} \sim 2\times 10^6$ m. The Mach number is well above the ion acoustic critical Mach number and, in any case, ion acoustic waves are unlikely to propagate unless the electrons become hotter than the ions. The shock is therefore likely to be turbulent. It also likely to be very thin because all of the microscopic scales are very much smaller than its radius.

Travel time~ $1.5\times 10^{13}/ \sim 1.5\times 10^4 \sim 10^9$ s ~ 30 yr. So should arrive in 2007. The current estimate is 2001-2003.