Part II

Hydrostatics

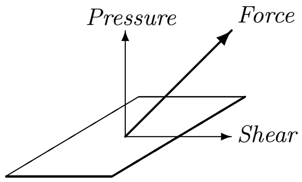
4

Fluids at rest

If the Sun did not shine, if no heat were generated inside the Earth and no energy radiated into space, all the winds in the air and the currents in the sea would die away, and the air and water on the planet would in the end come to rest in equilibrium with gravity. What distribution of pressure and density would we then find in the sea and the atmosphere?

Consider a fluid, be it air or water, under the influence of gravity. In the absence of external driving forces or time-dependent boundary conditions, and in the presence of dissipative contact forces, such a system must eventually reach a state of *hydrostatic equilibrium*, where nothing moves anymore anywhere and all fields become constant in time. This must be first approximation to the sea, the atmosphere, the interior of a planet or a star.

In hydrostatic equilibrium and more generally in mechanical equilibrium there is everywhere a balance between contact forces, such as pressure, having zero range, and body forces such as gravity with infinite range. Contact interactions between material bodies or parts of the same body take place across *contact surfaces*. A contact force acting on a tiny piece of a surface may take any direction. Its component orthogonal to the surface is called a *pressure force* whereas the tangential component is called a *shear force*. Solids and fluids in motion can sustain shear forces, whereas fluids at rest cannot. The wind will not move your parked car because of (shear) friction between the wheels and the road, but it will certainly sail your boat away if not properly moored.



The force on a small piece of a surface can always be resolved in a normal pressure force and a tangential shear force.

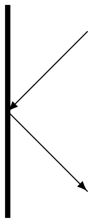
In this chapter we shall establish the basic concepts and formalism for pressure in hydrostatic equilibrium and apply it to the sea and the atmosphere. Along the way we shall recapitulate basic thermodynamics. In the following three chapters we shall continue the study of hydrostatic equilibrium for fishes, icebergs and ships, the interior of planets and stars, and the shapes of fluid bodies.

4.1 Pressure

A fluid at rest can as mentioned only sustain pressure forces. If shear forces arise, the fluid will tend to flow towards a new equilibrium without shear. This expresses the most essential property of fluids and is essentially a definition of what constitutes a macroscopic fluid.

Microscopic origin of pressure

At the microscopic level, fluid pressure on a container wall arises from the incessant molecular bombardment of the wall. Consider for example a gas in which the molecules move with some typical velocity v. A gas molecule impinging the wall tends to be thrown back again and thus gives up at most twice its momentum component normal to the wall. If the gas density is ρ, we may estimate the mass of the gas that hits a small area A of the wall in a small time interval dt to be of the order of $dM \sim \rho\, A\, v dt$. In this estimate we have disregarded that the molecules in fact move in random directions with random velocities. Multiplying with the velocity v and dividing by dt, we get a very rough estimate of the amount of momentum transferred to the wall area A per unit of time, *i.e.* the force $\mathcal{F} = d\mathcal{P}/dt \sim \rho\, A v^2$. Since the force is proportional to the area, it makes good sense to introduce the force per unit of area, $p = \mathcal{F}/A \sim \rho\, v^2$, also called the *pressure*.



A molecule elastically reflected by a wall transfers twice its normal momentum component to the wall.

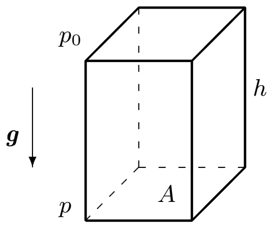
The unit of pressure is pascal ($\mathsf{Pa} = \mathsf{N/m^2} = \mathsf{kg/m/s^2}$). The air pressure at sea level is a little more than 10^5 Pa = 1 bar. The standard pressure at sea level is defined to be 1 atm = 1013.250 millibar. Modern meteorologists tend to quote air pressure in hectopascals (hPa) rather than in millibars.

> **Example 4.1.1:** Taking the density of air at sea level to be $\rho \approx 1\ \mathsf{kg/m^3}$ and the molecular velocity to be of the order of the velocity of sound $v \approx 330\ \mathsf{m/s}$, our estimate yields $p \approx 10^5\ \mathsf{Pa} \approx 1\ \mathsf{atm}$ which is very close to the pressure at sea level. In view of the roughness of the estimate this result must be considered entirely fortuitous.

Pressure field

So far we have only defined the pressure acting on the walls of a container. Is it meaningful to speak about pressure in the middle of a fluid far from containing walls? We could of course put in a tiny manometer to measure the pressure, but then we would just obtain the pressure acting on the surface of the manometer, which is another “wall”, and not the true internal pressure in the fluid.

In spite of these conceptual difficulties we shall imagine that there is a well-defined pressure field $p(\boldsymbol{x})$ everywhere in the fluid and that it acts along the normal to any surface, real or imaginary, in the fluid (Pascal’s law). From a molecular point of view, the estimate of the momentum transferred from the material on one side of the surface to the other takes the same form as before.



A column of sea water. The pressure difference between bottom and top must carry the weight of the water in the box.

Incompressible sea

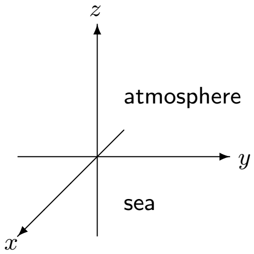
The mass of the water in all the seas is $M_{\rm sea} \approx 1.7 \times 10^{21}$ kg which is 1/3500 of the Earth itself. Water is a nearly incompressible fluid with almost constant density $\rho_0 \approx 10^3$ kg/m^3, making the volume of all the seas 1.7×10^9 km^3. The surface area of the globe is 0.51×10^9 km^2, so if no continents rose out of the sea, it would cover the planet to a uniform depth of about 3.3 km.

Consider now a vertical box in the sea with cross-sectional area A and height h. In mechanical equilibrium the difference between the pressure p at the bottom and the pressure p_0 at the top of this box must balance the total weight of the water in the box, or

$$\rho_0 A h g_0 = pA - p_0 A \ . \tag{4-1}$$

If this equation were not fulfilled, the total force on the column of water would not vanish, and it would have to move. Dividing by the area A the pressure difference between bottom and top of the box becomes,

$$p - p_0 = \rho_0 g_0 h \ . \tag{4-2}$$



The flat-earth coordinate system.

In a *flat-earth coordinate system* with vertical z-axis and the surface of the sea at $z = 0$, we find by setting $h = -z$,

$$\boxed{p = p_0 - \rho_0 g_0 z \ ,} \tag{4-3}$$

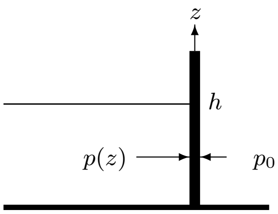
where p_0 is the surface pressure. The pressure thus rises linearly with depth $-z$ as we dive into the sea. The scale of the pressure increase per unit of depth in the sea becomes $\rho_0 g_0 \approx 100$ hPa/m, corresponding to about 1 atm/10 m. At the deepest point in the sea, $z \approx -11$ km the pressure is more than 1000 atm. The assumption of constant gravity is well justified even to this depth (it changes by less than 0.5 %), whereas the density of water changes by about 5 % (see section 4.3).

The linear rise of the pressure with depth for an incompressible fluid at rest in constant gravity may immediately be used to calculate the total pressure force on horizontal and vertical container walls. Curved and non-vertical walls require a slightly stronger mathematical formalism which we shall soon set up.

Example 4.1.2: Water is stemmed up behind a sluice gate of width L to height h. On the water surface and on the outer side of the gate there is atmospheric pressure p_0. On the inside of the gate the pressure is $p(z) = p_0 + \rho_0 g_0 (h - z)$, so that the total force on the gate becomes

$$\mathcal{F} = \int_0^h (p(z) - p_0)\, L dz = \frac{1}{2} \rho_0 g_0 L h^2 \ . \tag{4-4}$$

Because the pressure rises linearly with depth, this result could have been calculated without an explicit integral. The total force is simply the product of the area of the sluice gate Lh with the average pressure excess $\langle p - p_0 \rangle = \frac{1}{2} \rho_0 g_0 h$ acting on the gate.



Water stemmed up behind a sluice gate. The pressure varies linearly with height z over the bottom.

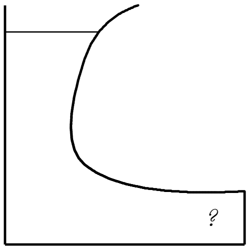
Incompressible atmosphere?

The mass of the atmosphere is about $M_{\text{atm}} \approx 5 \times 10^{18}$ kg, which is smaller than a millionth of the mass of the Earth. Since air is highly compressible, it makes little sense to use the above expression (4-3) for the pressure at constant density (except for very small values of z). If we anyway do so, we find a pressure which drops linearly with height and reaches zero at

$$z = h_0 = \frac{p_0}{\rho_0 g_0} \ . \tag{4-5}$$

Using $p_0 = 1$ atm and air density $\rho_0 = 1.2$ kg/m^3 we get $h_0 = 8.6$ km, which is a tiny bit lower than the height of Mount Everest. This is of course meaningless, since climbers have reached the summit of that mountain without oxygen masks. But as we shall see, this height is nevertheless the correct scale for major changes in the atmospheric properties.

Paradox of hydrostatics

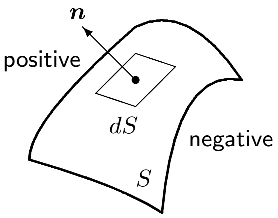


Paradox: the pressure along the vertical wall of the "boot" rises linearly because it has to carry the weight of the water above, but what about the pressure in the tip of the "toe"?

The linear rise of the pressure with depth may, as we have seen, be used to calculate the total pressure force on any vertical container wall. For a curved container wall, like that of an open vase, there seems to be no problem, except handling the necessary mathematics. But if the water column does not reach all the way to the surface, as when you fill a boot with water, what is then the pressure at the flat horizontal bottom? Will it be constant along the bottom as in the open sea, or will it vary? And if it is constant, what is it "up against", since there is only a short column of water above?

The quick answer to this paradox is that the pressure is indeed constant along the horizontal bottom. For if the pressure were lower in the "toe" than in the "heel", there would be unbalanced horizontal pressure forces directed towards the toe acting on a horizontal box of water. But that is not allowed in complete mechanical equilibrium. The only possible conclusion is that the material of the "boot" must supply the necessary forces to compensate for the missing weight of the water column.

Oriented surfaces

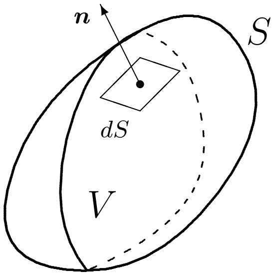


All normals to an oriented open surface have a consistent orientation with common positive and negative sides.

To establish a concise mathematical formalism for pressure we consider a surface S that divides a body into two parts. This surface needs not be a real surface where material properties change dramatically but may just be an imaginary surface separating two parts of the same body from each other. A tiny surface element is characterized by its area dS and the direction of its normal $\boldsymbol{n}$. It is convenient to combine these in the vector surface element

$$d\boldsymbol{S} = (dS_x, dS_y, dS_z) = \boldsymbol{n}\, dS \ . \tag{4-6}$$

There is nothing intrinsic in a surface which defines the orientation of the normal, *i.e.* whether the normal is really $\boldsymbol{n}$ and not $-\boldsymbol{n}$. A choice must, however, be made, and having done that, one may call the side of the surface element into which the normal points, positive (and the other of course negative). Usually neighboring surface elements are required to be oriented consistently, *i.e.* with the same positive sides. By universal convention the normal of a closed surface is chosen to be directed out of the enclosed volume, so that the enclosed volume always lies at the negative side of its surface.



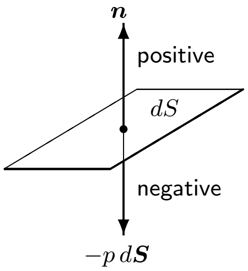
A volume V defined by the closed surface S has all normals oriented towards the outside.

Formal definition of pressure

The contact forces due to microscopic molecular interactions of electromagnetic origin have a finite range at the molecular scale, but zero range at macroscopic distances. Across a tiny but still macroscopic piece of surface, the number of neighboring molecules participating in the interaction as well as the force they exert may for this reason be expected to be proportional to the area of the surface. In a fluid at rest the only contact force is the pressure force acting along the normal to the surface. The force exerted by the material at the positive side of a surface element on the material at the negative side must therefore be of the form

$$\boxed{d\boldsymbol{\mathcal{F}} = -p\,d\boldsymbol{S}\ ,} \tag{4-7}$$

where the coefficient of proportionality p is called the *pressure*. Convention dictates that a positive pressure exerts a force directed *towards* the material on the negative side, and that explains the minus sign.



The force on a vector surface element under positive pressure is directed against the normal.

The total pressure force acting on a surface S is obtained by summing up all the little vector contributions from each surface element,

$$\boldsymbol{\mathcal{F}} = -\int_S p\,d\boldsymbol{S}\ . \tag{4-8}$$

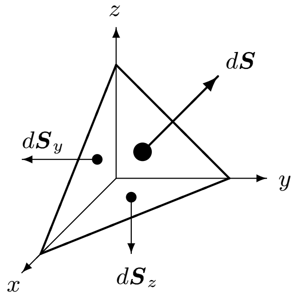
This is the force which acts on the cork in the champagne bottle, moves the pistons in the cylinders of your car engine, breaks a dam, and sends off a bullet from a canon. It is also this force that lifts fishes, ships, and balloons and thereby cancels their weight so that they are able to float (see chapter 5).

Same pressure in all directions?

Newton's third law guarantees that the material on the negative side of a surface element reacts with an equal and opposite force $-d\boldsymbol{\mathcal{F}} = -p(-d\boldsymbol{S})$ on the material on the positive side (provided there is no surface tension). Since the surface vector seen from the negative side is $-d\boldsymbol{S}$, the above relation shows that the pressure also has the value p on the negative side of the surface. This is part of a much stronger result, called Pascal's law, which we shall prove below: *the pressure in a fluid at rest is independent of the direction of the surface element on which it acts.* It implies that pressure cannot depend on the normal $\boldsymbol{n}$, but only on the location $\boldsymbol{x}$ of a surface element, and is therefore a true (scalar) field $p(\boldsymbol{x})$.

Blaise Pascal (1623–1662). *French mathematician and physicist. Founded probability theory. Constructed what may be viewed as the first digital calculator. He spent his later years with religious thinking in the Cistercian abbey of Port-Royal. More than one property of pressure goes under the name of Pascal's law.*

The simple reason for pressure being the same in all directions in hydrostatic equilibrium is that the pressure acts on the surface of a body whereas a body force by definition acts on the volume. If we let the body shrink, the contribution from the body force will vanish faster than the contribution from the surface force because the volume vanishes faster than the surface area. In the limit of vanishing body size only the surface force is left, but it must then itself vanish in mechanical (and thus hydrostatic) equilibrium where the total force on all parts of a body must be zero. This argument will now be fleshed out in mathematical detail.



A body in the shape of a tetrahedron. The vector normals to the sides are all pointing out of the body ($d\boldsymbol{S}_x$ is hidden from view). Any body shape can be built up from sufficiently many and sufficiently small tetrahedrons.

∗ Proof of Pascal's law

Assume first that the pressure is actually different in different directions. We shall then show that for physical reasons this assumption cannot be maintained. Consider a tiny body in the shape of a tetrahedron with three sides parallel to the coordinate planes. The total pressure force acting on the body is

$$d\boldsymbol{\mathcal{F}} = -p d\boldsymbol{S} - p_x d\boldsymbol{S}_x - p_y d\boldsymbol{S}_y - p_z d\boldsymbol{S}_z \ , \tag{4-9}$$

where we have denoted the pressures acting on the different faces of the tetrahedron by p, p_x, p_y, and p_z and the outwards pointing normals by $d\boldsymbol{S}$, $d\boldsymbol{S}_x$, $d\boldsymbol{S}_y$, and $d\boldsymbol{S}_z$. It is sufficient to consider infinitesimal bodies of this kind, because an arbitrary body shape can be put together from these. Each of the three triangles making up the sides of the tetrahedron is in fact the projection of the front face onto that plane. By elementary geometry the areas of the three projected triangles are dS_x, dS_y and dS_z, so that their vector surface elements become $d\boldsymbol{S}_x = (-dS_x, 0, 0)$, $d\boldsymbol{S}_y = (0, -dS_y, 0)$, and $d\boldsymbol{S}_z = (0, 0, -dS_z)$.

Inserting this in the above equation we find the total force

$$d\boldsymbol{\mathcal{F}} = ((p_x - p)dS_x, (p_y - p)dS_y, (p_z - p)dS_z) \ . \tag{4-10}$$

In mechanical (hydrostatic) equilibrium, which is all that we are concerned with here, the contact forces must balance body forces (gravity),

$$d\boldsymbol{\mathcal{F}} + \boldsymbol{f}\, dV = \boldsymbol{0} \ , \tag{4-11}$$

where $\boldsymbol{f}$ is the density of body forces. The idea is to show that for sufficiently small tetrahedrons the body forces can be neglected and the surface forces $d\boldsymbol{\mathcal{F}}$ must consequently vanish.

Consider now a geometrically congruent tetrahedron with all lengths scaled by a factor λ. Since the volume scales as the third power of λ whereas the surface areas only scale as the second power, the hydrostatic equation for the scaled tetrahedron becomes $\lambda^2 d\boldsymbol{\mathcal{F}} + \lambda^3 \boldsymbol{f} dV = \boldsymbol{0}$ or $d\boldsymbol{\mathcal{F}} + \lambda \boldsymbol{f} dV = \boldsymbol{0}$. In the limit of $\lambda \to 0$ it follows that the total contact force must vanish, *i.e.* $d\boldsymbol{\mathcal{F}} = 0$, and using (4-10) we find

$$\boxed{p_x = p_y = p_z = p \ ,} \tag{4-12}$$

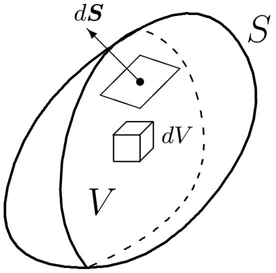
which is *Pascal's law.* As promised, the pressure must indeed be the same in all directions.

4.2 Hydrostatic equilibrium

In the preceding section we intuitively used that in a fluid at rest the weight of a vertical column of fluid should equal the difference in pressure forces between bottom and top of the column. We shall now generalize this to an arbitrary macroscopic volume of fluid, often called a *control volume*. The material in a control volume, fluid or solid or whatever, represents the most general "body" that can be constructed in continuum physics.

Up to this point we have studied only two kinds of forces that may act on the material in a control volume. One is a *body force* described by a force density field $\boldsymbol{f}$ caused by long-range interactions, here gravity $\boldsymbol{f} = \rho\boldsymbol{g}$. The other is a contact force, here the pressure field p, which has zero range and only acts on the surface of the control volume. The total force on the control volume V with surface S is the sum of two contributions

$$\boldsymbol{\mathcal{F}} = \int_V \boldsymbol{f}\, dV - \oint_S p\, d\boldsymbol{S} \ . \qquad (4\text{-}13)$$



A control volume V with its enclosing surface S, a volume element dV and a surface element $d\boldsymbol{S}$.

The first term is for the case of gravity just the weight of the fluid in the volume and the second is the *buoyancy* force. The circle in the symbol for the surface integral is only there to remind us that the surface is closed.

Global hydrostatic equilibrium equation

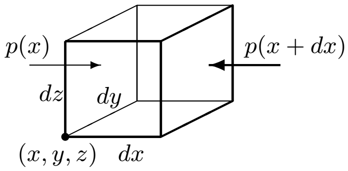
In hydrostatic equilibrium, the total force must vanish for any volume of fluid, $\boldsymbol{\mathcal{F}} = \boldsymbol{0}$, or

$$\boxed{\int_V \boldsymbol{f}\, dV - \oint_S p\, d\boldsymbol{S} = \boldsymbol{0}} \ . \qquad (4\text{-}14)$$

This is the *equation of global hydrostatic equilibrium*, which states that buoyancy must exactly balance the total volume force, *i.e.* the weight. If the cancellation is not exact, as for example when a small volume of water is heated or cooled relative to its surroundings, the fluid *must* start to move, either upwards if it weighs less than the buoyancy force or downwards if it weighs more.

The problem with the global equilibrium equation is that we have to know the fields $\boldsymbol{f}(\boldsymbol{x})$ and $p(\boldsymbol{x})$ in advance to calculate the integrals. Sometimes symmetry considerations can get us a long way. In constant gravity, the sea on the flat Earth ought to have the same properties for all x and y, suggesting that the pressure $p = p(z)$ can only depend on the depth z. This was in fact a tacit assumption used in deriving the pressure in the incompressible sea (4-3), and it is not difficult under this assumption to formally derive the same result from the equation of global equilibrium (4-14). But in general we need to establish a local form of the equations for hydrostatic equilibrium, valid in each point $\boldsymbol{x}$, to find the fields $\boldsymbol{f}(\boldsymbol{x})$ and $p(\boldsymbol{x})$.

Effective force on material particle



Pressure difference over a small rectangular box.

A material particle is like any other body subject to pressure from all sides, but being infinitesimal it is possible to derive a general expression for the resultant force. Let us choose a material particle in the shape of a small rectangular box with sides dx, dy, and dz, and thus a volume $dV = dxdydz$. Since the pressure is slightly different on opposite sides of the box the resultant pressure force is to leading approximation (in the x-direction)

$$d\mathcal{F}_x \approx (p(x,y,z) - p(x+dx,y,z))dydz \approx -\frac{\partial p}{\partial x}\,dxdydz \ .$$

Including the other coordinate directions we obtain

$$d\boldsymbol{\mathcal{F}} = -\boldsymbol{\nabla} p\, dV \ . \tag{4-15}$$

The resultant of all pressure forces acting on a tiny material particle is apparently equivalent to a volume force with a density equal to the negative gradient of the pressure. We shall see below that this result does not depend on the shape of the material particle.

If there is also a true volume force, $\boldsymbol{f}$, for example gravity ($\boldsymbol{f} = \rho\boldsymbol{g}$), acting on the material, the total force on a material particle becomes

$$\boxed{d\boldsymbol{\mathcal{F}} = \boldsymbol{f}^*\, dV \ ,} \tag{4-16}$$

where

$$\boxed{\boldsymbol{f}^* = \boldsymbol{f} - \boldsymbol{\nabla} p} \tag{4-17}$$

is called the *effective force density.*

Local hydrostatic equilibrium

In mechanical (*i.e.* hydrostatic) equilibrium, the total force on an arbitrary body has to vanish. Applying this to all the infinitesimal material particles in the body, it follows that the effective density of force must vanish everywhere, $\boldsymbol{f}^* = \boldsymbol{0}$, or

$$\boxed{\boldsymbol{f} - \boldsymbol{\nabla} p = \boldsymbol{0} \ .} \tag{4-18}$$

This is the *local equation of hydrostatic equilibrium.* It is a differential equation valid everywhere in a fluid at rest, and it encapsulates in an elegant way all the physics of hydrostatics.

The flat earth

Returning to the case of constant gravity in a flat-earth coordinate system we have $\boldsymbol{f} = \rho\boldsymbol{g}_0 = \rho(0,0,-g_0)$ and the local equilibrium equation takes the following form when written out in coordinates,

$$\frac{\partial p}{\partial x} = 0 \ , \tag{4-19a}$$

$$\frac{\partial p}{\partial y} = 0 \ , \tag{4-19b}$$

$$\frac{\partial p}{\partial z} = -\rho g_0 \ . \tag{4-19c}$$

The first two equations tell us that the pressure does not depend on x and y, but only on z. This confirms our previous intuition based on symmetry. It also resolves the hydrostatic paradox because we now know that independently of the shape of the container, the pressure will always be the same at a given depth in constant gravity. For the special case of constant density, $\rho(z) = \rho_0$, the last equation is immediately integrated to yield the previous result (4-3) for the pressure in the incompressible sea.

Gauss' theorem

Integrating the effective force over the volume V we get again the total force

$$\boldsymbol{\mathcal{F}} = \int_V \boldsymbol{f}^* \, dV = \int_V \boldsymbol{f} \, dV - \int_V \boldsymbol{\nabla} p \, dV \ ,$$

and comparing with (4-13) we obtain the relation,

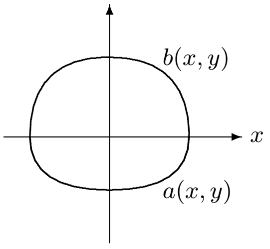
$$\boxed{\oint_S p \, d\boldsymbol{S} = \int_V \boldsymbol{\nabla} p \, dV \ .} \tag{4-20}$$

Since the pressure can be arbitrary, this must be a purely mathematical relationship between a closed surface integral over an arbitrary function $p(\boldsymbol{x})$ and the integral of its gradient $\boldsymbol{\nabla} p(\boldsymbol{x})$ over the enclosed volume. The equation is one version of *Gauss' theorem* which we shall meet in several different guises in the course of this book. Being a mathematical relationship, it is also valid when the volume becomes infinitesimal, confirming that the effective force on a material particle of any shape is $d\boldsymbol{\mathcal{F}} = -\boldsymbol{\nabla} p \, dV$.

Proof: Consider a volume V described by $a(x, y) \leq z \leq b(x, y)$ where $a(x, y)$ and $b(x, y)$ are two arbitrary functions defined in some area A of the xy-plane. We then find,

$$\begin{aligned} \int_V \nabla_z p \, dV &= \int_A dxdy \int_{a(x,y)}^{b(x,y)} \frac{\partial p}{\partial z} \, dz \\ &= \int_A dxdy (p(b(x, y), x, y) - p(a(x, y), x, y)) \\ &= \oint_S p(x, y, z) \, dS_z \ . \end{aligned}$$

Arbitrarily shaped volumes may be put together from such volumes.



A volume described by $a(x, y) \leq z \leq b(x, y)$.

∗ Gradient forces?

The local equation of hydrostatic equilibrium $\boldsymbol{f} = \boldsymbol{\nabla} p$ implies that the curl of the force density must vanish, $\boldsymbol{\nabla} \times \boldsymbol{f} = \boldsymbol{0}$ (see problem 2.14). In a gravitational field a material with constant density $\rho = \rho_0$, the force density $\boldsymbol{f} = \rho_0 \boldsymbol{g} = -\rho_0 \boldsymbol{\nabla}\Phi$ is evidently a gradient field. We shall see in section **??** that under quite general conditions which are often fulfilled the ratio $(\boldsymbol{\nabla} p)/\rho$ will always be a gradient field, even when ρ is not a constant, implying that the equation of hydrostatic equilibrium in a gravitational field, $\boldsymbol{\nabla} p = \rho \boldsymbol{g} = -\rho \boldsymbol{\nabla}\Phi$, can be fulfilled.

But what happens if a force, for example $\boldsymbol{f} = \boldsymbol{a} \times \boldsymbol{x}$, has a manifestly non-vanishing curl, here $\boldsymbol{\nabla} \times \boldsymbol{f} = 2\boldsymbol{a}$? In that case the only possible conclusion is that hydrostatic equilibrium cannot be established, and the fluid must start to move. This would for example happen in an electrically charged fluid with a magnetic field growing linearly with time. Such a magnetic field would generate a time-independent electric field with non-vanishing curl, and that would indeed accelerate the charged molecules, and thus the fluid.

4.3 The equation of state

Mass density, pressure and temperature are thermodynamic quantities that are well-defined as long as we consider spatial volumes containing large numbers of molecules and time intervals much longer than the intervals between molecular collisions. In ordinary equilibrium thermodynamics (see for example [6]), every thermodynamic quantity is assumed to take the same value everywhere in a macroscopic volume. Under very general conditions, there will exist a relation between these quantities,

$$\boxed{f(p, \rho, T) = 0\ ,} \tag{4-21}$$

called the *equation of state*. The actual form of the equation of state depends on the details of the molecular interactions, and the derivation of the equation of state from molecular properties of matter is a subject that falls outside the scope of continuum physics.

In the continuum description of fluids at rest these quantities are allowed to be fields that vary over space, and the equation of state should be understood as a local relation valid in each point of space,

$$f(p(\boldsymbol{x}), \rho(\boldsymbol{x}), T(\boldsymbol{x})) = 0\ . \tag{4-22}$$

The condition of constant density $\rho(\boldsymbol{x}) = \rho_0$ which we used to calculate the pressure in the sea (4-3) may be viewed as an extreme case of such an equation of state.

The ideal gas law

The oldest and most famous equation of state is the *ideal gas law*, which is usually presented in the form (credited to Clapeyron (1834))

$$pV = nRT \ , \tag{4-23}$$

where V is a (small) volume of gas, p its pressure, T its absolute temperature, n the number of moles in the volume, and $R = 8.31451(6)$ J/K/mol the molar gas constant. A gas obeying this law is said to be *ideal* or *perfect*. Using that $\rho = nM_{\mathsf{mol}}/V$ where M_{mol} is the molar mass, we obtain the ideal gas law in the form (4-21),

$$\boxed{\frac{p}{\rho} = \frac{RT}{M_{\mathsf{mol}}}} \ . \tag{4-24}$$

Benoit Paul Émile Clapeyron (1799-1864). *French engineer and physicist. Formulated the ideal gas law from previous work by Boyle, Mariotte, Charles, Gay-Lussac, and others. Contributed to early thermodynamics by developing on Carnot's work. Defined the concept of reversible transformations and formulated the first version of the Second Law of Thermodynamics. Established what is now called Clapeyron's formula for the latent heat in the change of state of a pure substance.*

The ideal gas law is also valid for mixtures of pure gases, such as air, for which one must use the average molar weight $M_{\text{air}} = 28.9635$ g/cm^3. Real air is well described by the ideal gas law, but in precise calculations it is necessary to include non-linear corrections (in p/T or T/V) and other corrections due to humidity (see for example [2]).

In section 4.5 we shall review the basic thermodynamics of ideal gases with constant heat capacity.

Isothermal hydrostatic equilibrium

All materials conduct heat to some extent. A small volume of a fluid at rest is always in thermal contact with the fluid surrounding it and will eventually obtain the same temperature as its environment. If the walls containing a fluid are held at a fixed temperature T_0, all of the fluid will eventually in arrive in *isothermal equilibrium* with this temperature everywhere, $T(\boldsymbol{x}) = T_0$, so that the equation of state simplifies to a relation between pressure and density,

$$f(p, \rho, T_0) = 0 \ . \tag{4-25}$$

A fluid is said to be in a *barotropic* state if its pressure like here is a function of the density alone, $p = p(\rho)$. This is evidently the case for an ideal isothermal gas which obeys $p/\rho = $ const. In section 4.5 we shall meet another barotropic state, the isentropic ideal gas, in which the pressure is proportional to a power of the density, $p \propto \rho^\gamma$.

Isothermal atmosphere

Everybody knows that the atmosphere is not at constant temperature, but if we nevertheless assume it to be, we obtain by combining the equation of hydrostatic equilibrium (4-19) with the ideal gas law (4-24)

$$\frac{dp}{dz} = -\rho g_0 = -\frac{M_{\mathsf{mol}}\, g_0}{RT_0} p \ . \tag{4-26}$$

With the initial condition $p = p_0$ for $z = 0$, this ordinary differential equation has the solution

$$p = p_0 e^{-z/h_0} \ , \tag{4-27}$$

where

$$h_0 = \frac{RT_0}{M_{\mathsf{mol}}\, g_0} = \frac{p_0}{\rho_0 g_0} \ . \tag{4-28}$$

In the last step we have used the ideal gas law at $z = 0$ to replace $RT_0/M_{\mathsf{mol}} = p_0/\rho_0$, making the expression for h_0 exactly the same as the scale height (4-5) of the incompressible atmosphere. In the isothermal atmosphere the pressure decreases exponentially with height with the characteristic length scale again set by h_0, but now the pressure at $h = h_0$ (cirka the top of Mount Everest) is finite and predicted to be 373 hPa, which is a little more than 1/3 of the pressure at sea level. This result is evidently much better than the vanishing pressure predicted by the incompressible model of the atmosphere.

4.4 Compressibility

Liquids tend to be rather incompressible, so that a homogenous liquid body may often be assumed to have constant density everywhere and at all time, $\rho(\boldsymbol{x}, t) = \rho_0$. Gases are on the other hand easily compressed with a density that grows with pressure. We shall now make a quantitative definition of compressibility, and then calculate the pressure in the sea under the assumption that water has constant compressibility.

Bulk modulus

Consider a fixed amount $M = \rho V$ of a barotropic fluid with pressure $p = p(\rho)$. When the fluid is compressed a bit, $\delta V < 0$, the pressure in the fluid must necessarily increase, $\delta p > 0$. If this were not the case, pumping your bicycle would take no effort and the piston would be sucked into the air chamber! Since a larger volume compresses proportionally more for a given pressure increase, we define the *bulk modulus* as the pressure increase δp per *fractional* decrease in volume $-\delta V/V$, or

$$K = \frac{\delta p}{-\delta V/V} = \frac{\delta p}{\delta\rho/\rho} = \rho\frac{dp}{d\rho} \ . \tag{4-29}$$

In the second step we have used the constancy of the mass to eliminate the volume, $V = M/\rho$, and in the last we have used that the fluid is barotropic. The bulk modulus is a measure of *incompressibility*, and the larger it is, the greater is the pressure increase that is needed to obtain a given fractional increase in density, $\delta\rho/\rho$. As a measure of *compressibility* one may take $1/K$.

For an isothermal ideal gas it follows from the equation of state (4-24) with $T = T_0$ that $K = p$. The bulk modulus of the material of the Earth is plotted in fig. 4.1 and is approximately $K \approx 4p$.

Material	K[GPa]
Plain steel	*205*
Nickel	*200*
Copper	*120*
Titanium	*110*
Brass	*100*
Zirconium	*95*
Cast iron	*90*
Aluminium	*70*
Magnesium	*45*
Lead	*13*

Typical values for the bulk modulus of various metals and alloys [3].

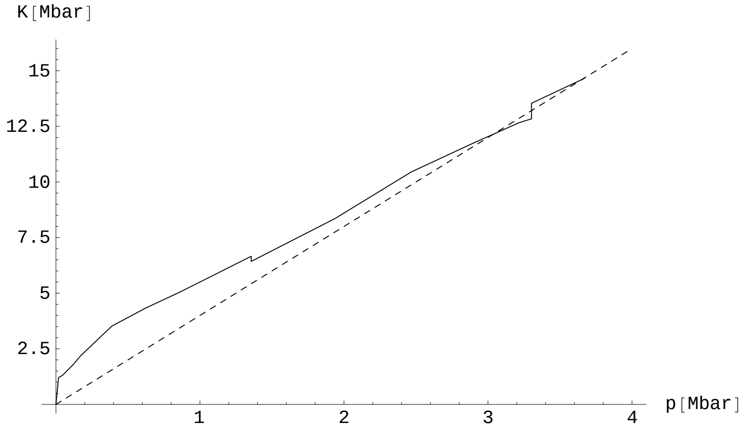


Figure 4.1: *Bulk modulus as a function of pressure in the Earth (data from [2]). The surface of the Earth is to the left and the centre to the right in this figure. The pressure is of course expected to rise as one approaches the centre of the Earth, but interestingly it turns out that the the bulk modulus varies approximately linearly with pressure, $K \approx \gamma p$, with $\gamma \approx 4$ (the dashed line). The dramatic change in density at the core/mantle boundary (see fig. 3.1) is barely visible in the bulk modulus.*

The definition of the bulk modulus shows that it is measured in the same units as pressure. For air at normal pressure we have $K \approx 1$ bar, and for sea water we have $K \approx 23$ kilobar, whereas typical values for metals are of the order of megabars (10^{11} Pa or 100 GPa). This means that it requires about a million atmospheres of pressure to squash iron. Pumping real iron takes more than muscles!

Sea with constant compressibility

The bulk modulus of water varies somewhat with pressure, doubling in value between 1 and 3000 atmospheres [2], but we shall nevertheless assume that it is constant, $K \approx 2.3$ GPa, throughout the sea in order to get an idea of how much sea water is compressed at the bottom. Solving the definition,

$$\rho \frac{dp}{d\rho} = K \tag{4-30}$$

for the pressure, we find $p = K \log \rho + \text{const}$ and inserting this into the hydrostatic equilibrium equation (4-19c), we obtain a differential equation for ρ,

$$\frac{d\rho}{dz} = -\frac{g_0}{K}\rho^2 \ . \tag{4-31}$$

Solving this equation with the initial condition $\rho = \rho_0$ for $z = 0$, we find

$$\frac{1}{\rho} = \frac{1}{\rho_0} + \frac{g_0 z}{K} \ .$$

Rearranging the terms we finally arrive at,

$$\rho = \frac{\rho_0}{1 + \dfrac{z}{h_1}} \ , \tag{4-32a}$$

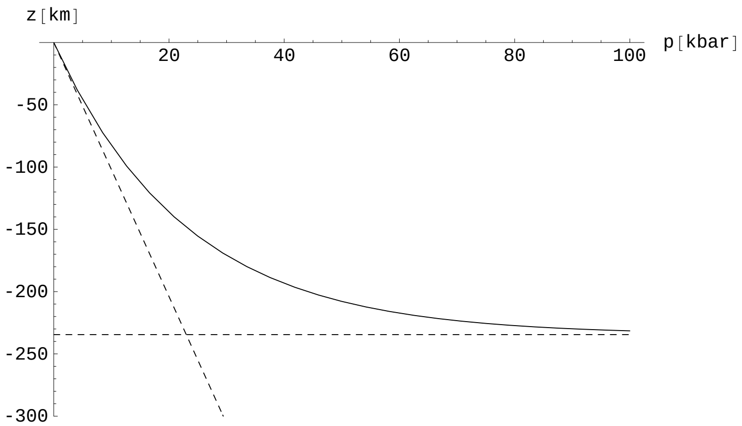


Figure 4.2: *Two models of the flat-earth pressure rise in the sea: constant density (dashed) and constant compressibility (fully drawn). In the latter case, there is an unphysical singularity at* $z = -235$ km.

where

$$h_1 = \frac{K}{\rho_0 g_0} = \frac{K}{p_0} h_0 \ , \tag{4-33}$$

is the characteristic scale for variations in density. The pressure becomes,

$$p = p_0 - K \log\left(1 + \frac{z}{h_1}\right) \ , \tag{4-34}$$

where p_0 is the surface pressure.

The weird thing is that at a depth of $z = -h_1$ both the density and pressure become infinite. The assumption of constant compressibility on a flat Earth apparently implies that the density and pressure will increase and reach an infinite value at a finite depth! This is of course nonsense. Many other properties of water break down long before, so one may use these expressions only as long as the depth is small in comparison with h_1. When $|z| \ll h_1$, we have

$$\rho \approx \rho_0 \left(1 - \frac{z}{h_1}\right) \ . \tag{4-35}$$

For water with $K \approx 23$ kilobar we get $h_1 = 235$ km, which is more than 20 times deeper than any of the seas, so the density of water only increases by at most $1/20 = 5\%$ in the deepest abyss. The pressure only increases by 4 atm (problem 4.4).

Example 4.4.1: A flat Earth made of cast iron with density 7.2 g/cm^3 would become infinite at a depth of 1275 km which is somewhere in the mantle. For all it is worth (since the material of the Earth does not have constant compressibility, fig. 4.1), the singularity disappears for a spherical Earth (problem 4.4).

4.5 Thermodynamics of ideal gases at rest

In the beginning of this chapter we analyzed a simple model of an ideal gas consisting of essentially non-interacting molecules. The pressure was estimated to be $p \sim \rho v^2$ where ρ is the mass density and v the typical molecular speed. The equation of state for the ideal gas (4-24) shows that $T \sim p/\rho \sim v^2$, or conversely that $v \propto \sqrt{T}$, so that the molecular speed only depends on the temperature and not on pressure or density. In this section we shall draw the consequences of this result and along the way recapitulate the conventional thermodynamics of ideal gases. For extensive treatments, see for example [68, 67].

Internal energy and heat capacity

Consider an amount of an ideal gas with a fixed number of molecules (and thus a fixed mass). Although the gas is macroscopically at rest, it contains an *internal energy* $\mathcal{U}$ due to the random molecular motion. Since the molecules do not interact, the internal energy is simply the sum of all the molecular kinetic energies, $\mathcal{U} = \sum \frac{1}{2} m v^2$, which is proportional to T, *i.e.*

$$\boxed{\mathcal{U} = C_V T \ ,} \tag{4-36}$$

where the constant of proportionality, C_V, is independent of the volume and the temperature. To see what this constant means we change the temperature of the gas by δT while keeping its volume fixed. Since no work is performed, and since energy is conserved, the necessary amount of heat is $\delta Q = \delta\mathcal{U} = C_V \delta T$. Consequently, the amount of heat which must be added per unit of temperature increase is $C_V = \delta Q/\delta T$, naturally called the *heat capacity at constant volume*[1].

If the pressure of the gas instead is kept constant while the temperature is raised by δT, we must also take into account that the volume expands by a certain amount δV and thereby performs work on the surroundings. Using the constancy of the pressure, the work performed by the gas becomes $p\delta V = \delta(pV) = nR\delta T$, where we in the last step used the ideal gas law (4-23). Since energy is conserved, the necessary amount of heat is now larger by this work, $\delta Q = \delta U + p\delta V = (C_V + nR)\delta T$, where we have also used (4-36). Consequently, the amount of heat which must be added per unit of increase in temperature, $C_p = \delta Q/\delta T$, is

$$\boxed{C_p = C_V + nR \ ,} \tag{4-37}$$

called the *heat capacity at constant pressure*. It is evidently, and for good reason, always larger than C_V.

---

[1] From the ideal gas law itself, it only follows by thermodynamical reasoning that $C_V = (\partial U/\partial T)_V$ is independent of the volume (see [68, p. 154]. The complete constancy of the heat capacity is an extra assumption which is well fulfilled for real gases under normal circumstances.

The adiabatic index

The dimensionless ratio of the heat capacities,

$$\gamma = \frac{C_p}{C_V} \ , \tag{4-38}$$

is called the *adiabatic index*. Using (4-37) we find

$$C_V = \frac{1}{\gamma - 1} nR \ , \qquad\qquad C_p = \frac{\gamma}{\gamma - 1} nR \ . \tag{4-39}$$

Given the adiabatic index, all thermodynamic quantities for n moles of an ideal gas are completely determined.

> Kinetic gas theory teaches us that $\gamma = 1 + 2/m$ where m is the number of motional degrees of freedom of the molecules. For monatomic gases the only degrees of freedom are the center of mass velocity, implying $m = 3$ and $\gamma = 5/3$. For diatomic gases such as air there are two extra rotational degrees of freedom for rotations orthogonal to the molecular axis (rotations around the molecular axis are not excited at normal temperatures), so that $m = 5$ and $\gamma = 7/5$. Finally, for multiatomic gases there are three extra rotational degrees of freedom, leading to $m = 6$ and $\gamma = 4/3$. These values agree decently with empirical results.

Entropy of an ideal gas

In general, when neither the volume nor the pressure are kept constant, the heat that must be added to the system in an infinitesimal process is,

$$\delta Q = \delta U + p\delta V = C_V \delta T + nRT \frac{\delta V}{V} \ . \tag{4-40}$$

It is a mathematical fact that there exists no function, $Q(T, V)$, which has this expression as differential (see problem 4.11). It may on the other hand be directly verified (by insertion) that

$$\delta \mathcal{S} = \frac{\delta Q}{T} = C_V \frac{\delta T}{T} + nR \frac{\delta V}{V} \ , \tag{4-41}$$

can be integrated to yield a function,

$$\boxed{\mathcal{S} = C_V \log T + nR \log V + \text{const} \ ,} \tag{4-42}$$

called the *entropy* of the amount of ideal gas. The entropy is only defined up to an arbitrary constant.

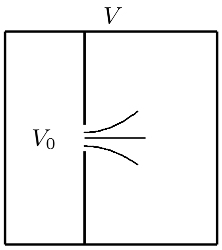
The entropy of the gas is, like its energy, an abstract quantity which cannot be directly measured. But since both quantities depend on the measurable thermodynamic quantities, ρ, p, and T, that characterize the state of the gas, we can calculate the value of energy and entropy in any state. But why bother to do so at all?

The reason is that the two major laws of thermodynamics are formulated for the energy and entropy. Both laws concern processes that may take place in an *isolated* system which is not allowed to exchange heat with the environment or perform work on it. The *First Law* states that the energy is unchanged under any process in an isolated system. This implies that the energy of an open system can only change by exchange of heat or work with the environment. We actually used this law implicitly in deriving the heat capacities and the entropy. The *Second Law* states that the entropy cannot decrease. In the real world, the entropy of an isolated system must in fact grow. Only if all the processes taking place in the system are completely *reversible* at all times, will the entropy stay constant. Reversibility is an ideal which can only be approached by very slow *quasistatic* processes, consisting of infinitely many infinitesimal reversible steps. Essentially all real-world processes are *irreversible* to some degree.

James Prescott Joule (1818-1889). *English physicist. Gifted experimenter who as the first demonstrated the equivalence of mechanical work and heat, a necessary step on the road to the First Law. Demonstrated (in continuation of earlier experiments by Gay-Lussac) that the irreversible expansion of a gas into vacuum does not change its temperature.*

Example 4.5.1 (Joule's expansion experiment): An isolated box of volume V contains an ideal gas in a walled-off compartment of volume V_0. When the wall is opened, the gas expands into vacuum and fills the full volume V. Since the box is isolated from the environment, and since the internal energy only depends on the temperature, it follows from the First Law that the temperature must be the same before and after the event. The change in entropy then becomes $\Delta\mathcal{S} = (C_V \log T + nR \log V) - (C_V T + nR \log V_0) = nR \log(V/V_0)$ which is evidently positive (because $V/V_0 > 1$). This result agrees with the Second Law, which thus appears to be unnecessary.

The strength of the Second Law becomes apparent when we ask the question of whether the air in the box could ever — perhaps by an unknown process to be discovered in the far future — by itself enter the compartment of volume V_0, leaving vacuum in the box around. Since that would entail a negative change in entropy, which is forbidden by the Second Law, it never happens. This is also why a scrambled egg never spontaneously unscrambles again.



A compartment of volume V_0 inside an isolated box of volume V. Initially, there is vacuum in the box, and the compartment contains an ideal gas. When the wall breaks, the gas expands by itself to fill the whole box. The reverse process never happens by itself.

Isentropic processes

Any process in an open system which does not exchange heat with the environment is said to be *adiabatic*. If the process is furthermore reversible, it follows that $\delta Q = 0$ in each infinitesimal step, so that the $\delta\mathcal{S} = \delta Q/T = 0$. The entropy (4-42) must in other words stay constant in any reversible, adiabatic process. Such a process is for this reason called *isentropic*.

By means of the adiabatic index (4-38) we may write the entropy (4-42) as,

$$\mathcal{S} = C_V \log\left(TV^{\gamma-1}\right) + \text{const} . \tag{4-43}$$

From this it follows that

$$T\,V^{\gamma-1} = \text{const} , \tag{4-44}$$

for any isentropic process in an ideal gas. Using the ideal gas law to eliminate $V \sim T/p$, this may be written equivalently as,

$$T^{\gamma} p^{1-\gamma} = \text{const} . \tag{4-45}$$

Eliminating instead $T \sim pV$, the isentropic condition takes its most common form,

$$\boxed{p\,V^\gamma = \text{const}\ .} \tag{4-46}$$

Notice that the constants are different in these three equations.

> **Example 4.5.2:** When the air in a bicycle pump is compressed from V_0 to V_1 (while you block the valve with your finger), the adiabatic law implies that $p_1 V_1^\gamma = p_0 V_0^\gamma$. For $p_0 = 1\ \mathsf{atm}$ and $V_1 = V_0/2$ we find $p_1 = 2.6\ \mathsf{atm}$. The temperature simultaneously rises about 100 degrees, but the hot air quickly becomes cold again during the backstroke. One may wonder why the fairly rapid compression stroke may be assumed to be reversible, but as long as the speed of the piston is much smaller than the velocity of sound, this is in fact a reasonable assumption. Conversely, we may conclude that the air expands with velocity close to the speed of sound when the wall is opened in example 4.5.1.

Isothermal versus isentropic bulk modulus

We have formerly seen that the bulk modulus of a strictly isothermal ideal gas with $p = \rho R T_0 / M_{\mathsf{mol}}$ is equal to the pressure,

$$K_T = \rho \left(\frac{\partial p}{\partial \rho}\right)_T = p\ . \tag{4-47}$$

Here the index T (in the usual thermodynamic way of writing derivatives) signals that the temperature must be held constant while we differentiate.

In terms of the mass density $\rho = M/V$, the isentropic condition may be written in any of three different ways (with three different constants),

$$\boxed{p\,\rho^{-\gamma} = \text{const}\ , \qquad T\rho^{1-\gamma} = \text{const}\ , \qquad T^\gamma p^{1-\gamma} = \text{const}\ .} \tag{4-48}$$

Using the first we find the isentropic bulk modulus of an ideal gas,

$$K_S = \rho \left(\frac{\partial p}{\partial \rho}\right)_S = \gamma\,p\ , \tag{4-49}$$

where the index S now signals that the entropy must be held constant. The distinction between the isothermal and isentropic bulk modulus is necessary in all materials, but for nearly incompressible liquids there is not a great difference between K_S and K_T.

> Among Isaac Newton's great achievements was the first calculation of the speed of sound in air, using essentially the ideal gas law with constant temperature. His result did not agree with experiment, because normal sound waves oscillate so rapidly that compression and expansion are essentially isentropic processes. In section 15.2 we shall find that the speed of sound is $c = \sqrt{K/\rho}$, such that the ratio between the isentropic and isothermal sound velocities is $c_S/c_T = \sqrt{\gamma}$. For air with $\gamma \approx 1.4$ this amounts to an 18% error in the sound velocity. Much later in 1799, Laplace derived the correct value for the speed of sound.

Specific quantities

In classical thermodynamics we always think of a macroscopic volume of matter with the same thermodynamic properties throughout the volume. Volume, mass, energy, entropy, and the specific heats are all *extensive* quantities, meaning that the amount of any such quantity in a composite system is the sum of the amounts in the parts. Pressure, temperature, and density are in contrast *intensive* quantities, that may not be added when a system is put together from its parts.

In continuum physics, an intensive quantity becomes a field that may vary from place to place, whereas an extensive quantity becomes an integral over the density of the quantity. Since a material particle with a fixed number of molecules has a fixed mass (subject to the reservations set down in chapter 1), the natural field to introduce for an extensive quantity like the energy is the *specific internal energy*, $u = d\mathcal{U}/dM$, which is the amount of energy per unit of mass in the neighborhood of a given point. The actual energy density becomes $d\mathcal{U}/dV = \rho\, u$, and the total energy in a volume

$$\mathcal{U} = \int_V \rho\, u \, dV \ . \tag{4-50}$$

The specific energy is an intensive quantity like temperature, pressure or density.

Similarly, we define the *specific heat* as the local heat capacity per unit of mass. Since the heat capacities (4-39) of an ideal gas are directly proportional to the mass $M = nM_{\mathsf{mol}}$, the specific heats of an ideal gas become,

$$c_V = \frac{1}{\gamma - 1}\frac{R}{M_{\mathsf{mol}}} \ , \qquad\qquad c_p = \frac{\gamma}{\gamma - 1}\frac{R}{M_{\mathsf{mol}}} \ . \tag{4-51}$$

They are constants which only depend on the properties of the gas. For air we have $c_V = 718$ J/K/kg and $c_p = 1005$ J/K/kg. From (4-36) we obtain after dividing by M,

$$u = c_V T \ . \tag{4-52}$$

The specific energy of an ideal gas is the specific heat times the absolute temperature.

Finally, we define the *specific entropy*, $s = d\mathcal{S}/dM$, from which the total entropy may be calculated as an integral,

$$\mathcal{S} = \int_V \rho\, s \, dV \ . \tag{4-53}$$

For an ideal gas, the specific entropy is obtained by dividing (4-43) by $M = nM_{\mathsf{mol}}$. It may be written in three different forms related by the ideal gas law,

$$s = c_V \log\left(T\rho^{1-\gamma}\right) + \text{const} \tag{4-54a}$$

$$= c_V \log\left(T^{\gamma} p^{1-\gamma}\right) + \text{const} \tag{4-54b}$$

$$= c_V \log\left(p\,\rho^{-\gamma}\right) + \text{const} \ . \tag{4-54c}$$

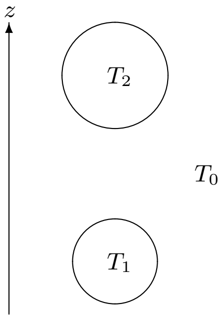
Notice, however, that the constants are different in the three cases.

4.6 The homentropic atmosphere

The assumption that the temperature is the same everywhere in the atmosphere is certainly wrong, as anyone who has ever flown in a modern passenger jet can testify. Temperature falls with height instead of staying constant. So the atmosphere is *not* in isothermal equilibrium, and this is perhaps not so surprising, since the "container walls" of the atmosphere, the ground and outer space, are not at the same temperature. There must be a heat flow through the atmosphere between the ground and outer space, maintained by an inflow of solar radiation and an outflow of geothermal energy. But air is a bad conductor of heat, so although heat conduction does play a role, it is not directly the cause of the temperature drop in the atmosphere.

Of much greater importance are the indirect effect of solar heating, the *convection* which creates air currents, winds, and local turbulence, continually mixing different layers of the atmosphere. The lower part of the atmosphere, the *troposphere*, is quite unruly and vertical mixing happens at time scales that are much shorter than the time scales necessary for reaching isothermal equilibrium. There is in fact no true hydrostatic equilibrium state for the real atmosphere. Even if we disregard large-scale winds and weather systems, horizontal and vertical mixing always takes place at small scales, and a realistic model of the atmosphere must take this into account.

A thought-experiment



Swapping air masses from different heights. If the air has temperature T_0 before the swap, the swapped air would be warmer $T_1 > T_0$ or colder $T_2 < T_0$.

Let us imagine that we take a small blob of air and exchange it with another blob of air of the same mass, but taken from a different height with a different volume and pressure. In order to fill out the correct volume, one air mass would have to be compressed and the other expanded. If this is done quickly, there will be no time for heat exchange with the surrounding air, and one air mass will consequently be heated up by isentropic compression and the other cooled down by isentropic expansion. If the atmosphere initially were in isothermal equilibrium, the temperature of the swapped air would not be the same as the temperature of the surrounding air, and the atmosphere would be brought out of equilibrium.

If, however, the surrounding air initially had a temperature distribution, such that the swapped air after the expansion and compression would arrive at the correct temperatures of the new surroundings, a kind of thermodynamic "equilibrium" could be established, in which the omnipresent local mixing had essentially no effect. Intuitively, it is reasonable to expect that the end result of fast local mixing and slow heat conduction might be precisely such a state. It should however not be forgotten that this state is not a true equilibrium state but rather a dynamically balanced state depending on the incessant motion in the atmosphere.

In this state, the movement of a small blob followed by compression or expansion is essentially a reversible, adiabatic process. This implies that the local entropy per unit of mass, the *specific entropy*, has to be the same throughout

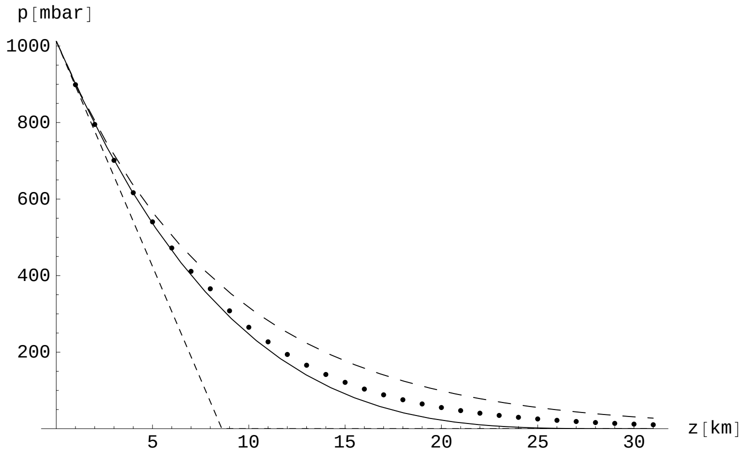


Figure 4.3: *Three different models for the atmospheric pressure: constant density (dashed), homentropic (fully drawn) and isothermal (large dashes), plotted together with the standard atmosphere data (dots)[3]. The parameters are $h_0 = 8.6$ km and $\gamma = 7/5$.*

the atmosphere. A gas in a state with constant specific entropy is said to be *homentropic*. In this state the right hand side any of the isentropic conditions (4-48) or (4-45) will be independent of the spatial position.

The temperature lapse rate

In order to calculate the pressure distribution in the homentropic atmosphere we differentiate the specific entropy (4-54)b after z, and find

$$\gamma \frac{1}{T}\frac{dT}{dz} + (1-\gamma)\frac{1}{p}\frac{dp}{dz} = 0 \ .$$

Combined with the hydrostatic equation (4-19c), we obtain

$$\frac{dT}{dz} = \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{dp}{dz} = -\left(1 - \frac{1}{\gamma}\right)\frac{g_0 M_{\mathsf{mol}}}{R} \ , \tag{4-55}$$

which by (4-51) may be written,

$$\boxed{\frac{dT}{dz} = -\frac{g_0}{c_p}} \ . \tag{4-56}$$

The great surprise is that the right hand side is independent of z. This means that the temperature in the homentropic atmosphere drops at a constant rate, also called the *atmospheric temperature lapse rate*.

With $M_{\mathsf{mol}} \approx 29$ g/mol and $\gamma \approx 7/5$ the numerical value of the temperature lapse rate becomes $dT/dz \approx 10$ K/km. Thus the temperature of the homentropic atmosphere will drop from, say, $+27^\circ$ centigrade at sea level to -60° at the top of Mount Everest.

Introducing the temperature T_0 at sea level, the above equation may be integrated, and the temperature at height z becomes

$$T(z) = T_0 \left(1 - \frac{z}{h_2}\right) , \tag{4-57}$$

where the scale is set by

$$h_2 = \frac{\gamma}{\gamma - 1} \frac{RT_0}{g_0 M_{\mathsf{mol}}} = \frac{\gamma}{\gamma - 1} h_0 . \tag{4-58}$$

At $z = h_2 \approx 30$ km the temperature has dropped to absolute zero, which is of course unphysical, since the atmosphere does not stop at this height. It is nevertheless a reasonable scale for the height of the atmosphere.

The pressure follows from (4-45)

$$p = p_0 \left(1 - \frac{z}{h_2}\right)^{\frac{\gamma}{\gamma-1}} , \tag{4-59}$$

and the density from (4-48)

$$\rho = \rho_0 \left(1 - \frac{z}{h_2}\right)^{\frac{1}{\gamma-1}} . \tag{4-60}$$

Both of these quantities vanish like the temperature for $z = h_2$. At the top of Mt. Everest the pressure is predicted to be 437 hPa.

Earth's atmosphere

In fig. 4.3 the various atmospheric models have been plotted together with the data for the standard atmosphere [3]. Even if the isentropic model gives the best fit, it fails at higher altitudes because some of the physical assumptions behind the derivation fail. The real atmosphere is in fact much more complicated than any of these models.

Standard quoted pressure at Mount Everest is 300 hPa. *Average temperature is* −36°C *in January but can fall to* −60°C. *In the summer it is* −19°C.

Water vapor is always present and will condense to clouds in rising currents of air. The latent heat of condensation heats up the air, so that the temperature lapse rate becomes smaller than 10 K/km, perhaps more like $6 - 7$ K/km, leading to a somewhat higher temperature at the top of Mount Everest. The clouds may eventually precipitate out as rain, and when the dried air afterwards descends again, for example on the lee side of a mountain, the air will heat up at a higher rate than it cooled during its ascent on the windward side and become quite hot, a phenomenon known as *föhn* in the Alps.

The fact that the temperature lapse rate is smaller in the real atmosphere than in the isentropic model has a bearing on the stability of the atmosphere. If a certain amount of air is transported to higher altitude without heat exchange and condensation of water vapor, it will behave like in the isentropic model and become cooler than the surrounding air. Consequently it will also be heavier than the surrounding air and tend to sink back to where it came from. Conversely, if the real temperature lapse becomes larger than in the isentropic model, the atmosphere becomes unstable and strong vertical currents may arise. This can, for example, happen in thunderstorms.

Problems

4.1 Consider a canal with a dock gate which is 12 m wide and has water depth 9 m on one side and 6 m on the other side.

a) Calculate the pressures in the water on both sides of the gate at a height z over the bottom of the canal.
b) Calculate the total force on the gate.
c) Calculate the total moment of force around the bottom of the gate.
d) Calculate the height over the bottom at which the total force acts.

4.2 Show that the pressure (4-3) can be derived from the equation of global equilibrium (4-14) under the assumption that $p = p(z)$.

4.3 An underwater lamp is covered by a hemispherical glass with a diameter of 30 cm and is placed at a depth of 3 m on the side of the pool. Calculate the total horizontal force from the water on the lamp, when there is air at normal pressure inside.

4.4 Calculate the change in pressure in the deepest sea, assuming a constant bulk modulus of water, $K = 2.3$ GPa.

4.4 Expanding the pressure (4-34) to second order in z/h_1 we find

$$p = p_0 \left(1 - \frac{z}{h_0} + \frac{z^2}{h_0 h_1}\right) . \qquad (4\text{-}61)$$

For $z = -11$ km and taking $h_0 = 10$ m and $h_1 = 235$ km, the last term becomes about 4 atm.

4.5 The equation of state for water is to a good approximation (for pressures up to 100,000 bar) given by

$$\frac{p + B}{p_0 + B} = \left(\frac{\rho}{\rho_0}\right)^n \qquad (4\text{-}62)$$

with $B = 3000$ bar, $n = 7$, $p_0 = 1$ bar and $\rho_0 = 1$ g/cm^3. a) Calculate the bulk modulus K for water. b) Calculate the density and pressure in the sea. c) What is the compression of the water at the deepest point in the sea?

4.6 Determine the form of the pressure across the core/mantle boundary when the bulk modulus is $K \approx \gamma p$ with $\gamma \approx 4$ throughout the Earth (see fig. 4.1).

4.7 Using a manometer, the pressure in an open container with liquid is found to be 1.6 bar at a height of 6 m over the bottom, and 2.8 bar at a height of 3 m. Determine a) the density of the liquid and b) the height of the liquid surface.

4.8 An open jar contains two non-mixable liquids with densities $\rho_1 > \rho_2$. The heavy layer has thickness h_1 and the light layer on top of it has thickness h_2. a) An open glass tube is lowered vertically into the liquids towards the bottom of the jar. Describe how high the liquids rise in the tube (disregarding capillary effects). b) The open tube is already placed in the container with its opening close to the bottom when the heavy fluid is poured in, followed by the light. How high will the heavy fluid rise in the tube?

4.9 A vertical plate is inserted into a liquid at rest with constant density ρ_0 in constant gravity g_0. Introduce a coordinate z going vertically down with the pressure defined to vanish for $z = 0$. In the following we denote the vertical area moments by,

$$I_n = \frac{1}{A}\int_A z^n \, dS \ , \qquad\qquad n = 1, 2, \dots \tag{4-63}$$

where dS is the surface element. The point $z_M = I_1$ is called the area center.

a) Calculate the pressure in the liquid.

b) Show that $I_2 \geq I_1^2$.

c) Calculate the total pressure force on the plate.

d) Calculate the total moment of force of the pressure forces around $z = 0$.

e) Show that the point of attack of the pressure forces is found below the area center $z_P \geq z_M$.

f) A thin isosceles triangle with height h and bottom length b is lowered into the liquid such that its top point is at $z = 0$. Calculate the area center and the point of attack of the pressure forces.

4.10 The equation of state due to van der Waals is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \tag{4-64}$$

where a and b are constants. It describes gases and their condensation into liquids. a) Calculate the isothermal bulk modulus. b) Under which conditions can it become negative, and what does it mean?

4.11 a) Show that for a function $Q = Q(T, V)$ the differential takes the form $dQ = AdT + BdV$ where $\partial A/\partial V = \partial B/\partial T$. b) Prove that this is not fulfilled for (4-40).