2 Space and time

In classical Newtonian physics space is absolute and eternal, obeying the rules of Euclidean geometry everywhere. It is the perfect stage on which all physical phenomena play out. Time is equally uniform and absolute all over space from the beginning to the end, and matter has no influence on the properties of space and time. Rulers to measure length never stretch, clocks to measure time never lose a tick, and both can be put to work anywhere from the deepest levels of matter to the farthest reaches of outer space. It is a clockwork universe, orderly, rigorous, and deterministic.

This semblance of perfection was shattered in the beginning of the twentieth century by the theories of relativity and quantum mechanics. Space and time became totally intertwined with each other and with matter, and the determinism of classical physics was replaced by the still-disturbing quantum indeterminism.

Relativity and quantum mechanics are both theories of extremes. Although they in principle apply to the bulk of all physical phenomena, their special features become dominant only at velocities approaching the velocity of light in the case of relativity, or length scales approaching the size of atoms in the case of quantum mechanics. Newtonian space and time remains a valid, if not "true", conceptual framework over the vast ranges of length and velocity scales covered by classical continuum physics.

In this chapter the basic ideas behind space and time are introduced in a way which emphasizes the operational aspects of physical concepts. Care is exerted in order that the concepts defined here should remain valid in more advanced theories. A certain familiarity with Euclidean geometry in Cartesian coordinates is assumed, and the chapter serves in most respects to define the mathematical notation to be used in the remainder of the book. It may be sampled at leisure, as the need arises.

2.1 Reference frames

Physics is a quantitative discipline using mathematics to relate measurable quantities expressed in terms of real numbers. In formulating the laws of nature, undefined mathematical primitives — for example the points, lines and circles of Euclidean geometry — are not particularly useful, and such concepts have for this reason been eliminated and replaced by numerical representations everywhere in physics. This necessitates a specification of the practical procedures by which these numbers are obtained in an experiment, for example, what units are being used.

Behind every law of nature and every formula in physics, there is a framework of procedural descriptions, a *reference frame*, supplying an operational meaning to all quantities. Part of the art of doing physics lies in comprehending this — often tacitly understood — infrastructure to the mathematical formalism. The reference frame always involves physical objects — balances to measure mass, clocks to measure time, and rulers to measure length — that are not directly a part of the mathematical formalism. Precisely because they *are* physical objects, they can at least in principle be handed over or copied, and thereby shared among experimenters. This is what is really meant by the objectivity of physics.

The system of units, the *Système Internationale (SI)*, is today fixed by international agreement. But even if our common frame of reference is thus defined by social convention, physics is nevertheless objective. In principle our frames of reference could be shared with any other being in the universe.

The unit of mass, the kilogram, is defined by a prototype stored by the *International Bureau of Weights and Measures* near Paris, France. Copies of this prototype and balances for weighing them can be made to a precision of one part in 10^9 [2].

2.2 Time

Time is the number you read on your *clock*. There is no better definition. Clocks are physical objects which may be shared, compared, copied, and synchronized to create an objective meaning of *time*. Most clocks, whether they are grandfather clocks based on a swinging pendulum or oscillating quartz crystals, are based on periodic physical systems that return to the same state again and again, and time intervals are simply measured by counting periods. There are also aperiodic clocks, for example hour glasses, and clocks based on radioactive elements. It is especially the latter that allow time to be measured on geological scales. Beyond that, the concept of time becomes increasingly more theory-laden.

Like all macroscopic physical systems, clocks are subject to small fluctuations in the way they run. The most stable clocks are those that keep time best with respect to copies of themselves as well as with clocks built on other principles. Grandfather clocks are much less stable than maritime chronometers that in turn are less stable than quartz clocks. The international frame of reference for time is always based on the currently most stable clocks.

> Formerly the unit of time, the *second*, was defined as 1/86,400 of a mean solar day, but the Earth's rotation is not that stable, and since 1966 the second has been defined by international agreement as the duration of 9,192,631,770 oscillations of the microwave radiation absorbed in a certain hyperfine transition in the cesium-133 atom. A beam of cesium-133 atoms is used to stabilize a quartz oscillator at the right frequency by a resonance method, so what we call an atomic clock is really an atomically stabilized quartz clock. The intrinsic relative precision in this time standard is about 4×10^{-14}, or about one second in a million years [2].

In the extreme mathematical limit, time may be assumed to be a real number, say t, and in Newtonian physics its value is assumed to be universally known.

2.3 Space

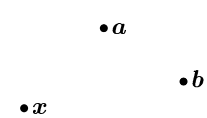
It is a mysterious and so far unexplained fact that physical *space* has three dimensions, which means that it takes exactly three real numbers — say x_1, x_2 and x_3 — to locate a point in space. These numbers are called the *coordinates* of the point, and the reference frame for coordinates is called the *coordinate system*. It must contain all the operational specifications for locating a point given the coordinates, and conversely obtaining the coordinates given the location. In this way we have relegated all philosophical questions regarding the *real* nature of points and of space to the operational procedures contained in the reference frame.

> On Earth everybody navigates by means of a geographical system, in which a point is characterized by latitude, longitude and height. The geographical coordinate system is based on agreed-upon fixed points on Earth: the north pole, Greenwich near London, and the average sea level. The modern Global Positioning System uses "fixed" points in the sky in the form of satellites, and the coordinates of any point on Earth is determined from differences in the time-of-flight of radio signals.

It is convenient to collect the coordinates x_1, x_2, and x_3 of a point in a single object, a triplet of real numbers

$$\boldsymbol{x} = (x_1, x_2, x_3) \ , \tag{2-1}$$

called the *position* of the point in the coordinate system [1]. The triplet notation is just a notational convenience, so a function of the position $f(\boldsymbol{x})$ is completely equivalent to a function of the three coordinates $f(x_1, x_2, x_3)$.



Points may be visualized as dots on a piece of paper.

[1] In almost all modern textbooks it is customary to use boldface notation for triplets of real numbers ("vectors") and we shall also do so here. In calculations with pencil on paper many different notations are used to distinguish such a symbol from other uses, for example a bar ($\overline{x}$), an arrow ($\vec{x}$), or underlining ($\underline{x}$).

There is nothing sacred about the names of the coordinates. In physics and especially in practical calculations, the coordinate variables are often renamed $x_1 \to x$, $x_2 \to y$ and $x_3 \to z$, so that the general point becomes $\boldsymbol{x} \to (x, y, z)$. It is also customary to write $\boldsymbol{a} = (a_x, a_y, a_z)$ for a general triplet, with the coordinate labels used as indices instead of 1, 2 and 3. It is of course of no importance whether the range of indices is labeled x, y, z or 1, 2, or 3 or something else, as long as there are three of them.

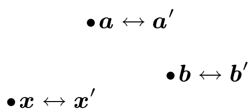
Coordinate transformations

Having located a point by a set of coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ in one coordinate system, the coordinates $\boldsymbol{x}' = (x'_1, x'_2, x'_3)$ of the exact same point in another coordinate system must be calculable from the first

$$
\begin{aligned}
x'_1 &= f_1(x_1, x_2, x_3) \ , \\
x'_2 &= f_2(x_1, x_2, x_3) \ , \\
x'_3 &= f_3(x_1, x_2, x_3) \ ,
\end{aligned}
$$

In triplet notation this is written

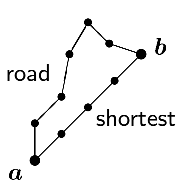
$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}) \ . \tag{2-2}$$



In different coordinate systems the same points have different coordinates.

This postulate reflects that physical reality is unique and that different coordinate systems are just different ways of representing the same physical space in terms of real numbers. Conversely, every such one-to-one mapping of the coordinates defines another coordinate system. The study of *coordinate transformations* is central to analytic geometry and permits characterization of geometric quantities by the way they transform rather than in abstract terms (see section 2.5).

Length

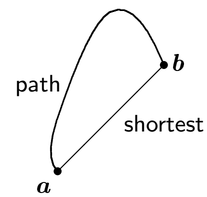


The length of the road between $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is measured by counting steps along the road. The distance is the length of the shortest road.*

From the earliest times humans have measured the *length* of a road between two points, say $\boldsymbol{a}$ and $\boldsymbol{b}$, by counting the number of steps it takes to walk along this road. In order to communicate to others the length of a road, the count of steps must be accompanied by a clear definition of a step, for example in terms of an agreed-upon unit of length.

Originally the units of length — inch, foot, span, and fathom — were directly related to the human body, but increasing precision in technology demanded better-defined units. In 1793 the meter was introduced as a ten millionth of the distance from equator to pole on Earth, and until far into the twentieth century a unique "normal meter" was stored in Paris. Later the meter became defined as a certain number of wave lengths of a certain spectral line in krypton-86, an isotope of a noble gas which can be found anywhere on Earth. Since 1983 the meter has been defined by international convention to be the distance traveled by light in exactly 1/299,792,458 of a second [2]. The problem of measuring lengths has thus been transferred to the problem of measuring time, which makes sense because the precision of the time standard is at least a thousand times better than any proper length standard.

This method for determining length may be refined to any desired practical precision by using very short steps. In the extreme mathematical limit, the steps become infinitesimally small, and the road becomes a continuous path. The shortest such path is called a *geodesic* and represents the "straightest line" between the points. Airplanes and ships travel along geodesics, *i.e.* great circles on the spherical surface of the Earth.



In the mathematical limit the shortest continuous path connecting $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is called the geodesic ("straight line").*

Distance

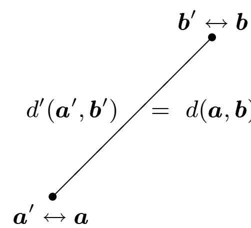
The *distance* between to points is defined to be the length of the shortest path between the points. Since the points are completely defined by their coordinates, $\boldsymbol{a}$ and $\boldsymbol{b}$ (relative to the coordinate system), the distance must be simply a real function $d(\boldsymbol{a}, \boldsymbol{b})$ of the two sets of coordinates. This function is not completely general; the definition of length in terms of numbers of steps implies that certain inequalities are fulfilled by the distance function.

From the definition it is clear that the distance between two points must be the same in all coordinate systems, because it can in principle be determined by laying out rulers between points without any reference to coordinate systems. The actual distance function $d'(\boldsymbol{a}', \boldsymbol{b}')$ in a new coordinate system may be different from the old, $d(\boldsymbol{a}, \boldsymbol{b})$, but the numerical value must be the same

$$d'(\boldsymbol{a}', \boldsymbol{b}') = d(\boldsymbol{a}, \boldsymbol{b}) \; , \tag{2-3}$$

where $\boldsymbol{a}' = \boldsymbol{f}(\boldsymbol{a})$ and $\boldsymbol{b}' = \boldsymbol{f}(\boldsymbol{b})$ are calculated by the coordinate transformation (2-2). Knowing the distance function $d(\boldsymbol{a}, \boldsymbol{b})$ in one coordinate system, it may be calculated in any other coordinate system by means of the appropriate coordinate transformation.

This expresses the *invariance of the distance* under all coordinate transformations. In the same way as (2-2) may be viewed mathematically as a definition of what is meant by "another coordinate system", the equation (2-3) may be viewed mathematically as a definition of what is meant by distance in an arbitrary other coordinate system.



Invariance of the distance.

Cartesian coordinate systems

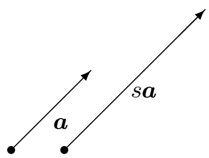
It is another fundamental physical fact that it is possible (within limited regions of space and time) to construct coordinate systems, in which the distance between any two points, $\boldsymbol{a}$ and $\boldsymbol{b}$, is given by the expression

$$d(\boldsymbol{a}, \boldsymbol{b}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2} \; . \tag{2-4}$$

Such coordinate systems are called *Cartesian* and were first analyzed by Descartes. The distance function implies that space is *Euclidean* and therefore has all the properties one learns about in elementary geometry.

René Descartes or Renatus Cartesius (1596–1650). *French scientist and philosopher, father of analytic geometry. Developed a theory of mechanical philosophy, later to be superseded by Newton's work. Confronted with doubts about reality, he saw thought as the only argument for existence: "I think, therefore I am".*

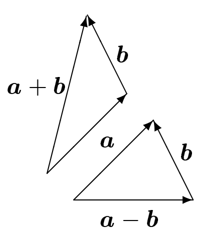
2.4 Vector calculus



Geometric scaling of a vector.

Triplets of real numbers play a central role in everything that follows, and it is convenient immediately to introduce a set of algebraic rules for these objects. We shall see below (section 2.6) that vectors in Cartesian coordinate systems are triplets that transform in a special way under coordinate transformations.

Algebraic rules



Geometric addition and subtraction of vector.

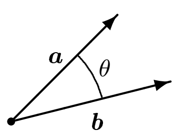
The following operations endow triplets with the properties of the familiar geometric vectors. Visualization on paper is of course as useful as ever, so we shall also draw triplets and illustrate their properties by means of arrows.

Linear operations: Linear operations lie at the core of triplet algebra,

$$s\,\boldsymbol{a} = (sa_1, sa_2, sa_3) \qquad \text{(scaling)}\ , \qquad (2\text{-}5)$$

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \qquad \text{(addition)}\ , \qquad (2\text{-}6)$$

$$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3) \qquad \text{(subtraction)}\ . \qquad (2\text{-}7)$$



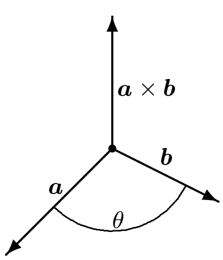
The dotproduct of two vectors is $|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$ where θ is the angle between them.

These rules tell us that the set of all triplets, also called $\mathbb{R}^3$, mathematically is a 3-dimensional *vector space*. A straight line with origin $\boldsymbol{a}$ and direction $\boldsymbol{b}$ is described by the linear function $\boldsymbol{a} + \boldsymbol{b}\,s$ with $-\infty < s < \infty$.

Bilinear products: There are three different bilinear products of triplets, of which the two first are well-known from ordinary vector calculus,

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \text{(dot product)}\ , \qquad (2\text{-}8)$$

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \qquad \text{(cross product)}\ . \qquad (2\text{-}9)$$



The cross product of two vectors is a vector orthogonal to both of length $|\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta$, here drawn using a right-hand rule.

Two triplets are said to be *orthogonal* when their dot product vanishes. Notice that the cross product is defined entirely in terms of the coordinates, and that we do not in the rule itself distinguish between left-handed and right-handed coordinate systems. Whether you use your right or left hand when you draw a cross product on paper does not matter for the triplet product rule, as long as you consistently use the same hand for all such drawings.

The last one,

$$\boldsymbol{a}\boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1, b_2, b_3) = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix} \qquad \text{(tensor product)}\ , \qquad (2\text{-}10)$$

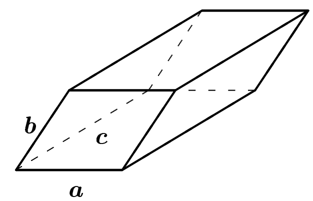
called the *tensor product*, is unusual in that it produces a 3×3 matrix from two triplets, but otherwise it is perfectly well-defined and useful to have around. It is nothing but an ordinary matrix product of a column-matrix and a row-matrix, also called the *direct product* and sometimes in the older literature the *dyadic product*. In section 2.6 we shall introduce more general geometric objects, called tensors, of which the simplest are matrices of this kind. The tensor product, and tensors in general, cannot be given a simple visualization on paper.

Volume product: The trilinear product of three triplets obtained by combining the cross product and the dot product is called the *volume product*,

$$\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1 \ . \tag{2-11}$$

The right hand side clearly shows that the volume product equals the determinant of the matrix constructed from the three vectors,

$$\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{(volume product)} \ . \tag{2-12}$$



Three vectors spanning a parallelepiped.

The volume product is antisymmetric under exchange of any pair of vectors. Its value is the (signed) volume of the parallelepiped spanned by the vectors.

Square and norm: The square and the norm are standard definitions

$$\boldsymbol{a}^2 = \boldsymbol{a} \cdot \boldsymbol{a} = a_1^2 + a_2^2 + a_3^3 \qquad \text{(square)} \ , \tag{2-13}$$

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a}^2} = \sqrt{a_1^2 + a_2^2 + a_3^3} \qquad \text{(norm or length)} \ . \tag{2-14}$$

This definition of the norm is closely related to the form of the Cartesian distance (2-4) which may now be written $d(\boldsymbol{a}, \boldsymbol{b}) = |\boldsymbol{a} - \boldsymbol{b}|$.

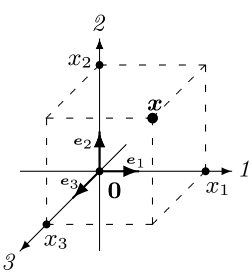
Basis vectors

The *coordinate axes* of a Cartesian coordinate system are straight lines with a common origin $\mathbf{0} = (0, 0, 0)$ and directions,

$$\boldsymbol{e}_1 = (1, 0, 0) \ , \tag{2-15a}$$

$$\boldsymbol{e}_2 = (0, 1, 0) \ , \tag{2-15b}$$

$$\boldsymbol{e}_3 = (0, 0, 1) \ . \tag{2-15c}$$



Visualization of the Cartesian coordinate system.

These triplets are called the *basis* vectors of the coordinate system[2], or just the basis, and every position $\boldsymbol{x}$ may trivially be written as a linear combination of the basis vectors with the coordinates as coefficients,

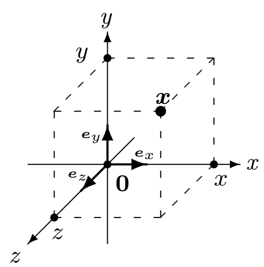
$$\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + x_3 \boldsymbol{e}_3 \ . \tag{2-16}$$

The basis vectors are *normalized* and mutually *orthogonal*,

$$|\boldsymbol{e}_1| = |\boldsymbol{e}_2| = |\boldsymbol{e}_3| = 1 \ , \tag{2-17}$$

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0 \ . \tag{2-18}$$

[2]In some texts the basis vectors are symbolized by the coordinate label with a hat above: $\hat{1}$, $\hat{2}$, and $\hat{3}$.



A Cartesian coordinate system with axes labeled x, y and z.

Using these relations and (2-16) we find

$$x_1 = \boldsymbol{e}_1 \cdot \boldsymbol{x} \ , \tag{2-19a}$$

$$x_2 = \boldsymbol{e}_2 \cdot \boldsymbol{x} \ , \tag{2-19b}$$

$$x_3 = \boldsymbol{e}_3 \cdot \boldsymbol{x} \ , \tag{2-19c}$$

showing that the coordinates of a point may be understood as the normal projections of the point on the axes of the coordinate system.

Combining (2-16) with (2-19) we obtain the identity

$$\boldsymbol{e}_1(\boldsymbol{e}_1 \cdot \boldsymbol{x}) + \boldsymbol{e}_2(\boldsymbol{e}_2 \cdot \boldsymbol{x}) + \boldsymbol{e}_3(\boldsymbol{e}_3 \cdot \boldsymbol{x}) = \boldsymbol{x} \ ,$$

valid for all $\boldsymbol{x}$. Since this is a linear identity, we may remove $\boldsymbol{x}$ and express this *completeness* relation in a compact form by means of the tensor product (2-10)

$$\boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3 = \mathbf{1} \ , \tag{2-20}$$

where on the right hand side the symbol $\mathbf{1}$ stands for the 3×3 unit matrix[3].

We emphasize that the handedness of the coordinate system has not entered the formalism. Correspondingly, the volume of the unit cube,

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 \times \boldsymbol{e}_3 = +1 \ , \tag{2-21}$$

is always positive, independently of whether you call the hand you write with the left or the right!

Index notation

Triplet notation for vectors is sufficient in most areas of physics, because physical quantities are mostly scalars (*i.e.* single numbers like mass) or vectors such as velocity, but sometimes it is necessary to use a more powerful and transparent notation which generalizes better to more complex expressions and quantities. It is called *index notation* or *tensor notation*, and consists in all simplicity in writing out the coordinate indices explicitly wherever they occur. Instead of thinking of a position as a triplet $\boldsymbol{x}$, we think of it as the set of coordinates x_i with the index i running implicitly over the coordinate labels, $i = 1, 2, 3$ or $i = x, y, z$, without having to state it every time.

Triplet and index notations coexist quite peacefully as witnessed by the linear operations

$$(s\boldsymbol{a})_i = s a_i \ , \tag{2-22}$$

$$(\boldsymbol{a} + \boldsymbol{b})_i = a_i + b_i \ , \tag{2-23}$$

$$(\boldsymbol{a} - \boldsymbol{b})_i = a_i - b_i \ . \tag{2-24}$$

[3] In order to distinguish a matrix from a triplet, the matrix symbol will be written in heavy unslanted boldface. The distinction is not particularly visible in print. With pencil on paper, 3×3 matrices are sometimes marked with a double bar ($\overline{\overline{1}}$) or a double arrow ($\overleftrightarrow{1}$).

For the scalar product we let the sum range implicitly over the coordinate labels,

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_i a_i b_i \ . \tag{2-25}$$

In full-fledged tensor calculus the summation symbol is left out and understood as implicitly present for all indices that occur precisely twice in a term, but we shall refrain from doing so here.

The nine scalar products of basis vectors has two indices that each run implicitly over the three coordinate labels, and is written

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \ . \tag{2-26}$$

The expression δ_{ij} is nothing but the unit matrix in index notation,

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} \end{cases} , \tag{2-27}$$

Leopold Kronecker (1823–1891). *German mathematician, contributed to the theory of elliptic functions, algebraic equations, and algebraic numbers.*

also called the *Kronecker delta*. This is the first example of a true *tensor* of rank 2. Another is the tensor product (2-10) of two vectors, which takes the form

$$(\boldsymbol{ab})_{ij} = a_i b_j \ . \tag{2-28}$$

It is in fact not enough for a tensor just to have two (or more) indices, but it suffices for now. In section 2.6 we shall see what really characterizes tensors.

The volume product (2-11) becomes a triple sum over indices,

$$\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \sum_{ijk} \epsilon_{ijk}\, a_i b_j c_k \ , \tag{2-29}$$

with 27 coefficients,

$$\epsilon_{ijk} = \begin{cases} +1 & ijk = 123, 231, 312 \ , \\ -1 & ijk = 132, 213, 321 \ , \\ \ \ 0 & \text{otherwise} \ . \end{cases} \tag{2-30}$$

The symbol ϵ_{ijk} is in fact a tensor of third rank, called the *Levi-Civita symbol.*

Tullio Levi-Civita (1873–1941). *Italian mathematician, contributed to differential calculus, relavitivy, and founded (with Ricci) tensor analysis in curved space.*

Finally, the cross product (2-9) may be written as a double sum over two indices of the form,

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \sum_{jk} \epsilon_{ijk} a_j b_k \ . \tag{2-31}$$

Mostly we shall avoid this complicated notation, although it does come in handy in some situations.

Derivatives

Various types of derivatives involving triplets may be defined, namely

$$\frac{\partial \boldsymbol{a}}{\partial s} = \left(\frac{\partial a_1}{\partial s}, \frac{\partial a_2}{\partial s}, \frac{\partial a_3}{\partial s}\right) \qquad \text{(scalar derivative)} , \tag{2-32}$$

$$\frac{\partial}{\partial \boldsymbol{a}} = \left(\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}\right) \qquad \text{(vector derivative)} . \tag{2-33}$$

In the first line, the derivative of a triplet after a parameter is defined. In the second line, a symbolic notation is introduced for the three derivatives after a triplet's coordinates (see problem 2.8 for simple uses of this notation).

In Cartesian coordinates a special symbol is introduced for the triplet of spatial derivatives, called the *gradient* operator or *nabla*,

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) , \tag{2-34}$$

which in index notation becomes,

$$\nabla_i = \frac{\partial}{\partial x_i} . \tag{2-35}$$

This illustrates the difference between the use of the coordinate *label* i as index on the gradient components ∇_i and the use of a coordinate *name* x_i as in the partial derivatives. The distinction becomes quite subtle but no less important when the coordinate names are also used as labels, as in $\nabla_x = \partial/\partial x$ etc.

Divergence and curl

In continuum physics we shall often meet vector fields of the form

$$\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), v_2(\boldsymbol{x}), v_3(\boldsymbol{x})) . \tag{2-36}$$

Using the gradient operator as the left hand component in the dot-product (2-8) we obtain a new field, called the *divergence* of $\boldsymbol{v}$,

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \nabla_1 v_1 + \nabla_2 v_2 + \nabla_3 v_3 = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} . \tag{2-37}$$

Here we have, as we shall often do in the following, left out the explicit dependence on the spatial variable $\boldsymbol{x}$.

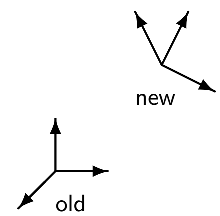
Similarly, if we use the gradient operator as the left hand component in the cross-product (2-9) we obtain another vector field,

$$\boldsymbol{\nabla} \times \boldsymbol{v} = (\nabla_2 v_3 - \nabla_3 v_2, \nabla_3 v_1 - \nabla_1 v_3, \nabla_1 v_2 - \nabla_2 v_1) , \tag{2-38}$$

called the *curl* [4] of $\boldsymbol{v}$. Trilinear combinations of these operators obey important identities (see problem 2.14).

[4] In the older literature the curl $\boldsymbol{\nabla} \times \boldsymbol{v}$ is often denoted by $\mathsf{curl}\,\boldsymbol{v}$ or $\mathsf{rot}\,\boldsymbol{v}$, and the divergence $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ by $\mathsf{div}\,\boldsymbol{v}$.

2.5 Cartesian coordinate transformations



The old and the new Cartesian systems.

The same Euclidean world may be described geometrically by different observers with different reference frames. Each observer constructs his own preferred Cartesian coordinate system and determines all positions relative to that. Every observer thinks that his basis vectors have the simple form (2-15) and satisfy the same orthogonality and completeness relations. Every observer believes he is right-handed. How can they ever agree on anything with such a self-centered view of the world?

The answer is — as explained in section 2.3 — that the two descriptions are related by a coordinate transformation (2-2). Since the distance between any two points is independent of the coordinate system, the shortest paths must coincide, and straight lines must be mapped onto straight lines by any Cartesian coordinate transformation. Seen from one Cartesian coordinate system, which we shall call the "old", the axes of another Cartesian coordinate system, called the "new", will therefore also appear to be straight lines with a common origin. Furthermore, since the scalar product of two vectors can be expressed in terms of the norm (problem 2.4), it must like distance be independent of the specific coordinate system, so that the new axes will also appear to be orthogonal in the geometry of the old coordinate system. Different observers will thus agree that their respective coordinate systems are indeed Cartesian.

Simple transformations

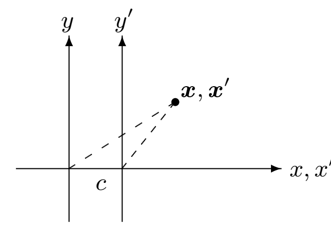
We begin the analysis of coordinate transformations with the familiar elementary transformations: translation, rotation, and reflection. These transformations are of the general form (2-2), expressing the coordinates of a geometrical point in the new system as a function of the coordinates of the same point in the old. The simple transformations are related to a special choice of coordinate axes, and similar simple transformations may be defined for other choices.

Simple translation: A simple *translation* of the origin of coordinates along the x-axis by a constant amount c is given by

$$x' = x - c \ , \qquad \text{(2-39a)}$$

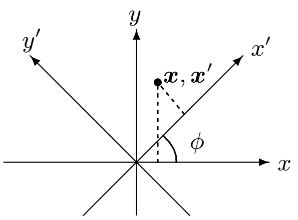
$$y' = y \ , \qquad \text{(2-39b)}$$

$$z' = z \ . \qquad \text{(2-39c)}$$



Simple translation of the coordinate system by c along the x-axis.

The axes of the new coordinate system are in this case parallel with the axes of the old.



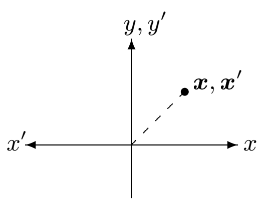
Simple rotation of the coordinate system around the z-axis (pointing out of the paper) through an angle ϕ.

Simple rotation: A simple *rotation* of the coordinate system through an angle ϕ around the z-axis corresponds to the transformation

$$x' = x\cos\phi + y\sin\phi \ , \tag{2-40a}$$
$$y' = -x\sin\phi + y\cos\phi \ , \tag{2-40b}$$
$$z' = z \ . \tag{2-40c}$$

The signs of the terms in the first two lines may be verified by ordinary geometrical considerations.

Simple reflection: A simple *reflection* in the yz-plane is described by



A simple reflection in the yz-plane.

$$\begin{aligned} x' &= -x \ , \\ y' &= y \ , \\ z' &= z \ . \end{aligned} \tag{2-41}$$

A simple reflection always transforms a right-handed coordinate system into a left-handed one, and conversely, independently of which hand you may claim to be the right one.

General transformations

Let the new Cartesian coordinate system be characterized (in the old) by the origin $\boldsymbol{c}$ and the three orthogonal and normalized basis vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$, satisfying the usual relations (see fig. 2.1)

$$\left.\begin{array}{c} |\boldsymbol{a}_1| = |\boldsymbol{a}_2| = |\boldsymbol{a}_3| = 1 \\ \boldsymbol{a}_1\cdot\boldsymbol{a}_2 = \boldsymbol{a}_2\cdot\boldsymbol{a}_3 = \boldsymbol{a}_3\cdot\boldsymbol{a}_1 = 0 \end{array}\right\} . \tag{2-42}$$

A position $\boldsymbol{x}' = (x_1', x_2', x_3')$ in the new coordinate system must then correspond to the (old) position,

$$\boldsymbol{x} = \boldsymbol{c} + x_1'\boldsymbol{a}_1 + x_2'\boldsymbol{a}_2 + x_3'\boldsymbol{a}_3 \ . \tag{2-43}$$

The new coordinates are obtained by multiplying from the left with the new basis vectors and using orthonormality (2-42)

$$\begin{aligned} x_1' &= \boldsymbol{a}_1\cdot(\boldsymbol{x}-\boldsymbol{c}) = a_{11}(x_1-c_1) + a_{12}(x_2-c_2) + a_{13}(x_3-c_3) \ , \\ x_2' &= \boldsymbol{a}_2\cdot(\boldsymbol{x}-\boldsymbol{c}) = a_{21}(x_1-c_1) + a_{22}(x_2-c_2) + a_{23}(x_3-c_3) \ , \\ x_3' &= \boldsymbol{a}_3\cdot(\boldsymbol{x}-\boldsymbol{c}) = a_{31}(x_1-c_1) + a_{32}(x_2-c_2) + a_{33}(x_3-c_3) \ . \end{aligned}$$

where $a_{ij} = (\boldsymbol{a}_i)_j$ are the coordinates of the new basis vectors. This is the most general coordinate transformation between any two Cartesian coordinate systems. It is not very difficult to show that the most general transformation may be composed from a sequence of simple transformations (problem 2.20).

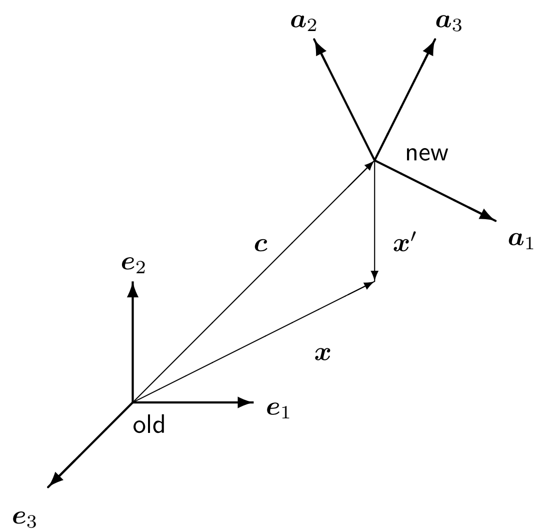


Figure 2.1: *Relations between the old and the new Cartesian systems.*

Using index notation, the general coordinate transformation may be written,

$$\boxed{x'_i = \sum_j a_{ij}(x_j - c_j)\ .} \tag{2-44}$$

It is characterized by the *translation vector* $\boldsymbol{c} = \{c_i\}$ and the *transformation matrix* $\mathbf{A} = (\boldsymbol{a}_i)_j = \{a_{ij}\}$ having the new basis vectors as rows. In matrix notation the transformation becomes even more compact[5],

$$\boldsymbol{x}' = \mathbf{A} \cdot (\boldsymbol{x} - \boldsymbol{c})\ . \tag{2-45}$$

The transformation matrix for a simple translation along the x-axis (2-39) is just the unit matrix, $\mathbf{A} = \mathbf{1}$, whereas for a simple rotation around the z-axis (2-40) we obtain the non-trivial matrix,

$$\mathbf{A} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\ . \tag{2-46}$$

A simple reflection in the yz-plane (2-41) is characterized by a diagonal transformation matrix with $(-1, 1, 1)$ along the diagonal.

The orthogonality of the new basis vectors implies that

$$\boldsymbol{a}_i \cdot \boldsymbol{a}_j = \sum_k a_{ik}a_{jk} = \delta_{ij}\ . \tag{2-47}$$

[5]In ordinary mathematical matrix calculus one would not use the dot to indicate multiplication (nor to indicate a scalar product), but this notation is quite natural for three-dimensional vectors and matrices that we encounter so often in physics.

On matrix form this is becomes

$$\mathbf{A} \cdot \mathbf{A}^\top = \mathbf{1} \tag{2-48}$$

where $(\mathbf{A}^\top)_{ij} = (\boldsymbol{a}_j)_i = a_{ji}$ is the transposed matrix having the new basis vectors as columns, and $\mathbf{1}$ is as before the 3×3 unit matrix.

The transposed matrix has the same determinant and the determinant of a product of matrices is the product of the determinants. Taking the determinant of (2-48) we obtain $(\det \mathbf{A})^2 = 1$, or

$$\det \mathbf{A} = \pm 1 \ . \tag{2-49}$$

The transformation matrices are thus divided into two completely separate classes, those with determinant $+1$, called *rotations* or sometimes *proper rotations*, and those with determinant -1, generically called reflections. Since the simple reflection (2-41) has determinant -1, all reflections may be composed of a simple reflection followed by a rotation.

2.6 Scalars, vectors, and tensors

Various *geometric quantities*, scalars, vectors and tensors, may now be classified according to their behavior under pure rotations. Further subclassification according to the behavior under reflection and translation is also possible.

Classification under rotation

When you rotate the coordinate system the world stays the same; it is only the way you describe that changes. Some geometrical quantities, for example the distance between two points, are unaffected by a rotation; others like the coordinates of your current position will change.

Scalars: A single quantity S is called a *scalar*, if it is invariant under rotations

$$\boxed{S' = S} \tag{2-50}$$

Thus the distance, the norm and the dotproduct are scalars. In physics the natural constants, material constants, as well as mass and charge of particles are scalars.

Vectors: Any triplet of quantities, $\boldsymbol{V}$, is called a *vector*, if it transforms under rotations according to

$$\boxed{V'_i = \sum_j a_{ij} V_j} \ , \tag{2-51}$$

or equivalently in matrix form,

$$\boldsymbol{V}' = \mathbf{A} \cdot \boldsymbol{V} \ . \tag{2-52}$$

In physics, velocity, acceleration, momentum, force, and many other quantities are vectors in this sense. The coordinates $\boldsymbol{x}$ of a point may also be called a vector according to this definition, but that is only correct in linear coordinate systems, in particular the Cartesian ones, and would be very wrong in curvilinear coordinates or curved spaces.

This definition shows that triplets must have special transformation properties to qualify as vectors. A triplet containing your weight, your height, and your age, is not a vector but a collection of three scalars.

Tensors: Using the vector transformation (2-51), the tensor product of two vectors $\boldsymbol{V}\,\boldsymbol{W}$, is found to transform according to the rule,

$$(\boldsymbol{V}'\boldsymbol{W}')_{ij} = V_i' W_j' = \left(\sum_k a_{ik} V_k\right)\left(\sum_l a_{jl} W_l\right) = \sum_{kl} a_{ik} a_{jl} V_k W_l \ .$$

In the last step we have reordered the sums into a convenient form.

More generally, any set of 9 quantities arranged in a matrix

$$\mathbf{T} = \{T_{ij}\} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \tag{2-53}$$

is called a *tensor* of rank 2, provided it obeys the transformation law,

$$\boxed{T_{ij}' = \sum_{kl} a_{ik} a_{jl} T_{kl} \ ,} \tag{2-54}$$

which in matrix form may be written,

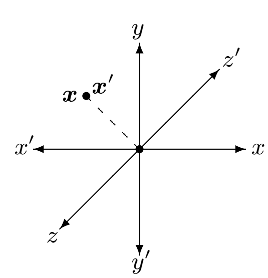
$$\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^\top \ . \tag{2-55}$$

In physics, the moment of inertia of an extended body and the quadrupole moment of a charge distribution are well-known tensors of second rank.

Similarly, tensors of higher rank may be constructed. A tensor of rank r has r indices and is a collection of 3^r quantities that transform as the direct product of r vectors. We have so far only met one third rank tensor, the Levi-Civita symbol (2-30) (see problem 2.22).

∗ Classification under reflection

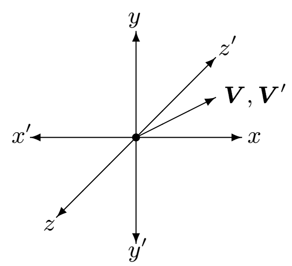
Instead of a simple reflection in the yz-plane, we shall classify quantities according to their behavior under a complete reflection of the coordinate system through its origin



A complete reflection of the coordinate system in the origin. A rotation through π around the x-axis converts this to a simple reflection in the yz-plane.

$$\boldsymbol{x}' = -\boldsymbol{x} \ . \tag{2-56}$$

Geometrically, the reflection in the origin may be viewed as a composite of simple reflections along the three coordinate axes, or as a simple reflection of a coordinate axis followed by a simple rotation through π around the same axis.



A polar vector retains its geometric placement under a reflection of the coordinate system in the origin.

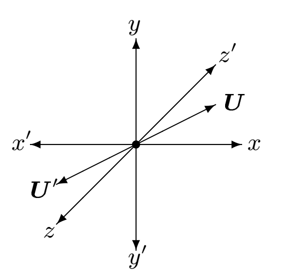
Polar vectors: A vector which obeys the usual transformation equation (2-51) under rotations as well as under reflections is called a *polar* vector. Under a reflection in the origin, the coordinates of a polar vector change sign just like the coordinates of a point, *i.e.* $\boldsymbol{V}' = -\boldsymbol{V}$. Since the coordinate axes all reverse direction, the geometrical position in space of a polar vector is unchanged by a reflection of the coordinate system, and the vector may faithfully be represented by an arrow, also under reflections. In physics, acceleration, force, velocity, and momentum are all polar vectors.

Axial vectors: There is, however, another possibility. The cross product of two polar vectors, $\boldsymbol{U} = \boldsymbol{V} \times \boldsymbol{W}$, behaves differently than a polar vector under a reflection. According to our rules for calculating the cross product, which are the same in all coordinate systems, we find

$$\boldsymbol{U}' = \boldsymbol{V}' \times \boldsymbol{W}' = (-\boldsymbol{V}) \times (-\boldsymbol{W}) = \boldsymbol{V} \times \boldsymbol{W} = \boldsymbol{U} \ , \tag{2-57}$$

without the expected change of sign. Since $\boldsymbol{U}$ behaves normally under rotations with determinant $+1$, we conclude that the missing minus sign is associated with any transformation with determinant -1, that is with any reflection. Generalizing, we define an *axial* vector $\boldsymbol{U}$ as a set of three quantities, transforming according to the rule

$$\boxed{U_i' = \det \mathbf{A} \sum_j a_{ij} U_j} \ , \tag{2-58}$$



Geometrically, an axial vector has its geometric direction reversed under a reflection of the coordinate system in the origin because it has the same coordinates in the reflected system as in the original.

under a general coordinate transformation. The extra determinant will for an axial vector eliminate the sign change otherwise associated with a reflection in the origin, $\mathbf{A} = -\mathbf{1}$. All quantities defined with a single cross product are axial vectors. In physics, angular momentum, moment of force, magnetic dipole moments, and the magnetic field itself, are axial vectors.

The direction of an axial vector depends on what we choose to be right and left. It is for this reason wrong to think of an axial vector as an arrow in space. Geometrically, it has magnitude and direction, but not *sense*, meaning that the positive direction of an axial vector is not a geometric property, but fixed by convention, and changes under a reflection of the coordinate system. For consistency, all humans have agreed that one coordinate system, and all coordinate systems that are obtained from it by proper rotation, are right-handed, whereas coordinate systems that are related to it by reflection, are left-handed. We do not know whether non-human aliens would have adopted the same convention, but if we should meet such beings we would be able to find the transformation between our reference frames and theirs.

The basis vectors of the old coordinate system have the coordinates, $\boldsymbol{e}'_i = -\boldsymbol{e}_i$, in the new (reflected) coordinate system. Basis vectors are proper geometric quantities and always transform as polar vectors under reflection. One may easily get confused by the fact that the new basis vectors in the new system have (by definition) the same coordinates $\boldsymbol{e}_i$ as the old basis vectors in the old system, but it would be a mistake to take this to mean that the basis vectors are axial.

Pseudo-scalars: The volume product of three polar vectors $P = \boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$ is a scalar quantity which changes sign under a reflection of the coordinate system because the cross product is an axial vector which does not change sign. More generally a *pseudoscalar* transforms like

$$\boxed{P' = \det \mathbf{A} \ \ P \ ,} \tag{2-59}$$

under an arbitrary rotation or reflection.

The sign of a pseudoscalar is not absolute, but depends on the handedness of the coordinate system, and thus on convention. One might think that physics had no use for such quantities, because after all physics itself does not depend on coordinate systems, only its mathematical description does. Nevertheless, magnetic charge, if it were ever found, would be pseudoscalar, and more importantly, some of the familiar elementary particles, for example the pi-mesons, are described by pseudoscalar fields.

Axial vectors are likewise called pseudovectors, and one may similarly define pseudotensors of higher rank. The Levi-Civita symbol is a pseudo-tensor of third rank (problem 2.22).

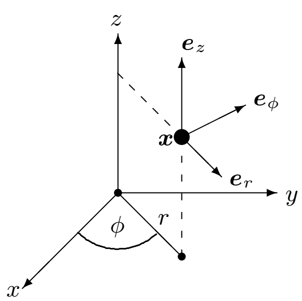
∗ Classification under translation

A true vector may always be viewed as the difference between two positions $\boldsymbol{v} = \boldsymbol{b} - \boldsymbol{a}$, and is thus invariant under a pure translation $\boldsymbol{x} \to \boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{c}$. Such vectors are called *proper*. Triplets that transform as vectors under rotations but change under translations, like the position $\boldsymbol{x}$ itself, are called *improper*. In physics electric dipole moments are improper polar vectors whereas angular momentum, momentum of force, and magnetic dipole moments are improper axial vectors.

* 2.7 Curvilinear coordinates

The distance between two points Euclidean space takes the simplest form (2-4) in Cartesian coordinates. The geometry of concrete physical problems may make non-Cartesian coordinates more suitable as a basis for analysis, even if the distance becomes more complicated in the new coordinates. Since the new coordinates are non-linear functions of the Cartesian coordinates, they define three sets of intersecting curves, and are for this reason called *curvilinear coordinates.*

At a deeper level, it is often the *symmetry* of a physical problem that points to the most convenient choice of coordinates. Cartesian coordinates are well suited to problems with translational invariance, cylindrical coordinates for problems that are invariant under rotations around a fixed axis, and spherical coordinates for problems that are invariant or partially invariant under arbitrary rotations. Elliptic and hyperbolic coordinates are also of importance but will not be discussed here (see [18, p. 455]).



Cylindrical coordinates and basis vectors.

Cylindrical coordinates

The relation between Cartesian coordinates x, y, z and cylindrical coordinates r, ϕ, z is given by

$$x = r\cos\phi \ , \tag{2-60a}$$

$$y = r\sin\phi \ , \tag{2-60b}$$

$$z = z \tag{2-60c}$$

with the range of variation $0 \leq r < \infty$ and $0 \leq \phi < 2\pi$. The two first equations simply define polar coordinates in the xy-plane[6]. The last is rather trivial but included to emphasize that this is a transformation in 3-dimensional space.

Curvilinear basis: The *curvilinear basis vectors* are defined from the tangent vectors, obtained by differentiating the Cartesian position after the cylindrical coordinates,

$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\cos\phi, \sin\phi, 0) \ , \tag{2-61a}$$

$$\boldsymbol{e}_\phi = \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0) \ , \tag{2-61b}$$

$$\boldsymbol{e}_z = \frac{\partial \boldsymbol{x}}{\partial z} = (0, 0, 1) \ . \tag{2-61c}$$

As may be directly verified, they are orthogonal and normalized everywhere, and thus define a local curvilinear basis with an orientation that changes from place

[6]Some texts use Θ instead of ϕ as the conventional name for the polar angle in the plane. Various arguments can be given one way or the other by comparing with spherical coordinates. But what's in a name? A polar angle by any name still works as sweet.

to place. An arbitrary vector field may therefore be resolved in this basis

$$\boldsymbol{V} = \boldsymbol{e}_r V_r + \boldsymbol{e}_\phi V_\phi + \boldsymbol{e}_z V_z \ , \tag{2-62}$$

where the vector coordinates

$$V_r = \boldsymbol{V} \cdot \boldsymbol{e}_r \ , \quad V_\phi = \boldsymbol{V} \cdot \boldsymbol{e}_\phi \ , \quad V_z = \boldsymbol{V} \cdot \boldsymbol{e}_z \ , \tag{2-63}$$

are the projections of $\boldsymbol{V}$ on the local basis vectors.

Resolution of the gradient: The derivatives after the cylindrical coordinates are found by differentiation through the Cartesian coordinates

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \ , \\
\frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y} \ .
\end{aligned}$$

From these relations we may calculate the projections of the gradient operator $\boldsymbol{\nabla} = (\partial_x, \partial_y, \partial_z)$ on the cylindrical basis, and we obtain

$$\nabla_r = \boldsymbol{e}_r \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial r} \ , \tag{2-64a}$$

$$\nabla_\phi = \boldsymbol{e}_\phi \cdot \boldsymbol{\nabla} = \frac{1}{r}\frac{\partial}{\partial \phi} \ , \tag{2-64b}$$

$$\nabla_z = \boldsymbol{e}_z \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial z} \ . \tag{2-64c}$$

Conversely, the gradient may be resolved on the basis

$$\boxed{\boldsymbol{\nabla} = \boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \boldsymbol{e}_z \frac{\partial}{\partial z}} \ . \tag{2-65}$$

Together with the non-vanishing derivatives of the basis vectors

$$\frac{\partial \boldsymbol{e}_r}{\partial \phi} = \boldsymbol{e}_\phi \ , \tag{2-66a}$$

$$\frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\boldsymbol{e}_r \ , \tag{2-66b}$$

we are now in possession of all the necessary tools for calculating in cylindric coordinates.

The Laplacian: An operator which often occurs in differential equations is the *Laplace operator* or *Laplacian*,

$$\boldsymbol{\nabla}^2 = \nabla_x^2 + \nabla_y^2 + \nabla_z^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \ . \tag{2-67}$$

In cylindrical coordinates this operator takes a different form, which may be found by squaring the resolution of the gradient (2-65). Keeping track of the order of the operators and basis vectors we get

$$\begin{aligned}\boldsymbol{\nabla}^2 &= (\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\cdot(\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\\ &= (\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\cdot\boldsymbol{e}_r\nabla_r\\ &+ (\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\cdot\boldsymbol{e}_\phi\nabla_\phi\\ &+ (\boldsymbol{e}_r\nabla_r + \boldsymbol{e}_\phi\nabla_\phi + \boldsymbol{e}_z\nabla_z)\cdot\boldsymbol{e}_z\nabla_z\\ &= \nabla_r^2 + \frac{1}{r}\nabla_r + \nabla_\phi^2 + \nabla_z^2\ .\end{aligned}$$

In the second line we have distributed the first factor on the terms of the second, and in going to the last line we have furthermore distributed the terms of the first factor, using the orthogonality of the basis and taking into account that differentiation after ϕ may change the basis vectors according to (2-66).

Finally, using (2-64) we arrive the cylindrical Laplacian,

$$\boxed{\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}}\ , \tag{2-68}$$

expressed in terms of the usual partial derivatives.

Spherical coordinates

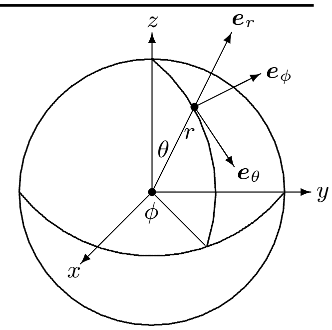
The treatment of spherical coordinates follows much the same pattern as cylindrical coordinates. Spherical or polar coordinates consist of the radial distance r, the polar angle θ and the azimuthal angle ϕ. If the z-axis is chosen as polar axis and the x-axis as the origin for the azimuthal angle, the transformation from spherical to Cartesian coordinates becomes,

$$x = r\sin\theta\cos\phi\ , \tag{2-69a}$$
$$y = r\sin\theta\sin\phi\ , \tag{2-69b}$$
$$z = r\cos\theta\ . \tag{2-69c}$$

The domain of variation for the spherical coordinates is $0 \le r < \infty$, $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

Curvilinear basis: The normalized tangent vectors along the directions of the spherical coordinate are,

$$\boldsymbol{e}_r = \frac{\partial\boldsymbol{x}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\ , \tag{2-70a}$$
$$\boldsymbol{e}_\theta = \frac{1}{r}\frac{\partial\boldsymbol{x}}{\partial\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)\ , \tag{2-70b}$$
$$\boldsymbol{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial\boldsymbol{x}}{\partial\phi} = (-\sin\phi, \cos\phi, 0)\ . \tag{2-70c}$$



Spherical coordinates and their basis vectors.

They are orthogonal, so that an arbitrary vector field may be resolved after these directions,

$$\boldsymbol{V} = \boldsymbol{e}_r V_r + \boldsymbol{e}_\theta V_\theta + \boldsymbol{e}_\phi V_\phi \tag{2-71}$$

with $V_a = \boldsymbol{e}_a \cdot \boldsymbol{V}$ for $a = r, \theta, \phi$.

Resolution of the gradient: The gradient operator may also be resolved on the basis,

$$\boxed{\boldsymbol{\nabla} = \boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi} \ , \tag{2-72}$$

where

$$\nabla_r = \boldsymbol{e}_r \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial r} \ , \tag{2-73a}$$

$$\nabla_\theta = \boldsymbol{e}_\theta \cdot \boldsymbol{\nabla} = \frac{1}{r}\frac{\partial}{\partial \theta} \ , \tag{2-73b}$$

$$\nabla_\phi = \boldsymbol{e}_\phi \cdot \boldsymbol{\nabla} = \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi} \ . \tag{2-73c}$$

The non-vanishing derivatives of the basis vectors are

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, \qquad \frac{\partial \boldsymbol{e}_r}{\partial \phi} = \sin\theta \, \boldsymbol{e}_\phi \ , \tag{2-74a}$$

$$\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r, \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial \phi} = \cos\theta \boldsymbol{e}_\phi \ , \tag{2-74b}$$

$$\frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\sin\theta \, \boldsymbol{e}_r - \cos\theta \, \boldsymbol{e}_\theta \ . \tag{2-74c}$$

These are all the relations necessary for calculations in spherical coordinates.

The Laplacian: The Laplacian (2-67) becomes in this case

$$\begin{aligned}
\boldsymbol{\nabla}^2 &= (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \\
&= (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot \boldsymbol{e}_r \nabla_r \\
&+ (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot \boldsymbol{e}_\theta \nabla_\theta \\
&+ (\boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\theta \nabla_\theta + \boldsymbol{e}_\phi \nabla_\phi) \cdot \boldsymbol{e}_\phi \nabla_\phi \\
&= \left(\nabla_r^2 + \frac{1}{r}\nabla_r + \frac{\sin\theta}{r\sin\theta}\nabla_r\right) + \left(\nabla_\theta^2 + \frac{\cos\theta}{r\sin\theta}\nabla_\theta\right) + \nabla_\phi^2 \ .
\end{aligned}$$

In the last step we have used the orthogonality and the derivatives (2-74). Finally, using (2-73) this becomes

$$\boxed{\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}} \ . \tag{2-75}$$

in standard notation.

Problems

2.1 Let $\boldsymbol{a} = (2, 3, -6)$ and $\boldsymbol{b} = (3, -4, 0)$. Calculate

a) the lengths of the vectors,

b) the dot product,

c) the cross product,

d) and the tensor product.

2.2 Are the vectors $\boldsymbol{a} = (3, 1, -2)$, $\boldsymbol{b} = (4, -1, -1)$ and $\boldsymbol{c} = (1, -2, 1)$ linearly dependent (meaning that there exists a non-trivial set of coefficients such that $\alpha\boldsymbol{a}+\beta\boldsymbol{b}+\gamma\boldsymbol{c} = \boldsymbol{0}$)?

2.3 Calculate the distance between two points on Earth in terms of longitude α, latitude β and height h over the average sea level. Try to write this elegantly, using rotational invariance of the distance function.

2.4 Show that

$$|\boldsymbol{a}+\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\,\boldsymbol{a}\cdot\boldsymbol{b}\ , \tag{2-76a}$$

$$|\boldsymbol{a}\cdot\boldsymbol{b}| \le |\boldsymbol{a}|\,|\boldsymbol{b}|\ , \tag{2-76b}$$

$$|\boldsymbol{a}+\boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|\ , \tag{2-76c}$$

2.5 Show that the mixed product is a determinant

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\ . \tag{2-77}$$

Use this to show that

$$(\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c})\,(\boldsymbol{d}\times\boldsymbol{e}\cdot\boldsymbol{f}) = \begin{vmatrix} \boldsymbol{a}\cdot\boldsymbol{d} & \boldsymbol{a}\cdot\boldsymbol{e} & \boldsymbol{a}\cdot\boldsymbol{f} \\ \boldsymbol{b}\cdot\boldsymbol{d} & \boldsymbol{b}\cdot\boldsymbol{e} & \boldsymbol{b}\cdot\boldsymbol{f} \\ \boldsymbol{c}\cdot\boldsymbol{d} & \boldsymbol{c}\cdot\boldsymbol{e} & \boldsymbol{c}\cdot\boldsymbol{f} \end{vmatrix}\ . \tag{2-78}$$

2.6 Show that

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c} = \boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{a} = \boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{b}\ , \tag{2-79}$$

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{c})\,\boldsymbol{a}\ , \tag{2-80}$$

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c})\ , \tag{2-81}$$

$$|\boldsymbol{a}\times\boldsymbol{b}|^2 = |\boldsymbol{a}|^2\,|\boldsymbol{b}|^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2\ . \tag{2-82}$$

2.7 Show that with the normal definition of the matrix product the following relations make sense for the tensor product

$$(\boldsymbol{a}\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c}) \tag{2-83}$$

$$\boldsymbol{a}\cdot(\boldsymbol{b}\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \tag{2-84}$$

Do not get confused. This notation is sometimes quite useful although it fails for complex expressions.

2.8 Show that

$$\frac{\partial(\boldsymbol{a}\cdot\boldsymbol{b})}{\partial\boldsymbol{a}} = \boldsymbol{b} \tag{2-85}$$

and that

$$\frac{\partial\,|\boldsymbol{a}|}{\partial\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}\ . \tag{2-86}$$

2.9 Show that

$$(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{d} = (\boldsymbol{a}\cdot\boldsymbol{d})\boldsymbol{b}\times\boldsymbol{c} + (\boldsymbol{b}\cdot\boldsymbol{d})\boldsymbol{c}\times\boldsymbol{a} + (\boldsymbol{c}\cdot\boldsymbol{d})\boldsymbol{a}\times\boldsymbol{b} \tag{2-87}$$

for arbitrary vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$.

2.10 Show that

$$\sum_i \delta_{ii} = 3 \tag{2-88}$$

$$\sum_j \delta_{ij}\delta_{jk} = \delta_{ik} \tag{2-89}$$

2.11 Show that

$$\sum_i \nabla_i x_i = 3\ , \tag{2-90}$$

$$\nabla_i x_j = \delta_{ij}\ , \tag{2-91}$$

$$\nabla_i\nabla_j(x_k x_l) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\ , \tag{2-92}$$

2.12 Show that the Levi-Civita symbol is completely antisymmetric in all three indices,

$$\epsilon_{ijk} = -\epsilon_{ikj} = -\epsilon_{jik} = -\epsilon_{kji} \tag{2-93}$$

2.13 Show that the product of two Levi-Civita symbols is (see problem 2.5)

$$\begin{aligned}\epsilon_{ijk}\ \epsilon_{lmn} &= \begin{vmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{vmatrix}\\ &= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km}\\ &\quad - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn}\end{aligned} \tag{2-94}$$

and from this

$$\sum_k \epsilon_{ijk}\ \epsilon_{lmk} = \begin{vmatrix}\delta_{il} & \delta_{im}\\ \delta_{jl} & \delta_{jm}\end{vmatrix} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\ , \tag{2-95}$$

$$\sum_{jk} \epsilon_{ijk}\ \epsilon_{ljk} = 2\delta_{il}\ , \tag{2-96}$$

$$\sum_{ijk} \epsilon_{ijk}\ \epsilon_{ijk} = 6\ . \tag{2-97}$$

2.14 Prove the following relations involving the nabla operator (here Φ is a scalar field and $\boldsymbol{v}$ a vector field),

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = \boldsymbol{0} \;, \tag{2-98}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \Phi) = 0 \;, \tag{2-99}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{2-100}$$

Where in these relations does it make sense to remove the parentheses?

2.15 Show that the trace (sum of the diagonal elements) of a tensor is invariant under a rotation.

2.16 Show that the Kronecker delta is a second rank tensor.

2.17 Show that if $W_i = \sum_j T_{ij} V_j$ and if it is known that $\boldsymbol{W}$ is a vector for all vectors $\boldsymbol{V}$, then T_{ij} must be a tensor.

2.18 Show that the distance $|\boldsymbol{x} - \boldsymbol{y}|$ is invariant under any transformation between Cartesian coordinate systems.

* **2.19** Consider two Cartesian coordinate systems and make no assumptions about the transformation $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ between them. Show that the invariance of the distance,

$$\left|\boldsymbol{x}' - \boldsymbol{y}'\right| = |\boldsymbol{x} - \boldsymbol{y}| \;, \tag{2-101}$$

implies that the transformation is of the form $\boldsymbol{x}' = \mathbf{A} \cdot \boldsymbol{x} + \boldsymbol{b}$ where $\mathbf{A}$ is an orthogonal matrix.

* **2.20** Show that the general Cartesian coordinate transformation may be built up from a combination of simple translations, rotations and reflections.

* **2.21** Show that under a simple rotation, a tensor T_{ij} transforms into

$$T'_{xx} = \cos\phi(T_{xx}\cos\phi + T_{xy}\sin\phi) + \sin\phi(T_{yx}\cos\phi + T_{yy}\sin\phi) \;, \tag{2-102a}$$

$$T'_{xy} = \cos\phi(-T_{xx}\sin\phi + T_{xy}\cos\phi) + \sin\phi(-T_{yx}\sin\phi + T_{yy}\cos\phi) \;, \tag{2-102b}$$

$$T'_{xz} = \cos\phi T_{xz} + \sin\phi T_{yz} \;, \tag{2-102c}$$

$$T'_{yx} = -\sin\phi(T_{xx}\cos\phi + T_{xy}\sin\phi) + \cos\phi(T_{yx}\cos\phi + T_{yy}\sin\phi) \;, \tag{2-102d}$$

$$T'_{yy} = -\sin\phi(-T_{xx}\sin\phi + T_{xy}\cos\phi) + \cos\phi(-T_{yx}\sin\phi + T_{yy}\cos\phi) \;, \tag{2-102e}$$

$$T'_{yz} = -\sin\phi T_{xz} + \cos\phi T_{yz} \;, \tag{2-102f}$$

$$T'_{zx} = T_{zx}\cos\phi + T_{zy}\sin\phi \;, \tag{2-102g}$$

$$T'_{zy} = -T_{zx}\sin\phi + T_{zy}\cos\phi \;. \tag{2-102h}$$

$$T'_{zz} = T_{zz} \tag{2-102i}$$

∗ **2.22** Show that

a) the Levi-Civita symbol satisfies

$$\sum_{lmn} a_{il} a_{jm} a_{kn} \epsilon_{lmn} = \det \mathbf{A}\, \epsilon_{ijk} \tag{2-103}$$

where $\mathbf{A}$ is an arbitrary matrix.

b) the Levi-Civita symbol (which by the definition of the cross product must be invariant, $\epsilon'_{ijk} = \epsilon_{ijk}$) obeys the rule

$$\epsilon'_{ijk} = \epsilon_{ijk} = \det \mathbf{A} \sum_{lmn} a_{il} a_{jm} a_{kn} \epsilon_{lmn} \tag{2-104}$$

for an arbitrary coordinate transformation (which has $\det \mathbf{A} = \pm 1$).

c) the cross product of two vectors $\boldsymbol{W} = \boldsymbol{U} \times \boldsymbol{V}$ must transform like

$$W'_i = \det \mathbf{A} \sum_j a_{ij} W_j \tag{2-105}$$