# 12
# Solids at rest

Flagpoles, bridges, houses, and towers are built from elastic materials, and are designed to stay in one place with at most small excursions away from equilibrium due to wind and water currents. Ships, airplanes and space shuttles are designed to move around, and their structural integrity depends crucially on the elastic properties of the materials from which they are made. In fact almost all human construction depends on elasticity for stability and ability to withstand external stresses.

It can come as no surprise that the theory of static elastic deformation, *elastostatics*, is a huge engineering subject. Engineers must know the deformation and internal stresses in their constructions in order to predict risk of failure and set safety limits, and that is only possible if the elastic properties of the building materials are known, and if they are able to solve the equations of elastostatics, or at least get decent approximations to them. Today computers aid the engineers in getting precise numeric solutions to the equations of elastostatics and allow them to build critical structures, such as submarines, supertankers, airplanes, and space vehicles, in which over-dimensioning of safety limits is deleterious to fuel consumption as well as to construction costs.

In this chapter we shall develop the theory of *elastostatics* for bodies made from isotropic materials and apply it to generic cases. In elastostatics, field equations and boundary conditions are essential and in many respects similar to the equations of electrostatics and magnetostatics, and to the equations of stationary fluid flow which will be taken up in later chapters. We shall mainly consider simple body geometries, in which analytic solutions can be obtained. In the following chapter the basic technique behind numeric solution of the equations of elastostatics is presented and applied to a relatively simple two-dimensional problem.

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## 12.1 Equations of elastostatics

The fundamental equations for elastostatics are obtained by combining the results of the preceding three chapters

$$f_i + \sum_j \nabla_j \sigma_{ij} = 0 \,, \qquad \text{Mechanical equilibrium (9-19)} \qquad \text{(12-1a)}$$

$$u_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i) \,, \qquad \text{Cauchy's strain tensor (10-17)} \qquad \text{(12-1b)}$$

$$\sigma_{ij} = 2\mu\, u_{ij} + \lambda \delta_{ij} \sum_k u_{kk} \,, \qquad \text{Hooke's law (11-8)} \qquad \text{(12-1c)}$$

Inserting the last into the first, we obtain the equilibrium equation for the strain tensor

$$2\mu \sum_j \nabla_j u_{ij} + \lambda \nabla_i \sum_j u_{jj} = -f_i \,. \qquad \text{(12-2)}$$

The derivatives appearing here refer in principle to the true position of the material, but under the assumption of small displacement gradients they may be replaced by the derivatives after the original position (see problem 12.11). Finally, inserting Cauchy's strain tensor into this equation, we obtain

$$\mu \sum_j \nabla_j \nabla_j u_i + \mu \nabla_i \sum_j \nabla_j u_j + \lambda \nabla_i \sum_j \nabla_j u_j = -f_i \,.$$

Rewriting this equation in vector notation, we arrive at *Navier's equation of equilibrium*, also called the Navier-Cauchy equation,

Claude Louis Marie Henri Navier (1785–1836). *French engineer, worked on applied mechanics, elasticity, fluid mechanics, and suspension bridges. Formulated the first version of the elastic equilibrium equation in 1821, a year before Cauchy gave it its final form.*

$$\boxed{\mu \boldsymbol{\nabla}^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla}\, \boldsymbol{\nabla} \cdot \boldsymbol{u} = -\boldsymbol{f}} \,. \qquad \text{(12-3)}$$

After all the troubles with tensor notation, we end up with a relatively simple field equation for the vector displacement field! One should, however, not be taken in by its apparent simplicity. There is a surprising richness hidden in its compact form.

The equilibrium equation completely determines the behavior of the displacement field *inside* a body. The only remaining freedom lies in the different boundary conditions that may be applied to the surface of the body. Typically, either the displacement itself or the stress vector are prescribed at the surface. That may, however, not completely determine the displacement field everywhere, because as we have seen in the discussion of uniform deformation in section 11.3, there may still be room for rigid body displacements consistent with the boundary conditions. More generally, the linearity of the equilibrium equation permits us to *superpose* solutions to it. If you, for example, both compress and stretch a body uniformly, the displacement field for the combined operation is the sum of the respective displacement fields, (11-24) and (11-23).

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## Estimates

Confronted with partial differential equations, it is always useful to get a rough idea of the order of magnitude of a particular solution. Imagine, for example, that a body made from elastic material is subjected to surface stresses of a typical magnitude $P$ and no body forces. A rough guess on order of magnitude of the stresses in the body is then also $\sigma_{ij} \approx P$. Assuming that the material is not exceptional, so that the elastic moduli $\lambda$, $\mu$, $E$, and $K$ are all of the same magnitude, one may estimate the deformation from Hooke's law (11-8) to be of the order of $u_{ij} \approx P/E$. Since deformation is calculated from gradients of the displacement field, the variation in displacement across a body of typical size $L$ may be estimated to be of the order of $u_i \approx L u_{ij} \approx LP/E$.

**Example 12.1.1:** Standing with your full weight of 70 kg on the seat of a chair supported by four wooden legs, each 7 $\mathsf{cm}^2$ in cross section and 50 cm long, you exert a stress $P \approx 250$ kPa $= 2.5$ bar on the legs. Taking $E \approx 10^9$ Pa, the deformation will be about 0.00025 and the maximal displacement about 0.125 mm. The squashing of the legs of the chair due to your weight is barely visible and the extra squashing due to their own weight even smaller. For comfortable seating, however, the seat of a chair is usually constructed as a kind of membrane which deforms much more under your weight.

In mechanical equilibrium (12-1a), there is balance between local changes in stress and body forces. This allows us to estimate the change in stress over a distance $L$ due to for example gravity of magnitude $f = \rho g$ to be $\sigma_{ij} \approx \rho g L$. The corresponding variation in strain becomes $u_{ij} \approx L\rho g/E$ for non-exceptional materials. Since $u_{ij}$ is dimensionless, it is convenient to define the quantity

$$D \sim \frac{E}{\rho g} , \tag{12-4}$$

so that $u_{ij} \approx L/D$. The quantity $D$ has dimension of length and sets the scale for major changes in deformation (of order unity). Small deformations require $L \ll D$. Finally, we estimate the variation in the displacement over a distance $L$ to be of magnitude $u_i \approx L u_{ij} \approx L^2/D$, depending quadratically on $L$.

**Example 12.1.2:** How tall can a solid steel tower be at the surface of Earth, when it is required that the strain may not surpass 1 % anywhere? Taking $E \approx 2 \times 10^{11}$ Pa and $\rho \approx 7 \times 10^3$ kg/m$^3$, we find $D \approx 3,000$ km and estimate the maximal height to be $L \approx |u_{ij}| D \approx 30$ km for $|u_{ij}| = 1$ %. The top of the tower will settle by about $u_i \approx 300$ m under its weight, and the pressure at the bottom will be about $\sigma_{ij} \approx \rho_0 g_0 L \approx 20$ kilobar, comparable to the pressure at the bottom of the deepest sea trench.

It should be emphasized that these estimates just aim at getting the right orders of magnitude of the fields, and that there may be special circumstances in a particular problem, which invalidate them. If that is the case, or if precision is needed, there is no way around analytic or numeric calculation.

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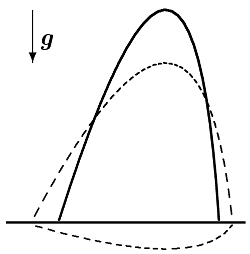
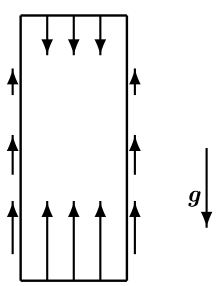



*Settling of a body under the influence of gravity.*

## 12.2 Standing up to gravity

Solid objects, be it mountains, bridges, houses, or coffee cups, standing on a surface are deformed by gravity, and deform in turn by their weight the supporting surface. Intuition tells us that gravity makes such objects settle towards the ground and squashes their material so that it bulges out horizontally, unless prevented by constraining walls. In a fluid at rest, each horizontal surface element has to carry the weight of the column of fluid above it, and this determines the pressure in the fluid. In a solid at rest, this is more or less also the case, except that shear elastic stresses in the material are able distribute the vertical load in the horizontal directions.

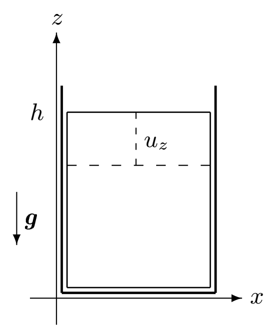



*Shear stresses may aid in carrying the weight of a vertical column of elastic material.*

### Uniform settling

An infinitely extended slab of homogeneous and isotropic elastic material placed on a horizontal surface is a kind of "elastic sea", which like the fluid sea may be assumed to have the same properties everywhere in a horizontal plane. In a flat-Earth coordinate system, where gravity is given by $\boldsymbol{g} = (0, 0, -g_0)$, we expect a uniformly vertical displacement, which only depends on the $z$-coordinate,

$$\boldsymbol{u} = (0, 0, u_z(z)) = u_z(z)\,\boldsymbol{e}_z \ . \tag{12-5}$$




*Elastic "sea" of material undergoing a uniform downwards displacement because of gravity. The container has fixed, slippery walls.*

In order to realize this "elastic sea" in a finite system, it must be surrounded by fixed, vertical, and slippery walls. The vertical walls forbid horizontal but allow vertical displacement, and at the bottom, $z = 0$, we place a horizontal supporting surface which forbids vertical but allows horizontal displacement. On the top, $z = h$, the elastic material is left free to move without any external forces acting on it.

The only non-vanishing strain is $u_{zz} = \nabla_z u_z$. From Hooke's law (11-9), we obtain the non-vanishing stresses

$$\sigma_{xx} = \sigma_{yy} = \lambda u_{zz} \ , \quad \sigma_{zz} = (\lambda + 2\mu) u_{zz} \ , \tag{12-6}$$

and Cauchy's equilibrium equation (12-1a) simplifies in this case to

$$\nabla_z \sigma_{zz} = \rho_0 g_0 \ . \tag{12-7}$$

Using the boundary condition $\sigma_{zz} = 0$ at $z = h$, this equation may immediately be integrated to

$$\sigma_{zz} = -\rho_0 g_0 (h - z) \ . \tag{12-8}$$

The vertical pressure $p_z = -\sigma_{zz} = \rho_0 g_0 (h - z)$ is positive and rises linearly with depth $h - z$, just as in the fluid sea. It balances everywhere the full weight of the material above, but this was expected since there are no shear stresses to distribute the vertical load. The horizontal pressures $p_x = p_y = p_z \lambda/(\lambda + 2\mu)$ are also positive but smaller than the vertical, because both $\lambda$ and $\mu$ are positive in normal materials. The horizontal pressures are eventually balanced by the fixed vertical walls.

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The strain

$$u_{zz} = \nabla_z u_z = -\frac{\rho_0 g_0}{\lambda + 2\mu}(h - z) \tag{12-9}$$

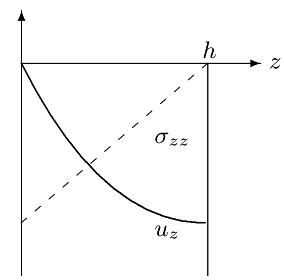
is negative, corresponding to a compression. The characteristic length scale for major deformation is in this case

$$D = \frac{\lambda + 2\mu}{\rho_0 g_0} = \frac{1-\nu}{(1+\nu)(1-2\nu)} \cdot \frac{E}{\rho_0 g_0} \ . \tag{12-10}$$

Integrating the strain with the boundary condition $u_z = 0$ for $z = 0$, we finally obtain

$$\boxed{u_z = -\frac{h^2 - (h-z)^2}{2D}} \ . \tag{12-11}$$

The displacement is always negative, largest in magnitude at the top, $z = h$, and varies quadratically with height $h$ at the top, as expected from the estimate in the preceding section.

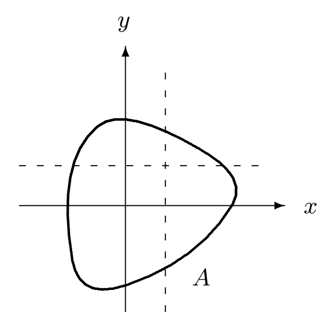


*Sketch of the displacement (solid curve) and stress (dashed) for the elastic "sea in a box".*

## Shear-free settling

What happens, if we remove the container walls? A fluid will of course spill out all over the place, whereas an elastic material is only expected to settle a bit more while bulging horizontally out where the walls were before. Jelly on a flat plate is perhaps the best image to have in mind.

In spite of the basic simplicity of the problem, which for physical reasons must have an answer, there seems to be no simple analytic solution. But if one cannot find the right solution to a problem, it is common practice — in physics as well as in politics — to redefine the problem to fit a solution which one *can* get! What we can get here, is a solution with no shear stresses (and strains), but the price to pay is that the vertical displacement will not vanish everywhere at the bottom of the container, as it ought to.



*Horizontal cross section of elastic block of "jelly". Straight lines running parallel with the axes of the coordinate system must cross the outer perimeter in at least two places.*

The equilibrium equations (12-1a) with all shear stresses set to zero, *i.e.* $\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$, simplify now to

$$\nabla_x \sigma_{xx} = 0 \ , \qquad \nabla_y \sigma_{yy} = 0 \ , \qquad \nabla_z \sigma_{zz} = \rho_0 g_0 \ , \tag{12-12}$$

The first equation says that $\sigma_{xx}$ does not depend on $x$, or in other words that $\sigma_{xx}$ is constant on straight lines parallel with the $x$-axis. But such lines must always cross the vertical sides, where the $x$-component of the stress vector, $\sigma_{xx} n_x$, has to vanish, and consequently we must have $\sigma_{xx} = 0$ everywhere. In the same way it follows that $\sigma_{yy} = 0$ everywhere. Finally, the third equation tells us that $\sigma_{zz}$ is linear in $z$, and using the condition that $\sigma_{zz} = 0$ for $z = h$ we find

$$\sigma_{zz} = -\rho_0 g_0 (h - z) \ , \tag{12-13}$$

implying that every column carries the weight of the material above it. This result was again to be expected because there are no shear stresses to redistribute the vertical load.

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From the inverse Hooke's law (11-17), the non-vanishing strain components are found to be

$$u_{xx} = u_{yy} = \nu \frac{\rho_0 g_0}{E}(h-z) \ , \qquad u_{zz} = -\frac{\rho_0 g_0}{E}(h-z) \ , \tag{12-14}$$

where $E$ is Young's modulus and $\nu$ Poisson's ratio. The typical scale of major deformation is in this case

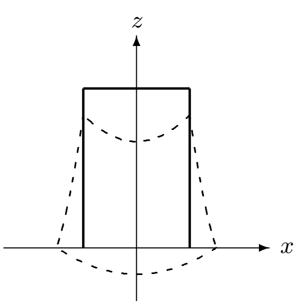
$$D = \frac{E}{\rho_0 g_0} \tag{12-15}$$

Using that $u_{xx} = \nabla_x u_x$, etc, the strains may be readily integrated, but in doing so, one must remember that all shear strains have to vanish. One may verify that the following displacement field leads to the strains above,

$$u_x = \frac{\nu}{D}(h-z)x \ , \tag{12-16a}$$

$$u_y = \frac{\nu}{D}(h-z)y \ , \tag{12-16b}$$

$$u_z = -\frac{1}{2D}\left(h^2 - (h-z)^2 + \nu(a^2 - x^2 - y^2)\right) \ , \tag{12-16c}$$




*Simple model for the gravitational settling of a block of elastic material ("jelly on a plate"). The model is not capable of fulfilling the boundary condition $u_z = 0$ at $z = 0$ and describes a cylindrical block which partly settles into the supporting surface.*

where $a$ is a constant. The peculiar term in the last expression is forced upon us by the requirement of vanishing shear stresses. The most general solution of the same kind is obtained by adding translations of the body in the $xy$-plane and a rotation of the body around the $z$-axis.

The trouble with the solution is that we cannot impose flatness at the bottom, *i.e.* $u_z = 0$ for $z = 0$, as we would like to do. The vertical displacement is negative everywhere except at the circle $x^2 + y^2 = a^2$, where it vanishes. In the $xy$-plane, however, the horizontal displacement represents a uniform expansion in all horizontal directions with a $z$-dependent scale factor, which vanishes on top and is maximal at the bottom. Instead of describing the deformation of a block of material sitting on a hard and flat horizontal surface, we have obtained a solution which seems to describe a cylindrical block sinking into the supporting surface. We shall not speculate into what kind of material would allow this solution to be realized.

There can be only one explanation, namely that the initial assumption about the shear-free stress tensor is wrong. What seems to be needed to get a solution respecting that the supporting surface is hard and flat, is an extra vertical pressure distribution from the supporting surface which is able to "shore up" the sagging underside of the shear-free solution and make it flat. We expect that this extra pressure will be accompanied by shear stresses, enabling the inner part of the block at $x = 0$ to carry more than its share of the weight of the material above it, and the outer part at $x = a$ to carry less (see section 14.3).

The extra stress distribution can presumably only exert influence on the deformation up to a height of the same order of magnitude as the horizontal dimensions of the block. For a tall block, the shear-free solution may thus be expected to be valid everywhere except in a region near the bottom of the same height as the horizontal dimensions of the block.

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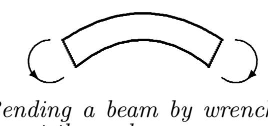
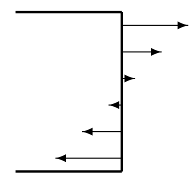
## 12.3 Bending a beam

Sticks, girders, masts, towers, planks, poles, and pipes are all examples of a generic object, which we shall call a *beam*. Geometrically, a beam consists of a bundle of straight parallel lines, or *rays*, covering the same cross section in any plane orthogonal to the lines. Physically, the beam is assumed to be made from homogeneous and isotropic elastic material.



*Bending a beam by wrenching at the ends.*

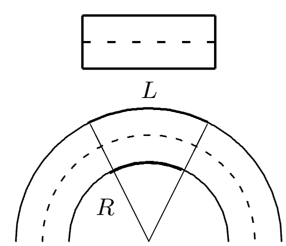
There are many ways to bend a beam. It may be fixed in one end and bent like a horizontal flagpole, or like a fishing rod with a fish at the end. It may be weighed down in the middle like a bridge, but the cleanest way is probably to grab it close to the ends and wrench it like a pencil so that it gets a uniformly curved shape.



*A bending couple may be created by varying normal stresses applied to a terminal.*

Ideally, in *pure bending*, external stresses should not be applied to the sides of the beam, but only to the terminal cross sections, and on the average these stresses should neither stretch nor compress the beam, but only provide external couples at the terminals. It should be noted that such couples do not require shear stresses, but may be created by normal stresses alone which vary in strength over the terminal cross sections. If you try, you will realize that it is in fact rather hard to bend a pencil in this way. Bending a rubber eraser by pressing it between two fingers is somewhat easier, but tends to add longitudinal compression as well.

The bending of the beam is also assumed to be *uniform*, such that the physical conditions, stresses and strains, will be the same everywhere along the beam. This is only possible, if the originally straight beam of length $L$ is deformed to become a small section of a (huge) circular ring of radius $R$ with every ray becoming part of a perfect circle. In that case, it is sufficient to consider just a small slice of the beam in order to understand uniform bending for a beam of any length.




*In uniform bending, the bent beam becomes a part of a circular ring (without twist).*

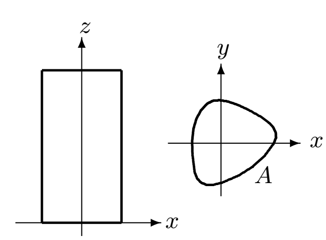
### Centering the beam

In a Cartesian coordinate system, we align the undeformed beam with the $z$-axis, and for a beam of length $L$, we put the terminal cross sections at $z = 0$ and $z = L$. The cross section $A$ in the $xy$-plane may be of arbitrary shape.

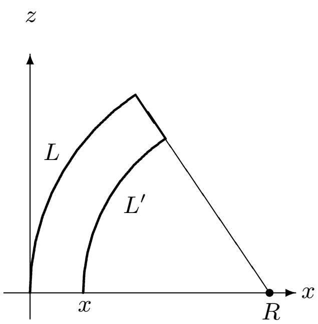
We may without loss of generality position the coordinate system in the $xy$-plane such that its origin coincides with the *center* of the area, defined such that

$$\int_A x\,dxdy = \int_A y\,dxdy = 0 \ . \tag{12-17}$$

In other words, we require that the $z$-axis, $x = y = 0$, coincides with the *central ray* of the unbent beam. Finally, we fix the remaining freedom by requiring the central ray after bending to be part of a circle in the $xz$-plane with radius $R$ and its center on the $x$-axis at $x = R$. The radius $R$ is obviously the length scale for major deformation.

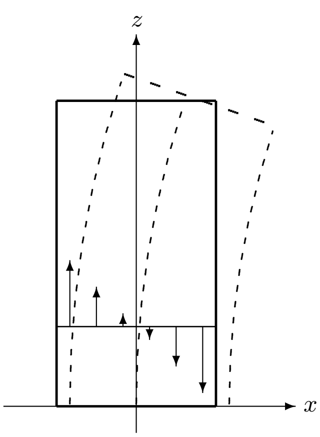



*The unbent beam is aligned with the $z$-axis, and its cross section, $A$, in the $xy$-plane is the same for all $z$. The $z$-axis goes through the center of the cross section.*

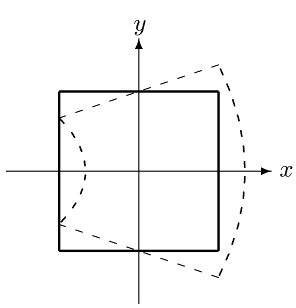



*The length of the arc at $x$ must satisfy*

$$\frac{L'}{R-x} = \frac{L}{R} \ .$$




*Sketch of the bending of a beam towards positive $x$-values. The arrows indicate the vertical strain.*




*Sketch of the deformation in the $xy$-plane of a beam with rectangular cross section (exaggerated). This peculiar deformation may easily be observed by bending a rubber eraser.*

## Shear-free solution

What happens in the beam when it is bent depends on the way the stresses are distributed on its terminals. In the simplest case, we may view the beam as a loose bundle of thin elastic strings that do not interact with each other, but get stretched or compressed individually according to their position in the beam without generating shear stresses. Let us fix the central string so that it does not change its length $L$, when bent into a circle. A simple geometric construction then shows, that a nearby ray in position $x$ will change its length to $L' = L(1 - x/R)$, and consequently experience a longitudinal strain,

$$u_{zz} = \frac{L' - L}{L} = -\frac{x}{R} \tag{12-18}$$

For negative $x$ the material of the beam is being stretched vertically, and compressed for positive $x$.

Under the assumption that the bending is done without shear and that there are no forces acting on the sides of the beam, it follows as in the preceding section that $\sigma_{xx} = \sigma_{yy} = 0$. The only non-vanishing stress is $\sigma_{zz} = Eu_{zz}$, and the non-vanishing strains are as before found from the inverted Hooke's law (11-17),

$$u_{xx} = u_{yy} = -\nu u_{zz} = \nu \frac{x}{R} \ , \tag{12-19}$$

where $\nu$ Poisson's ratio. This shows that the material is being stretched horizontally for $x < 0$ and compressed for $x > 0$.

Using that $u_{xx} = \nabla_x u_x$, etc, a particular solution is found to be

$$\boxed{\begin{aligned} u_x &= \frac{1}{2R}\left(z^2 + \nu\left(x^2 - y^2\right)\right) \ , \\ u_y &= \frac{\nu}{R} xy \ , \\ u_z &= -\frac{1}{R} xz \ . \end{aligned}} \tag{12-20}$$

The quadratic terms depending on $y$ and $z$ in $u_x$ are as in the preceding section forced upon us by the requirement of no shear stresses (and strains). In order for displacement gradients to be small, all dimensions of the beam have to be small compared to $R$.

## Forces and couples

The only non-vanishing stress component is as mentioned before

$$\sigma_{zz} = Eu_{zz} = -\frac{E}{R} x \ . \tag{12-21}$$

It is a tension for negative $x$, and we consequently expect the material of the beam to first break down at the extreme point of the cross section opposite to the direction of bending, as common experience also tells us.

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The total force acting on a cross section vanishes

$$\mathcal{F}_z = \int_A \sigma_{zz}\, dS_z = -\frac{E}{R}\int_A x\, dxdy = 0 \ , \tag{12-22}$$

because of the centering of the beam (12-17). The non-vanishing moments of the longitudinal stress are

$$\mathcal{M}_x = \int_A y\sigma_{zz}\, dS_z = -\frac{E}{R}\int_A xy\, dxdy \ , \tag{12-23}$$

$$\mathcal{M}_y = -\int_A x\sigma_{zz}\, dS_z = \frac{E}{R}\int_A x^2\, dxdy \ . \tag{12-24}$$

If the undeformed beam is mirror symmetric under reflection in the $xz$-plane or in the $yz$-plane, then $\int_A xy\, dxdy = 0$, so that $M_x = 0$.

## The Bernoulli-Euler law

The above expression for $\mathcal{M}_y$ is an example of the *Bernoulli-Euler* law, which may also be written

$$\boxed{R = \frac{EI}{\mathcal{M}}} \ , \tag{12-25}$$

where

$$I = \int_A r_\perp^2\, dA \tag{12-26}$$

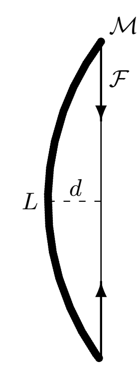
is the "moment of inertia" of the beam cross section around the direction of $\mathcal{M}$ (analogous to eq. (5-28)). In this form, it expresses the bending radius $R$ of a beam in terms of the applied couple $\mathcal{M}$. For a rectangular beam with $A = a \times b$, we get

$$I = \int_A x^2\, dxdy = b\int_{-a/2}^{a/2} x^2 dx = \frac{a^3 b}{12} \ . \tag{12-27}$$

For a circular beam with radius $a$, the moment of inertia becomes

$$I = \int_A x^2\, dxdy = \int_0^a dr \int_0^{2\pi} r d\phi\, r^2 \cos^2\phi = \frac{\pi}{4}a^4$$

In many engineering applications this law is enough to give a reasonable idea of how much a beam is deformed by applied moments.




*Steel bow under tension. The moment can be estimated as $\mathcal{M} \approx \mathcal{F}d$.*

**Example 12.3.1:**

A steel bow is constructed from a rectangular beam of length $L = 2$m with dimensions $a = 10$ mm and $b = 20$ mm. The bow is bent such that its radius of curvature becomes $R = 5$ m, and the string distance from the bow becomes $d = L^2/8R = 10$ cm. The moment of inertia is $I \approx 1.6 \times 10^{-9}$ m$^4$ and the terminal moment is $\mathcal{M} = EI/R \approx 65$ Nm$^2$ for $E = 2 \times 10^{11}$ Pa. The force acting on the bow string is $\mathcal{F} \approx \mathcal{M}/d \approx 650$ N, corresponding to a weight of 65 kg. It takes a Little John's strength to shoot with this bow.

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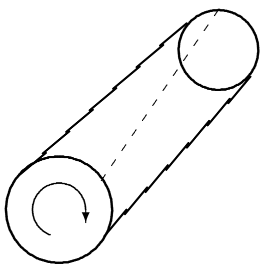
## 12.4 Twisting a shaft

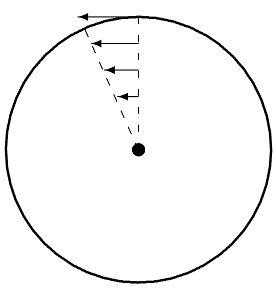


*Pure torsion consists in rotating every cross section by a fixed amount per unit of length.*

The drive shaft in your car connects the gear box with the differential, and transmits engine power to the wheels. In characterizing engine performance, maximum torque is often quoted, because it creates the largest shear force between wheels and road and therefore maximal acceleration, barring wheel-spin. Although the shaft is made from steel, it will nevertheless undergo a tiny deformation, a *torsion* or twist.

### Pure torsion

The shaft is assumed to be a circular beam with radius $a$ and its axis coinciding with the $z$-axis. The deformation is said to be a *pure torsion*, if the shaft's material is rotated by a constant amount $\tau$ per unit of length, such that a given cross section at $z$ is rotated by the angle $\tau z$ relative to the cross section at $z = 0$. The constant $\tau$ which measures the rotation angle per unit of length is sometimes called the torsion angle.



*The displacement field for a rotation through a tiny angle $\tau z$ (exaggerated here) is purely tangential and grows linearly with the radial distance.*

The uniform nature of pure torsion allows us to consider a small slice of the shaft of length $L$ which is only twisted through a tiny angle, so that $\tau L \ll 1$. Since the physical conditions are the same in all such slices, we can later put them together to make a shaft of any length. To lowest order in the angle $\tau z$, the displacement in the slice becomes

$$\boldsymbol{u} = \tau z \boldsymbol{e}_z \times \boldsymbol{x} = \tau z(-y, x, 0) \ . \tag{12-28}$$

Not surprisingly, it is purely tangential and is always much smaller than the radius, $a$.

### Strain and stress

From Cauchy's strain tensor, we see that the only non-vanishing strains are

$$u_{xz} = -\frac{1}{2}\tau y \ , \quad u_{yz} = \frac{1}{2}\tau x \ . \tag{12-29}$$

For the strains to be small, we must also require $|\tau|\, a \ll 1$, or in other words that the twist must be small over a length of the shaft comparable to its radius. The corresponding stresses are obtained from Hooke's law (11-8)

$$\sigma_{xz} = \sigma_{zx} = -\mu\tau y \ , \quad \sigma_{yz} = \sigma_{zy} = \mu\tau x \ , \tag{12-30}$$

and it is seen that the equilibrium equations

$$\nabla_z \sigma_{xz} = 0 \ , \quad \nabla_z \sigma_{yz} = 0 \ , \quad \nabla_x \sigma_{zx} + \nabla_y \sigma_{zy} = 0 \ ,$$

are trivially fulfilled by the solution. At the cylindrical surface of the shaft, the normal is $(x, y, 0)/a$, and the stress vector vanishes, *i.e.* $\sigma_{zx} x/a + \sigma_{zy} y/a = 0$, as it should when there are no external forces acting there. This solution was

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first obtained by Coulomb in 1787, whereas the corresponding solution for rods with non-circular cross section (see [10, p. 59] or [28, p. 109]) was obtained by Saint-Venant in 1855.

## Torque

In any cross section, we may calculate the total moment of force, also called the *torque*, that one part of the shaft imposes on the other. On the surface element $d\boldsymbol{S}$, it is given by the cross product $d\boldsymbol{\mathcal{M}} = \boldsymbol{x} \times \boldsymbol{\sigma} \cdot d\boldsymbol{S}$. Since the cross section lies in the $xy$-plane, the moment has only a $z$-component

$$\begin{aligned} \mathcal{M}_z &= \int_A (x\sigma_{yz} - y\sigma_{xz}) dS_z = \int_A \mu\tau(x^2+y^2)dxdy \\ &= \int_0^a \mu\tau r^2 \cdot 2\pi r dr = \frac{1}{2}\pi\mu\tau a^4 \ . \end{aligned} \tag{12-31}$$

The quantity

$$C = \frac{M_z}{\tau} = \frac{1}{2}\pi\mu a^4 \tag{12-32}$$

is called the *torsional rigidity* of the shaft and is measured in units of $\mathsf{N\ m}^2$. The torsional rigidity depends only the radius of the shaft and the shear modulus but not on the applied torque. Knowing the torsional rigidity, one may calculate the torsion angle given the torque, $\tau = \mathcal{M}_z/C$, and conversely.

If the shaft rotates with constant angular velocity $\Omega$, the material in the point $(x, y, z)$ will have velocity $\boldsymbol{v}(x, y) = \Omega(-y, x, 0)$. The shear stresses acting on an element of the cross section, $d\boldsymbol{S}$, will transmit a power (work per unit of time) of $dP = \boldsymbol{v} \cdot d\boldsymbol{\mathcal{F}} = \boldsymbol{v} \cdot \boldsymbol{\sigma} \cdot d\boldsymbol{S} = \boldsymbol{v} \cdot \boldsymbol{\sigma}_z dS_z$. Integrating over the cross section this becomes

$$P = \int_A \boldsymbol{v}(x,y) \cdot \boldsymbol{\sigma}_z \, dS_z = \int_A \Omega(x\sigma_{yz} - y\sigma_{xz}) dS_z = \Omega\mathcal{M}_z \ . \tag{12-33}$$

This relation is in fact generally true for rigid body rotation.

> **Example 12.4.1:** The typical torque delivered by a car engine can be of the order of 100 Nm. If the shaft rotates with $3,000$ rpm, corresponding to an angular velocity of $\Omega \approx 300$ rad/s, the transmitted power is about 30 kW, or 40 horsepower. For a drive shaft made of steel with radius 2 cm, the torsional rigidity is $C \approx 2 \times 10^4 Nm^2$. In direct drive without gearing, the torsion angle becomes $\tau \approx 0.005$ rad/m. For a car with rear-wheel drive, the length of the drive shaft may be about 2 m, and the total twist amounts to about 0.6 degrees. The maximal shear stress in the material is $\mu\tau a \approx 8 \times 10^6$ Pa $=$ 80 bar at the rim of the shaft.

In order to realize a pure torsion, the correct stress distribution (12-30) must be applied to the ends of the shaft. Applying a different stress distribution, for example grabbing one end of the shaft with a monkey-wrench, leads to a different solution near the end. The pure torsion solution should however be a valid approximation far away from the ends.

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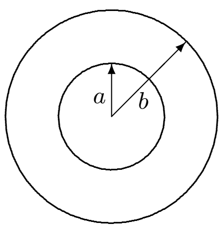
## 12.5 Tube under pressure



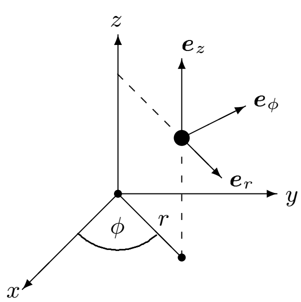

*Tube cross section.*

Elastic tubes carrying fluids under pressure are found everywhere, in living organisms and in machines, not to forget the short moments of intense pressure in the barrel of a gun or canon. How much does a tube expand because of the pressure, and how is the deformation distributed? What are the stresses in the material and where will it tend to break down?

### Uniform radial displacement

The ideal tube is a beam in the shape of a circular cylinder with inner radius $a$, outer radius $b$, and length $L$, made from homogeneous and isotropic elastic material. When subjected to a uniform internal pressure, the tube is expected to expand radially and maybe also contract longitudinally. The latter may be prevented by clamping the ends of the tube, so to simplify matters we shall assume that the displacement field is uniformly radial, of the form[1]

$$\boldsymbol{u} = u_r(r)\,\boldsymbol{e}_r \;, \tag{12-34}$$




*Cylindrical coordinates and basis vectors.*

where $u_r(r)$ is only a function of the radial distance $r = \sqrt{x^2+y^2}$, and $\boldsymbol{e}_r$ is the radial unit vector at the point $(x, y, z)$,

$$\boldsymbol{e}_r = \frac{(x, y, 0)}{r} \;. \tag{12-35}$$

It is tempting here to introduce true cylindrical coordinates, $(r, \phi, z)$, instead of the Cartesian coordinates, $(x, y, z)$, but although more systematic it would in fact make the following analysis harder. The only other element we need here is the angular unit vector

$$\boldsymbol{e}_\phi = \frac{(-y, x, 0)}{r} \;, \tag{12-36}$$

which is orthogonal to both $\boldsymbol{e}_r$ and $\boldsymbol{e}_z = (0, 0, 1)$. The three unit vectors form together a local orthogonal basis for cylindrical geometry (see appendix C for details).

### Displacement gradients

In Cartesian coordinates, the displacement field takes the form

$$u_x = \frac{x}{r} u_r(r) \tag{12-37}$$

$$u_y = \frac{y}{r} u_r(r) \tag{12-38}$$

[1] The index on $u_r$ is redundant and could be dropped, but we keep it systematically so as to remind ourselves that it is the radial component of the displacement field.

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It is then straightforward to calculate the non-vanishing displacement gradients

$$\nabla_x u_x = \frac{x^2}{r}\frac{d}{dr}\left(\frac{u_r}{r}\right) + \frac{u_r}{r} = \frac{x^2}{r^2}\frac{du_r}{dr} + \frac{y^2}{r^2}\frac{u_r}{r} \; , \tag{12-39a}$$

$$\nabla_y u_y = \frac{y^2}{r}\frac{d}{dr}\left(\frac{u_r}{r}\right) + \frac{u_r}{r} = \frac{y^2}{r^2}\frac{du_r}{dr} + \frac{x^2}{r^2}\frac{u_r}{r} \; , \tag{12-39b}$$

$$\nabla_x u_y = \nabla_y u_x = \frac{xy}{r}\frac{d}{dr}\left(\frac{u_r}{r}\right) = \frac{xy}{r^2}\frac{du_r}{dr} - \frac{xy}{r^2}\frac{u_r}{r} \; , \tag{12-39c}$$

where we have also used that $\partial r/\partial x = x/r$, etc. Adding the two first equations we get, for example the divergence of the displacement field

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{du_r}{dr} + \frac{u_r}{r} = \frac{1}{r}\frac{d(ru_r)}{dr} \; , \tag{12-40}$$

where the last expression is convenient in the following.

## Equilibrium equation

Since $\boldsymbol{\nabla} r = \boldsymbol{e}_r$, it follows from the radial assumption (12-34) that the displacement field may be written as the gradient of another field

$$\boldsymbol{u} = \boldsymbol{\nabla}\psi(r) \; , \qquad \psi(r) = \int u_r(r)\, dr \; . \tag{12-41}$$

But then the gradient of its divergence becomes

$$\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{\nabla}\boldsymbol{\nabla}^2\psi = \boldsymbol{\nabla}^2\boldsymbol{\nabla}\psi = \boldsymbol{\nabla}^2\boldsymbol{u} \; , \tag{12-42}$$

because the derivatives commute. The Navier-Cauchy equation (12-3) now takes the much simpler form

$$(2\mu + \lambda)\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u} = -\boldsymbol{f} \; , \tag{12-43}$$

and using the expression (12-40) for the divergence, this becomes

$$(2\mu + \lambda)\boldsymbol{e}_r \frac{d}{dr}\left(\frac{1}{r}\frac{d(ru_r)}{dr}\right) = -\boldsymbol{f} \; . \tag{12-44}$$

This shows that the body force density, if present, must also be radial,

$$\boldsymbol{f} = f_r(r)\boldsymbol{e}_r \; , \tag{12-45}$$

and we finally arrive at the ordinary second order differential equation in $r$,

$$\boxed{(\lambda + 2\mu)\frac{d}{dr}\left(\frac{1}{r}\frac{d(ru_r)}{dr}\right) = -f_r} \; . \tag{12-46}$$

Given a radial body force, this equation may be integrated to yield the radial displacement.

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## General solution without body forces

In the simplest case, $f_r = 0$, and we find immediately

$$\frac{1}{r}\frac{d(ru_r)}{dr} = 2A \ , \tag{12-47}$$

where $A$ is a constant. Integrating once more we obtain

$$\boxed{u_r(r) = Ar + \frac{B}{r}} \ , \tag{12-48}$$

where $B$ is another constant. These constants will be determined by the boundary conditions to be imposed on the tube.

## Strain and stress

Expressed in the tensor product notation (2-10), the displacement gradients (12-39) may be compactly written

$$\boldsymbol{\nabla u} = \frac{du_r}{dr}\boldsymbol{e}_r\boldsymbol{e}_r + \frac{u_r}{r}\boldsymbol{e}_\phi\boldsymbol{e}_\phi \ .$$

Since the right hand side is a symmetric matrix, it is identical to Cauchy's strain tensor, which accordingly has only two non-vanishing projections (in the sense of (10-22))

$$u_{rr} = \frac{du_r}{dr} = A - \frac{B}{r^2} \ , \tag{12-49}$$

$$u_{\phi\phi} = \frac{u_r}{r} = A + \frac{B}{r^2} \ . \tag{12-50}$$

Finally, the non-vanishing stress tensor components are found from Hooke's law (11-8) by projecting on the basis vectors

$$\sigma_{rr} = 2\mu u_{rr} + \lambda(u_{rr} + u_{\phi\phi}) = 2A(\lambda+\mu) - \frac{2\mu B}{r^2} \ , \tag{12-51a}$$

$$\sigma_{\phi\phi} = 2\mu u_{\phi\phi} + \lambda(u_{rr} + u_{\phi\phi}) = 2A(\lambda+\mu) + \frac{2\mu B}{r^2} \ , \tag{12-51b}$$

$$\sigma_{zz} = \lambda(u_{rr} + u_{\phi\phi}) = 2A\lambda \ . \tag{12-51c}$$

Here we have used that the trace of the strain tensor is independent of the basis, so that $\sum_k u_{kk} = u_{xx} + u_{yy} = u_{rr} + u_{\phi\phi}$. One should notice that a longitudinal stress, $\sigma_{zz}$, appears as a consequence of the fixed clamps on the ends of the cylinder.

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## Solution for the pressurized tube

The boundary conditions are $\sigma_{rr} = -P$ at the inside surface $r = a$ and $\sigma_{rr} = 0$ at the outside surface $r = b$. The minus sign may be a bit surprising, but remember that the normal to the inside surface of the tube is in the direction of $-\boldsymbol{e}_r$, so that the stress vector $\sigma_{rr}(-\boldsymbol{e}_r) = P\boldsymbol{e}_r$ points in the positive radial direction, as it should. Solving the boundary conditions for $A$ and $B$,

$$2A(\lambda+\mu) - \frac{2\mu B}{a^2} = -P \tag{12-52}$$

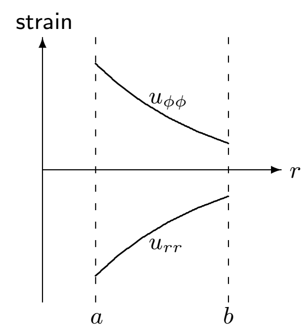
$$2A(\lambda+\mu) - \frac{2\mu B}{b^2} = 0 \tag{12-53}$$

we find the integration constants

$$A = \frac{1}{2(\lambda+\mu)}\frac{a^2}{b^2-a^2}P = (1+\nu)(1-2\nu)\frac{a^2}{b^2-a^2}\frac{P}{E}\ , \tag{12-54a}$$

$$B = \frac{1}{2\mu}\frac{a^2b^2}{b^2-a^2}P = (1+\nu)\frac{a^2b^2}{b^2-a^2}\frac{P}{E}\ , \tag{12-54b}$$

where $E$ is Young's modulus and $\nu$ Poisson's ratio.



*Sketch of strain components in the tube.*

**Displacement field:** The displacement field becomes

$$u_r(r) = (1+\nu)\frac{a^2}{b^2-a^2}\left((1-2\nu)r + \frac{b^2}{r}\right)\frac{P}{E}\ . \tag{12-55}$$

Since $\nu \leq 1/2$, it is always positive and monotonically decreasing. It reaches its maximum at the inner surface, $r = a$, confirming the intuition that the pressure in the tube should push the innermost material farthest away from its original position.

**Strain tensor:** The non-vanishing strain tensor components become

$$u_{rr} = (1+\nu)\frac{a^2}{b^2-a^2}\left(1-2\nu-\frac{b^2}{r^2}\right)\frac{P}{E}\ , \tag{12-56a}$$

$$u_{\phi\phi} = (1+\nu)\frac{a^2}{b^2-a^2}\left(1-2\nu+\frac{b^2}{r^2}\right)\frac{P}{E}\ , \tag{12-56b}$$

For normal materials with $0 < \nu \leq 1/2$, the radial strain $u_{rr}$ is negative, corresponding to a compression of the material, whereas the tangential strain $u_{\phi\phi}$ is always positive, corresponding to an extension. There is no longitudinal strain because of the clamping of the ends of the tube.

The scale of the strain is again set by the ratio $P/E$. For normal materials under normal pressures, for example an iron pipe with $E \approx 1$ Mbar subject to a water pressure of a few bars, the strain is only of the order of parts per million, whereas

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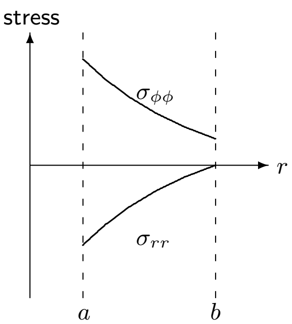
the strains in the walls of your garden hose or the arteries in your body are much larger. When the walls become thin, *i.e.* for $d = b - a \ll a$, the strains grow stronger because of the denominator $b^2 - a^2 \approx 2da$, and actually diverge towards infinity in the limit. This is in complete agreement with the feeling that a tube needs a certain thickness of its walls in order to withstand the inside pressure.




*Sketch of stress components in the tube.*

**Stress tensor:** The non-vanishing stress tensor components become

$$\sigma_{rr} = -\frac{a^2}{b^2 - a^2}\left(\frac{b^2}{r^2} - 1\right) P \ , \tag{12-57a}$$

$$\sigma_{\phi\phi} = \frac{a^2}{b^2 - a^2}\left(\frac{b^2}{r^2} + 1\right) P \ , \tag{12-57b}$$

$$\sigma_{zz} = 2\nu \frac{a^2}{b^2 - a^2} P \ . \tag{12-57c}$$

Notice that the transversal stresses, $\sigma_{rr}$ and $\sigma_{\phi\phi}$, are independent of the material properties of the tube, $E$ and $\nu$, and that the longitudinal stress, $\sigma_{zz}$, only depends on Poisson's ratio, $\nu$.

**Pressure:** The radial pressure, $p_r = -\sigma_{rr}$ can never become larger than $P$, because we may write

$$\frac{p_r}{P} = \frac{b^2 - r^2}{b^2 - a^2}\ \frac{a^2}{r^2} \ , \tag{12-58}$$

which is the product of two factors, both smaller than unity for $a < r < b$. The tangential pressure $p_\phi = -\sigma_{\phi\phi}$ and the longitudinal pressure $p_z = -\sigma_{zz}$ are both negative (tensions), and can become large for thin-walled tubes. The average pressure

$$p = \frac{1}{3}(p_r + p_\phi + p_z) = -\frac{2}{3}(1 + \nu)\frac{a^2}{b^2 - a^2} P \tag{12-59}$$

is also negative and like the longitudinal pressure constant everywhere in the material. Notice that the average pressure is not continuous with the pressure outside the tube. This confirms the suspicion voiced on page 146 that the average pressure may behave differently in a solid with shear stresses than the pressure in a fluid at rest, which has to be continuous at the boundaries.

**Blowup:** A tube under pressure develops cracks and eventually blows up if the material is extended beyond a certain limit. Compression doesn't matter, except for enormous pressures. The point where the tube breaks is primarily determined by the point of maximal tension. As we have seen, this occurs at the inside of the tube for $r = a$ in the tangential direction. A crack will develop where the material has a small weakness, and then the tube blows up from the inside!

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## Unclamped tube

The constancy of the longitudinal tension (12-57c) permits us to solve the case of an unclamped tube by superposing the above solution with the displacement field for uniform stretching (11-24). In the cylindrical basis the field of uniform stretching becomes (after interchanging $x$ and $z$)

$$u_r = -\nu r \frac{Q}{E} , \qquad u_z = z\frac{Q}{E} , \tag{12-60}$$

where $Q$ is the tension applied to the ends. Choosing $Q$ equal to the longitudinal tension (12-57c) in the clamped tube,

$$Q = 2\nu \frac{a^2}{b^2 - a^2} P , \tag{12-61}$$

and subtracting the stretching field from the clamped tube field (12-55), we get for the unclamped tube

$$u_r = \frac{a^2}{b^2 - a^2}\left((1-\nu)r + (1+\nu)\frac{b^2}{r}\right)\frac{P}{E} , \tag{12-62a}$$

$$u_z = -2\nu \frac{a^2}{b^2 - a^2} z \frac{P}{E} . \tag{12-62b}$$

The strains are likewise obtained from the clamped strains (12-56a) by subtracting the strains for uniform stretching, and we get

$$u_{rr} = \frac{a^2}{b^2 - a^2}\left(1 - \nu - (1+\nu)\frac{b^2}{r^2}\right)\frac{P}{E} , \tag{12-63a}$$

$$u_{\phi\phi} = \frac{a^2}{b^2 - a^2}\left(1 - \nu + (1+\nu)\frac{b^2}{r^2}\right)\frac{P}{E} , \tag{12-63b}$$

$$u_{zz} = -2\nu \frac{a^2}{b^2 - a^2}\frac{P}{E} . \tag{12-63c}$$

The radial and tangential stresses are the same as before and given by (12-57), except for the longitudinal stress which now vanishes, $\sigma_{zz} = 0$.

## Thin wall approximation

Most tubes have thin walls relative to their radius. Let us introduce the wall thickness, $d = b - a$, and the radial distance, $s = r - a$, from the inner wall. In the thin-wall approximation, these quantities are small compared to $a$, and we obtain the following expressions to leading order for the unclamped tube.

The radial displacement field is constant in the material whereas the longitudinal one is linear in $z$,

$$u_r \approx a\frac{a}{d}\frac{P}{E} , \qquad u_z \approx -z\nu\frac{a}{d}\frac{P}{E} . \tag{12-64}$$

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The corresponding strains become

$$u_{rr} \approx -\nu \frac{a}{d} \frac{P}{E} , \qquad u_{\phi\phi} \approx \frac{a}{d} \frac{P}{E} , \qquad u_{zz} \approx -2\nu \frac{a}{d} \frac{P}{E} . \tag{12-65a}$$

The strains all diverge for $d \to 0$, and the condition for small strains is now $P/E \ll d/a$. The ratio $a/d$ amplifies the strains beyond naive estimates. Finally, we get the non-vanishing stresses

$$\sigma_{rr} \approx -\left(1 - \frac{s}{d}\right) P , \tag{12-66a}$$

$$\sigma_{\phi\phi} \approx \frac{a}{d} P . \tag{12-66b}$$

The radial pressure $p_r = -\sigma_{rr}$ varies between 0 and $P$ as it should when $s$ ranges from 0 to $d$. It is always positive and of order $P$, whereas the tangential tension $p_\phi = -\sigma_{\phi\phi}$ diverges for $d \to 0$. Blowups always happen because the tangential tension becomes excessive.

## ∗ 12.6 Elastic spherical shell

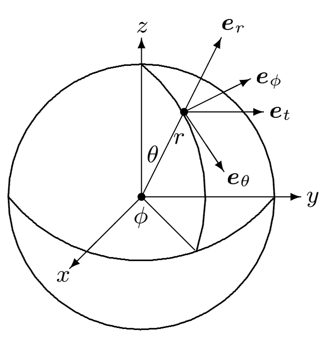
Elastic spheres are as interesting technologically as elastic tubes, from ball-bearings to deep-sea exploration. A spherically shaped vessel withstands external pressure better than any other shape. Nevertheless, in films of submarines going too deep, you see the rivets beginning to pop on the inside of the vessel. Why is that?

### Uniform radial displacement

In this case we shall imagine that a homogeneous, isotropic spherical shell with inner radius $a$ and outer radius $b$ is submitted to a constant pressure $P$ from the outside. The displacement field must for symmetry reasons be radial

$$\boldsymbol{u}(\boldsymbol{x}) = u_r(r)\boldsymbol{e}_r \tag{12-67}$$

where $\boldsymbol{e}_r = \boldsymbol{x}/r$ is the radial unit vector and $r = \sqrt{x^2 + y^2 + z^2}$ is the radial distance.




*Spherical coordinates and their basis vectors. The general tangential unit vector is called $\boldsymbol{e}_t$*

As for the pressurized tube, it is not necessary to introduce polar coordinates as such, but it is useful in addition to $\boldsymbol{e}_r$ to introduce a tangential unit vector $\boldsymbol{e}_t$ which is orthogonal to $\boldsymbol{e}_r$. On a sphere all directions orthogonal to the radius vector are equivalent, so we do not have to specify which of the two angular basis vectors it is. It may in fact be any linear combination of them.

### Equilibrium equation

Exactly as for the pressurized tube, we may use the spherical form to write the displacement field as a gradient, so that $\boldsymbol{\nabla}^2\boldsymbol{u} = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{u}$. The divergence is, however, different, and we find

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{u} &= \nabla_x\left(x\frac{u_r}{r}\right)+\nabla_y\left(y\frac{u_r}{r}\right)+\nabla_z\left(z\frac{u_r}{r}\right)\\ &= r\frac{d}{dr}\left(\frac{u_r}{r}\right)+3\frac{u_r}{r}=\frac{du_r}{dr}+2\frac{u_r}{r}\\ &= \frac{1}{r^2}\frac{d(r^2u_r)}{dr}\ .\end{aligned}$$

The Navier-Cauchy equation (12-3) now takes the form

$$\boxed{(\lambda+2\mu)\frac{d}{dr}\left(\frac{1}{r^2}\frac{d(r^2u_r)}{dr}\right)=-f_r}\ , \tag{12-68}$$

where $f_r$ is the necessarily radial body force.

The most general solution to this equation for $f_r=0$ is

$$u_r = Ar+\frac{B}{r^2} \tag{12-69}$$

where $A$ and $B$ are integration constants to be determined by the boundary conditions.

## Strains and stresses

The non-vanishing strain gradients are calculated using tensor notation for $u_i = x_iu_r/r$, and we find

$$\nabla_j u_i = \frac{x_ix_j}{r}\frac{d}{dr}\left(\frac{u_r}{r}\right)+\delta_{ij}\frac{u_r}{r}=\frac{x_ix_j}{r^2}\left(\frac{du_r}{dr}-\frac{u_r}{r}\right)+\delta_{ij}\frac{u_r}{r}\ .$$

By means of the tensor product (2-10) this may be written compactly in the form

$$\boldsymbol{\nabla}\boldsymbol{u}=\left(\frac{du_r}{dr}-\frac{u_r}{r}\right)\boldsymbol{e}_r\boldsymbol{e}_r+\frac{u_r}{r}\mathbf{1}\ . \tag{12-70}$$

Due to the symmetry of the right hand side, this is also the strain tensor, and projecting from both sides with $\boldsymbol{e}_r$ and $\boldsymbol{e}_t$, we obtain

$$u_{rr}=\frac{du_r}{dr}=A-2\frac{B}{r^3}\ ,\qquad u_{tt}=\frac{u_r}{r}=A+\frac{B}{r^3} \tag{12-71}$$

The corresponding stresses are obtained from (11-8) by projection from both sides

$$\sigma_{rr}=2\mu u_{rr}+\lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}=(2\mu+3\lambda)A-4\mu\frac{B}{r^3} \tag{12-72a}$$

$$\sigma_{tt}=2\mu u_{tt}+\lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}=(2\mu+3\lambda)A+2\mu\frac{B}{r^3} \tag{12-72b}$$

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Finally, the constants are determined from $\sigma_{rr}(a) = 0$ and $\sigma_{rr}(b) = -P$, and we get

$$A = -\frac{P}{2\mu + 3\lambda}\frac{b^3}{b^3 - a^3} , \qquad B = -\frac{P}{4\mu}\frac{a^3 b^3}{b^3 - a^3} . \tag{12-73}$$

The displacement field becomes

$$u_r = -\frac{b^3}{b^3 - a^3}\left((1 - 2\nu)r + \frac{1}{2}(1+\nu)\frac{a^3}{r^2}\right)\frac{P}{E} , \tag{12-74}$$

where $E$ is Young's modulus and $\nu$ Poisson's ratio. The radial and tangential strains now follow

$$u_{rr} = -\frac{b^3}{b^3 - a^3}\left(1 - 2\nu - (1+\nu)\frac{a^3}{r^3}\right)\frac{P}{E} , \tag{12-75a}$$

$$u_{tt} = -\frac{b^3}{b^3 - a^3}\left(1 - 2\nu + \frac{1}{2}(1+\nu)\frac{a^3}{r^3}\right)\frac{P}{E} . \tag{12-75b}$$

A positive external pressure will always compress the material in the tangential directions. In the radial direction the material is always expanded at the inside for $r = a$, although it may or may not turn into a compression further out, depending on the precise value of $\nu$ and the ratio $b/a$.

The shell always becomes thicker when it is compressed, because

$$u_r(b) - u_r(a) = \frac{1}{2}(1+\nu)\frac{ab(a+b)}{a^2 + ab + b^2}\frac{P}{E} \tag{12-76}$$

is positive. This is presumably why you are shown rivets jumping out of the hull in films of submarines going too deep. The harder rivets are literally being pulled out by the thickening of the wall caused by compression. A similar result holds for a cylindrical submarine (problem 12.5).

The stresses become

$$\sigma_{rr} = -\frac{b^3}{b^3 - a^3}\left(1 - \frac{a^3}{r^3}\right)P , \tag{12-77a}$$

$$\sigma_{tt} = -\frac{b^3}{b^3 - a^3}\left(1 + \frac{1}{2}\frac{a^3}{r^3}\right)P , \tag{12-77b}$$

and are completely independent of the properties of the material. Both of these correspond to positive pressures, in spite of the radial expansion at the inside.

## Thin shell

In the special case of a thin shell with thickness $d = b - a \ll a$ we find the strain tensor components

$$u_{rr} = \nu\frac{a}{d}\frac{P}{E} , \tag{12-78a}$$

$$u_{tt} = -\frac{1}{2}(1-\nu)\frac{a}{d}\frac{P}{E} , \tag{12-78b}$$

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which also shows that there is radial expansion and tangential compression for $\nu > 0$. The stresses become with $s = r - a$

$$\sigma_{rr} = -\frac{s}{d}\, P \tag{12-79a}$$

$$\sigma_{tt} = -\frac{a}{2d} P \tag{12-79b}$$

As for the elastic tube, the tangential compression diverges for $d \to 0$. Long before that, the spherical shell will probably buckle.

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## Problems

**12.1** Show that Navier's equation of equilibrium may be written as

$$\boldsymbol{\nabla}^2\boldsymbol{u} + \frac{1}{1-2\nu}\boldsymbol{\nabla}\,\boldsymbol{\nabla}\cdot\boldsymbol{u} = -\frac{1}{\mu}\boldsymbol{f} \;, \tag{12-80}$$

where $\nu$ is Poisson's ratio.

**12.2** A 100 m tall skyscraper covering an area of 10,000 $\mathsf{m}^2$ is supported by 10 steel pillars, each 10 m tall and 1 $\mathsf{m}^2$ in cross section. Assuming that the average density of the building is the same as that of water, calculate the deformation of the steel in the pillars.

**12.3** A certain gun has a steel barrel of length of 1 m, a bore diameter of 1 cm, and a muzzle velocity of 1 km/s. The expansion of the gases left by the explosion is assumed to be adiabatic with index $\gamma = 7/5$. a) Determine the motion of a bullet of mass $m = 5$ g through the barrel starting at rest, when the gases originally are contained in a volume of length 1 cm. b) Calculate the pressure just after the explosion and when the bullet leaves the muzzle. c) Calculate the initial and final temperatures, assuming the ideal gas law. d) Calculate the strains in the steel on the inside of the barrel with thickness $b - a = 5$ mm.

**12.4** Show that the most general solution to the uniform shear-free bending of a beam is

$$u_x = a_x - \phi_z y + \phi_y z - \alpha\nu x + \frac{1}{2}\beta_x\left(z^2 - \nu(x^2 - y^2)\right) - \beta_y \nu x y \;, \tag{12-81a}$$

$$u_y = a_y + \phi_z x - \phi_x z - \alpha\nu y + \frac{1}{2}\beta_y\left(z^2 - \nu(y^2 - x^2)\right) - \beta_x \nu x y \;, \tag{12-81b}$$

$$u_z = a_z - \phi_y x + \phi_x y + \alpha z - \beta_x x z - \beta_y y z \;, \tag{12-81c}$$

and interpret the coefficients.

**12.5** Calculate the displacement, strain and stress for an evacuated tube with fixed ends subject to an external pressure $P$.

**12.6** A massive cylindrical body with radius $a$ and constant density $\rho_0$ rotates around its axis with constant angular frequency $\Omega$. a) Find the centrifugal force density in cylindrical coordinates rotating with the cylinder. b) Calculate the displacement for the case where the ends of the cylinder are clamped to prevent change in length and the sides of the cylinder are free. c) Show that the tangential strain always corresponds to an expansion, whereas the radial strain corresponds to an expansion close to the center and a compression close to the rim. Find the point, where the radial strain vanishes. d) Where will the breakdown happen.

∗ **12.7** Analyze the symmetry arguments that lead to a radial displacement field for the cylindrical tube, in particular the role played by mirror symmetry.

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**12.8** A plane sheet of material with constant thickness is rolled up into a cylinder with inner radius $a$ and outer radius $b$. Calculate the displacement in the Eulerian formulation.

**12.9** Consider a beam with arbitrary cross section $A$, subjected to shear-free bending (12-20), but without the origin of the coordinate system at the center of the area. Determine the total contact force.

**12.10** Show that a shift in the $x$-coordinate, $x \to x + \alpha$, in the shear-free bending field (12-20) corresponds to adding in a uniform stretching deformation (and a simple translation).

* **12.11** The correct formulation of the equation of mechanical equilibrium for finite deformations in linear isotropic elastic materials is in terms of the actual position $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$

$$\sum_j \nabla'_j \sigma_{ij}(\boldsymbol{x}') = -f_i(\boldsymbol{x}') \ ,$$

where $\sigma_{ij}(\boldsymbol{x}') = 2\mu u_{ij}(\boldsymbol{x}) + \lambda\delta_{ij}\sum_k u_{kk}(\boldsymbol{x})$. Show that in terms of the reference position the field equation becomes

$$\sum_k D^{-1}_{jk}(\boldsymbol{x})\nabla_k(2\mu u_{ij}(\boldsymbol{x}) + \lambda\delta_{ij}\sum_k u_{kk}(\boldsymbol{x})) = -f_i(\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})) \qquad \text{(12-82)}$$

where $D^{-1}_{ij}(\boldsymbol{x})$ is the inverse of $D_{ij}(\boldsymbol{x}) = \delta_{ij} + \nabla_i u_j(\boldsymbol{x})$.

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