Answers to problems

WARNING: unfinished

An answer may not represent an explicit or complete solution to a problem, but only the final result and some useful hints how to get there. Some problems are not given an answer either because it is just too easy or because the result is already contained in the formulation of the problem.

1 Continuous matter

1.1

a) This is the usual binomial distribution

$$\mathcal{P}_M(n) = \binom{M}{n} p^n (1-p)^{M-n}$$

b) The average of n is

$$\langle n \rangle = \sum_{n=0}^{M} n\mathcal{P}_M(n) = pM \sum_{n=1}^{M} \mathcal{P}_{M-1}(n-1) = pM \sum_{n=0}^{M-1} \mathcal{P}_{M-1}(n) = pM$$

c) Similarly we get

$$\langle n(n-1) \rangle = \sum_{n=0}^{M} n(n-1)\mathcal{P}_M(n) = pM \sum_{n=1}^{M} (n-1)\mathcal{P}_{M-1}(n-1) = p^2 M(M-1)$$

from which we get

$$\langle (n - \langle n \rangle)^2 \rangle = \langle n(n-1) \rangle + \langle n \rangle - \langle n \rangle^2 = p(1-p)M$$

1.2 The surface is the difference between the cube and an inner cube of side length $M - 2$, so that $K = M^2 - (M-2)^3$.

1.3 If the sphere has radius R the number of molecules in the volume is $N = \frac{4}{3}\pi R^3/L_{\mathsf{mol}}^3$ whereas the number of molecules at the surface is $N_S = 4\pi R^2/L_{\mathsf{mol}}^2 = 4\pi(3N/4\pi)^{2/3}$. For $N \gg 1$ we have $N_S \ll N$. If the surface molecules are randomly in or out of the volume, one gets $\Delta N \approx \sqrt{N_S} = \sqrt{4\pi}(3N/4\pi)^{1/3}$.

1.4 The CMS-velocity is

$$\boldsymbol{v}_c = \frac{1}{N}\sum_n \boldsymbol{v}_n = \boldsymbol{v} + \frac{1}{N}\sum_n \boldsymbol{u}_n$$

The average is obviously $\boldsymbol{v}$ and its fluctuation is

$$\Delta v_c^2 = \langle(\boldsymbol{v}_c - \boldsymbol{v})^2\rangle = \frac{1}{N^2}\sum_{n,n'}\langle(\boldsymbol{v}_n - \boldsymbol{v})\cdot(\boldsymbol{v}_{n'} - \boldsymbol{v})\rangle = \frac{1}{N^2}\sum_{n,n'}\langle\boldsymbol{u}_n \cdot \boldsymbol{u}_{n'}\rangle = \frac{1}{N}v_0^2$$

2 Space and time

2.1

a) $|\boldsymbol{a}| = 7$, $|\boldsymbol{b}| = 5$

b) $\boldsymbol{a}\cdot\boldsymbol{b} = -6$

c) $\boldsymbol{a}\times\boldsymbol{b} = (-24, -18, -17)$

d) $\boldsymbol{a}\boldsymbol{b} = \begin{pmatrix} 6 & -8 & 0 \\ 9 & -12 & 0 \\ -18 & 24 & 0 \end{pmatrix}$

2.2 Yes, $\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{0}$.

2.3 In an Earth-centered Cartesian coordinate system with z-axis towards the North-pole at latitude $\beta = 90^\circ$ and x-axis towards Greenwich at longitude $\alpha = 0$ we have $x = (a+h)\cos\alpha\cos\beta$, $y = (a+h)\sin\alpha\cos\beta$, and $z = (a+h)\sin\beta$ where a is the sea-level radius of the Earth. Using the invariance of the distance the distance function becomes $d^2 = (a+h_1)^2 + (a+h_2)^2 - 2(a+h_1)(a+h_2)(\cos\beta_1\cos\beta_2\cos(\alpha_1-\alpha_2) + \sin\beta_1\sin\beta_2)$.

2.4

a) Use that $|\boldsymbol{a}|^2 = \boldsymbol{a}^2 = \boldsymbol{a}\cdot\boldsymbol{a}$.

b) Use that $0 \le |\boldsymbol{a} + s\boldsymbol{b}|^2 = |\boldsymbol{a}|^2 + s^2|\boldsymbol{b}|^2 + 2s\boldsymbol{a}\cdot\boldsymbol{b}$ and find the minimum of the right hand side (as a function of s).

2.5 Use the definition of a determinant and that the determinant of a product of matrices is the product of the determinant.

2.9 Assume first that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are linearly independent. In that case the cross-products on the right hand side will also be linearly independent, so that $\boldsymbol{d}$ can be resolved on this (skew) basis. Check that the dot-products of $\boldsymbol{d}$ with $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are correct. Finally, one must discuss the case where $\boldsymbol{a}$, $\boldsymbol{b}$ og $\boldsymbol{c}$ are linearly dependent.