Then we put

$$\boldsymbol{u}_L = \boldsymbol{\nabla}\psi \tag{13-A2}$$

$$\boldsymbol{u}_T = \boldsymbol{u} - \boldsymbol{u}_L \tag{13-A3}$$

Clearly, $\boldsymbol{\nabla} \times \boldsymbol{u}_L = \boldsymbol{0}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{u}_T = 0$.

14 Numeric elastostatics

14.3 We assume a linear combination

$$\nabla_x^+ f(x) = af(x) + bf(x + \Delta x) + cf(x + 2\Delta x)$$

Expand to second order and require the coefficient of $f(x)$ and $\nabla_x^2 f(x)$ to vanish and the coefficient of $\nabla_x f(x)$ to be 1, to get

$$\begin{aligned} a + b + c &= 0 \\ \frac{1}{2}b + 2c &= 0 \\ b + 2c &= 1 \end{aligned}$$

The solution is $a = -3/2$, $b = 2$, and $c = -1/2$.

15 Matter in motion

15.1 In a small interval of time,δt, a material particle with a small volume dV is displaced to fill out another volume dV', the size of which may be calculated from the Jacobi determinant of the infinitesimal mapping (15-2)

$$\frac{dV'}{dV} = \left|\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}}\right| = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 + \nabla_x v_x \delta t & \nabla_x v_y \delta t & \nabla_x v_z \delta t \\ \nabla_y v_x \delta t & 1 + \nabla_y v_y \delta t & \nabla_y v_z \delta t \\ \nabla_z v_x \delta t & \nabla_z v_y \delta t & 1 + \nabla_x v_z \delta t \end{vmatrix}.$$

To first order in δt, only the diagonal elements contribute to the determinant, and we find

$$\frac{dV'}{dV} \approx (1 + \nabla_x v_x \delta t)(1 + \nabla_y v_y \delta t)(1 + \nabla_z v_z \delta t) \tag{15-A1}$$

$$\approx 1 + \nabla_x v_x \delta t + \nabla_y v_y \delta t + \nabla_z v_z \delta t = 1 + \boldsymbol{\nabla} \cdot \boldsymbol{v}\, \delta t \tag{15-A2}$$

The change in volume is $\delta(dV) = dV' - dV$, and after dividing by δt the rate of change of such a *comoving* volume becomes

$$\boxed{\frac{D(dV)}{Dt} = \boldsymbol{\nabla} \cdot \boldsymbol{v}\, dV}\,. \tag{15-A3}$$

15.4 Leonardo's law tells us that $Av = A_1 v_1 + A_2 v_2$. The ratio of the rates in the two pipes is $A_1 v_1 / A_2 v_2 = 2$, and since $A_1/A_2 = 9/4$ we get $v_1/v_2 = 8/9$. The total rate is $Av = 3A_2 v_2$ so that $v_2/v = 4/3$ and $v_1/v = 32/27$.

15.5 Define the vector field

$$\boldsymbol{f}'(\boldsymbol{v}) = \frac{\partial f(\boldsymbol{v})}{\partial \boldsymbol{v}} \; . \tag{15-A4}$$

Then

$$\frac{\partial \rho}{\partial t} = -\frac{3}{t}\rho - \frac{M_0}{t^3}\frac{\boldsymbol{x}}{t^2} \cdot \boldsymbol{f}'\left(\frac{\boldsymbol{x}}{t}\right) \; , \tag{15-A5}$$

$$\rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{3}{t}\rho \; , \tag{15-A6}$$

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\rho = \frac{M_0}{t^3}\frac{\boldsymbol{x}}{t^2} \cdot \boldsymbol{f}'\left(\frac{\boldsymbol{x}}{t}\right) \; . \tag{15-A7}$$

The sum of the three right hand sides vanishes which means that the equation of continuity (15-27) is satisfied.

15.6 Differentiating through all the time-dependence, one gets

$$\frac{d\rho(\boldsymbol{x}(t),t)}{dt} = \frac{d\boldsymbol{x}(t)}{dt} \cdot \frac{\partial \rho(\boldsymbol{x},t)}{\partial \boldsymbol{x}} + \frac{\partial \rho(\boldsymbol{x},t)}{\partial t} = \boldsymbol{v}(\boldsymbol{x},t) \cdot \nabla \rho(\boldsymbol{x},t) + \frac{\partial \rho(\boldsymbol{x},t)}{\partial t} = \frac{D\rho}{Dt} \; .$$

15.7 a) Let Q be the total volume of flow in the stream. Then the average velocity in the x-direction is $v(x) = Q/h(x)d$. b) The inertial acceleration is estimated as $w = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \approx v dv/dx$. c) Constant acceleration implies $v \approx \sqrt{2wx}$ for a suitable choice of origin and orientation of the x-axis. Hence $h(x) \sim 1/\sqrt{x}$ is the shape of the curve.

15.8 a) Let Q be the total volume of flow in the stream. Then the average velocity in the x-direction is $v(x) = Q/\pi a(x)^2$. b) The inertial acceleration is estimated as $w = (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} \approx v dv/dx$. c) Constant acceleration implies $v \approx \sqrt{2wx}$ for a suitable choice of origin and orientation of the x-axis. Hence $a(x) \sim 1/x^{1/4}$ is the shape of the tube.

15.9 The local transport equations for mass and momentum are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = J$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v}) = \rho \boldsymbol{g}$$

such that the cosmological equations become

$$\dot{\rho} = -3H\rho + J \tag{15-A8}$$

$$\dot{H} + H^2 = J\frac{H}{\rho} - \frac{4\pi}{3}G\rho \tag{15-A9}$$

Clearly there is a solution

$$J = 3H\rho \tag{15-A10}$$

$$\rho = \frac{9H^2}{4\pi G} = 6\rho_c \tag{15-A11}$$

From $H = 1.78 \times 10^{18}\ \mathsf{s}^{-1}$ one gets $\rho = 3.4 \times 10^{-26}\ \mathsf{kg/m^3}$ and $J = 1.82 \times 10^{-43}\ \mathsf{kg/m^3 s}$, corresponding to the creation of 3 protons per cubic-kilometer per year. Not much!

