B

Newtonian particle mechanics

The equations of continuum mechanics are derived by a systematic application of Newton's laws for systems that nearly behave as if they consisted of idealized point particles. It is for this reason useful here to recapitulate the basic mechanics of point particles, and to derive the global laws that so often are found to be useful.

The global laws state for any collection of point particles that

- *the rate of change of momentum equals force*
- *the rate of change of angular momentum equals moment of force*
- *the rate of change of kinetic energy equals power*

Even if these laws are not sufficient to determine the dynamics of a physical system, they represent seven individual constraints on the motion of any system of point particles, independently of how complex it is. They are equally valid for continuous systems when it is taken properly into account that the number of particles in a body may change with time.

In the main text a certain familiarity with Newtonian mechanics is assumed throughout. This appendix only serves as a reminder and as refreshment. It can in no way substitute for a proper course on Newtonian mechanics.

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B.1 Dynamic equations

In Newtonian mechanics, a physical system or *body* is understood as a collection of a certain number N of point particles numbered $n = 1, 2, \ldots, N$. Each particle obeys Newton's second law,

$$m_n \frac{d^2 \boldsymbol{x}_n}{dt^2} = \boldsymbol{f}_n \ , \tag{B-1}$$

where m_n denotes the (constant) mass of the n'th particle, $\boldsymbol{x}_n$ its instantaneous position, and $\boldsymbol{f}_n$ the instantaneous force acting on the particle. Due to the mutual interactions between the particles, the forces may depend on the instantaneous positions and velocities of all the particles, including themselves,

$$\boldsymbol{f}_n = \boldsymbol{f}_n \left(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, \frac{d\boldsymbol{x}_1}{dt}, \ldots, \frac{d\boldsymbol{x}_N}{dt}, t \right) \ . \tag{B-2}$$

The forces will in general also depend on parameters describing the external influences from the system's environment, for example Earth's gravity. The explicit dependence on t in the last argument of the force usually derives from such time dependent external influences. It is, however, often possible to view the environment as just another collection of particles and include it in a larger *isolated* body without any influences from the environment.

The dynamics of a collection of particles thus becomes a web of coupled second order differential equations in time. In principle these equations may be solved numerically for all times t, given initial positions and velocities for all particles at a definite instant of time, say $t = t_0$. Unfortunately, the large number of molecules in any macroscopic body usually presents an insurmountable obstacle to such an endeavor. Even for smaller numbers of particles, *deterministic chaos* may effectively prevent any long-term numeric integration of the equations of motion.

B.2 Force and momentum

A number of quantities describe the system as a whole. The total mass of the system is defined to be

$$M = \sum_n m_n \ , \tag{B-3}$$

and the total force

$$\boldsymbol{\mathcal{F}} = \sum_n \boldsymbol{f}_n \ . \tag{B-4}$$

Notice that these are truly definitions. Nothing in Newton's laws tells us that it is physically meaningful to add masses of different particles, or worse, forces acting on different particles. As shown in problem B.1, there is nothing in the way for making a different definition of total force.

The choice made here is particularly convenient for particles moving in a constant field of gravity, such as we find on the surface of the Earth, because the gravitational force on a particle is directly proportional to the mass of the particle. With the above definitions, the total gravitational force, the weight, becomes proportional to the total mass. This additivity of weights, the observation that a volume of flour balances an equal volume of flour, independently of how it is subdivided into smaller volumes, goes back to the dawn of history.

Having made these definitions, the form of the equations of motion (B-1) tells us that we should also define the average of the particle positions weighted by the corresponding masses

$$\boldsymbol{x}_M = \frac{1}{M}\sum_n m_n \boldsymbol{x}_n \ . \tag{B-5}$$

For then the equations of motion imply that

$$M\frac{d^2\boldsymbol{x}_M}{dt^2} = \boldsymbol{\mathcal{F}} \ . \tag{B-6}$$

Formally, this equation is of the same form as Newton's second law for a single particle, so *the center of mass moves like a point particle under influence of the total force.* But before we get carried completely away, it should be remembered that the total force depends on the positions and velocities of all the particles, not just on the center of mass position $\boldsymbol{x}_M$ and its velocity $d\boldsymbol{x}_M/dt$. The above equation is in general *not* a solvable equation of motion for the center of mass.

There are, however, important exceptions. The state of a stiff body is characterized by the position and velocity of its center of mass, together with the body's orientation and its rate of change. If the total force on the body does not depend on the orientation, the above equation becomes truly an equation of motion for the center of mass. It is fairly easy to show that for a collection of spherically symmetric stiff bodies, the gravitational forces can only depend on the positions of the centers of mass, even if the bodies rotate. It was Newton's good fortune that planets and stars to a good approximation behave like point particles.

It is convenient to reformulate the above equation by defining the total *momentum* of the body,

$$\boldsymbol{\mathcal{P}} = \sum_n m_n \frac{d\boldsymbol{x}_n}{dt} \ , \tag{B-7}$$

Like the total force it is a purely *kinematic* quantity, depending only on the particle velocities, calculated as the sum over the individual momenta $m_n d\boldsymbol{x}_n/dt$ of each particle. The equations of motion (B-1) imply that the total momentum obeys the equation

$$\boxed{\frac{d\boldsymbol{\mathcal{P}}}{dt} = \boldsymbol{\mathcal{F}}} \ , \tag{B-8}$$

which is evidently equivalent to (B-6).

Again it should be remarked that this equation cannot be taken as an equation of motion, except in very special circumstances. It should rather be viewed as a *constraint* (or rather three since it is a vector equation) that follows from the true equations of motion, independently of what form the forces take. This constraint is particularly useful if the total momentum is known to be constant, or equivalently the center of mass has constant velocity, because then the total force must vanish.

B.3 Moment of force and angular momentum

Similarly, the total *moment of force* acting on the system is defined as

$$\boldsymbol{\mathcal{M}} = \sum_n \boldsymbol{x}_n \times \boldsymbol{f}_n \;, \tag{B-9}$$

Like the total force, it is a *dynamic* quantity calculated from the sum of the individual moments of force acting on the particles.

The corresponding kinematic quantity is the total *angular momentum*,

$$\boldsymbol{\mathcal{L}} = \sum_n \boldsymbol{x}_n \times m_n \frac{d\boldsymbol{x}_n}{dt} \;. \tag{B-10}$$

Differentiating after time we find

$$\frac{d\boldsymbol{\mathcal{L}}}{dt} = \sum_n m_n \left(\frac{d\boldsymbol{x}_n}{dt} \times \frac{d\boldsymbol{x}_n}{dt} + \boldsymbol{x}_n \times \frac{d^2\boldsymbol{x}_n}{dt^2} \right) \;.$$

The first term in the parenthesis vanishes because the cross product of a vector with itself always vanishes, and using the equations of motion in the second term, we obtain

$$\boxed{\frac{d\boldsymbol{\mathcal{L}}}{dt} = \boldsymbol{\mathcal{M}}} \;. \tag{B-11}$$

Like the equation for total momentum and total force, (B-8), this equation is also a constraint that must be fulfilled, independently of the nature of the forces acting on the particles. Angular momentum has to do with the state of rotation of the body as a whole. If the total angular momentum is known to be constant, as for a non-rotating body, the total moment of force must vanish. This is what lies behind the lever principle.

From the earliest times levers have been used to lift and move heavy weights, such as those found in stone age monuments. A primitive lever is simply a long stick with one end wedged under a heavy stone. Applying a small "mansized" force orthogonally to the other end of the stick, the product of the long arm and the small force translates into a much larger force at the end of the small arm wedged under the stone. The total moment vanishes, when the stick is not moving.

The moment of force and the angular momentum both depend explicitly on the choice of origin of the coordinate system. These quantities might as well have been calculated around any other fixed point $\boldsymbol{c}$, leading to

$$\boldsymbol{\mathcal{M}}(\boldsymbol{c}) = \sum_n (\boldsymbol{x}_n - \boldsymbol{c}) \times \boldsymbol{f}_n = \boldsymbol{\mathcal{M}} - \boldsymbol{c} \times \boldsymbol{\mathcal{F}} \ , \tag{B-12}$$

$$\boldsymbol{\mathcal{L}}(\boldsymbol{c}) = \sum_n (\boldsymbol{x}_n - \boldsymbol{c}) \times m_n \frac{d(\boldsymbol{x}_n - \boldsymbol{c})}{dt} = \boldsymbol{\mathcal{L}} - \boldsymbol{c} \times \boldsymbol{\mathcal{P}} \ . \tag{B-13}$$

This shows that if the total force vanishes, the total moment of force becomes independent of the choice of origin, and similarly if the total momentum vanishes, the total angular momentum will be independent of the choice of origin.

If a point $\boldsymbol{c}$ exists such that $\boldsymbol{\mathcal{M}}(\boldsymbol{c}) = \boldsymbol{0}$ we get $\boldsymbol{\mathcal{M}} = \boldsymbol{c} \times \boldsymbol{\mathcal{F}}$. In that case, the point $\boldsymbol{c}$ is called the *center of action* or *point of attack* for the total force $\boldsymbol{\mathcal{F}}$. In general there is no guarantee that a center of action exists, since it requires the total force $\boldsymbol{\mathcal{F}}$ to be orthogonal to the total moment $\boldsymbol{\mathcal{M}}$. Even if the center of action exists, it is not unique because any other point $\boldsymbol{c} + k\boldsymbol{\mathcal{F}}$ with arbitrary k is as good a center of action as $\boldsymbol{c}$. In constant gravity where $\boldsymbol{f}_n = m_n \boldsymbol{g}_0$, it follows immediately that the center of mass is also the center of action for gravity, or as it is mostly called, the center of gravity.

B.4 Power and kinetic energy

Forces generally perform work on the particles they act on. The total rate of work or *power* performed by the forces acting on all the particles making up a body is

$$P = \sum_n \boldsymbol{f}_n \cdot \frac{d\boldsymbol{x}_n}{dt} \ . \tag{B-14}$$

Notice that there is a dot-product between the force and the velocity. In non-anglosaxon countries this is called *effect* rather than power.

The corresponding kinematic quantity is the total *kinetic energy*,

$$\mathcal{T} = \frac{1}{2} \sum_n m_n \left(\frac{d\boldsymbol{x}_n}{dt} \right)^2 \ , \tag{B-15}$$

which is the sum of individual kinetic energies of each particle. Differentiating after time and making use of the equations of motion (B-1), we find

$$\boxed{\frac{d\mathcal{T}}{dt} = P} \ . \tag{B-16}$$

The rate of change of the kinetic energy equals the power.

B.5 Internal and external forces

The force acting on a particle may often be split into an internal part due to the other particles in the system, and an external part due to the system's environment,

$$\boldsymbol{f}_n = \boldsymbol{f}_n^{\text{int}} + \boldsymbol{f}_n^{\text{ext}} \ . \tag{B-17}$$

The internal forces, in particular gravitational forces, are often two-particle forces with $\boldsymbol{f}_{n,n'}$ denoting the force that particle n' exerts on particle n. The total internal force on particle n thus becomes

$$\boldsymbol{f}_n^{\text{int}} = \sum_{n'} \boldsymbol{f}_{n,n'} \ . \tag{B-18}$$

Most two-particle forces also obey Newton's third law, which states that the force from n' on n is equal and opposite to the force from n on n',

$$\boldsymbol{f}_{n,n'} = -\boldsymbol{f}_{n',n} \ . \tag{B-19}$$

Although the external forces may themselves stem from two-particle forces of this kind, this is ignored as long as the nature of the environment is unknown.

Strong theorems follow from the above assumptions about the form of the internal forces. The first is that the total internal force vanishes,

$$\boldsymbol{\mathcal{F}}^{\text{int}} = \sum_n \boldsymbol{f}_n^{\text{int}} = \sum_{n,n'} \boldsymbol{f}_{n,n'} = \boldsymbol{0} \ . \tag{B-20}$$

where we have used the antisymmetry (B-19). This expresses the simple fact that you cannot lift yourself by your bootstraps; baron Münchausen notwithstanding.

The momentum rate equation (B-8) thus takes the form

$$\frac{d\boldsymbol{\mathcal{P}}}{dt} = \boldsymbol{\mathcal{F}}^{\text{ext}} \ , \tag{B-21}$$

showing that it is sufficient to know the total external force acting on a system in order to calculate its rate of change of momentum. The details of the internal forces can be ignored as long as they are of two-particle kind and obey Newton's third law.

Under the same assumptions, the internal moment of force becomes

$$\boldsymbol{\mathcal{M}} = \sum_{n,n'} \boldsymbol{x}_n \times \boldsymbol{f}_{n,n'} = \frac{1}{2} \sum_{n,n'} (\boldsymbol{x}_n - \boldsymbol{x}_{n'}) \times \boldsymbol{f}_{n,n'} \ . \tag{B-22}$$

This does not in general vanish, except for the case of *central forces* where

$$\boldsymbol{f}_{n,n'} \sim \boldsymbol{x}_n - \boldsymbol{x}_{n'} \ . \tag{B-23}$$

Gravitational forces (and others) are of this kind. Provided the internal forces stem from central two-particle forces, the total moment of force equal the external moment, so that

$$\frac{d\boldsymbol{\mathcal{L}}}{dt} = \boldsymbol{\mathcal{M}}^{\text{ext}} \ . \tag{B-24}$$

This rule is, however, not on nearly the same sure footing as the corresponding equation for the momentum rate (B-21).

Finally, there is not much to be said about the kinetic energy rate (B-16), which in general has non-vanishing internal and external contributions.

B.6 Hierarchies of particle interactions

Under what circumstances can a collection of point particles itself be viewed as a point particle? The dynamics of the solar system may to a good approximation be described by a system of interacting point particles, although the planets and the sun in no way are pointlike at our own scale. At the scale of the whole universe, even galaxies are sometimes treated as point particles.

A point particle approximation may be in place as long as the internal cohesive forces that keep the interacting bodies together are much stronger than the external forces. In addition to mass and momentum, such a point particle may also have to be endowed with an intrinsic angular momentum (spin), and an intrinsic energy. The material world appears in this way as a hierarchy of approximately point-like interacting particles, from atoms to galaxies, at each level behaving as if they had no detailed internal structure. Corrections to the ideal point-likeness can later be applied to add more detail to this overall picture. Over the centuries this extremely reductionist method has shown itself to be very fruitful, but it is an open (scientific) question whether it can continue indefinitely.

Problems

B.1 Try to define the total force to be $\boldsymbol{\mathcal{F}}' = \sum_n m_n \boldsymbol{f}_n$ rather than (B-4), and investigate what this entails for the global properties of a system. Can you build a consistent mechanics on this definition?

B.2 Show that the total momentum is $\boldsymbol{\mathcal{P}} = M d\boldsymbol{x}_M/dt$ where $\boldsymbol{x}_M$ is the center of mass position.

