3 Buoyancy and stability

Fishes, whales, submarines, balloons and airships all owe their ability to float to *buoyancy*, the lifting power of water and air. The understanding of the physics of buoyancy goes back as far as antiquity and probably sprung from the interest in ships and shipbuilding in classic Greece. The basic principle is due to Archimedes. His famous Law states that the buoyancy force on a body is equal and oppositely directed to the weight of the fluid that the body displaces. Before his time it was thought that the shape of a body determined whether it would sink or float.

"Buoy" mostly pronounced "booe", probably of Germanic origin. A tethered floating object used to mark a location in the sea.

The shape of a floating body and its mass distribution do, however, determine whether it will float stably or capsize. Stability of floating bodies is of vital importance to shipbuilding — and to anyone who has ever tried to stand up in a small rowboat. Newtonian mechanics not only allows us to derive Archimedes' Law for the equilibrium of floating bodies, but also to characterize the deviations from equilibrium and calculate the restoring forces. Even if a body floating in or on water is in hydrostatic equilibrium, it will not be in complete mechanical balance in every orientation, because the center of mass of the body and the center of mass of the displaced water, also called the center of buoyancy, do not in general coincide.

Archimedes of Syracuse (287–212 BC). Greek mathematician, physicist and engineer. Discovered the formulae for area and volume of cylinders and spheres, and invented rudimentary infinitesimal calculus. Formulated the Law of the Lever, and wrote two volumes on hydrostatics titled *On Floating Bodies*, containing his Law of Buoyancy. Killed by a Roman soldier.

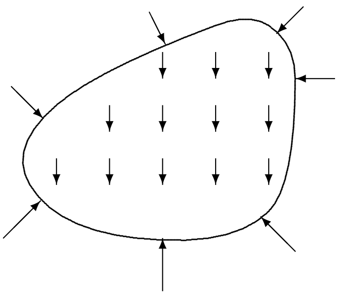
The mismatch between the centers of mass and buoyancy for a floating body creates a moment of force, which tends to rotate the body towards a stable equilibrium. For submerged bodies, submarines, fishes and balloons, the stable equilibrium will always have the center of gravity situated directly below the center of buoyancy. But for bodies floating stably on the surface, ships, ducks, and dumplings, the center of gravity is mostly found directly *above* the center of buoyancy. It is remarkable that such a configuration can be stable. The explanation is that when the surface ship is tilted away from equilibrium, the center of buoyancy moves instantly to reflect the new volume of displaced water. Provided the center of gravity does not lie too far above the center of buoyancy, this change in the displaced water creates a moment of force that counteracts the tilt.

3.1 Archimedes' principle

Mechanical equilibrium takes a slightly different form than global hydrostatic equilibrium (2.18) on page 27 when a body of another material is immersed in a fluid. If its material is incompressible, the body retains its shape and displaces an amount of fluid with exactly the same volume. If the body is compressible, like a rubber ball, the volume of displaced fluid will be smaller. The body may even take in fluid, like a sponge or the piece of bread you dunk into your coffee, but we shall disregard this possibility in the following.

A body which is partially immersed with a piece inside and another outside the fluid may formally be viewed as a body that is fully immersed in a fluid with properties that vary from place to place. This also covers the case where part of the body is in vacuum which may be thought of as a fluid with vanishing density and pressure.

Weight and buoyancy



Gravity pulls on a body over its entire volume while pressure only acts on the surface.

Let the actual, perhaps compressed, volume of the immersed body be V with surface S. In the field of gravity an unrestrained body with mass density ρ_{body} is subject to two forces: its weight

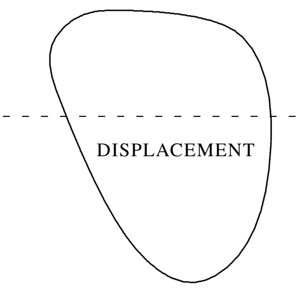
$$\mathcal{F}_G = \int_V \rho_{\text{body}}\, \boldsymbol{g}\, dV, \tag{3.1}$$

and the buoyancy due to pressure acting on its surface,

$$\mathcal{F}_B = -\oint_S p\, d\boldsymbol{S}. \tag{3.2}$$

If the total force $\mathcal{F} = \mathcal{F}_G + \mathcal{F}_B$ does not vanish, an unrestrained body will accelerate in the direction of $\mathcal{F}$ according to Newton's Second Law. Therefore, in mechanical equilibrium weight and buoyancy must precisely cancel each other at all times to guarantee that the body will remain in place.

Assuming that the body does not itself contribute to the field of gravity, the local balance of forces in the fluid, expressed by eq. (2.22) on page 28, will be the same as before the body was placed in the fluid. In particular the pressure in the fluid cannot depend on whether the volume V contains material that is different from the fluid itself. The pressure acting on the surface of the immersed body must for this reason be identical to the pressure on a body of fluid of the same shape, but then the global equilibrium condition (2.18) on page 27 for any volume of fluid tells us that $\mathcal{F}_G^{\text{fluid}} + \mathcal{F}_B = \boldsymbol{0}$, or



A body partially submerged in a liquid. The *displacement* is the amount of water that has been displaced by the body below the waterline. In equilibrium the weight of the displaced fluid equals the weight of the body.

$$\mathcal{F}_B = -\mathcal{F}_G^{\text{fluid}} = -\int_V \rho_{\text{fluid}}\, \boldsymbol{g}\, dV. \tag{3.3}$$

This theorem is indeed Archimedes' principle:

- *the force of buoyancy is equal and opposite to the weight of the displaced fluid.*

The total force on the body may now be written

$$\mathcal{F} = \mathcal{F}_G + \mathcal{F}_B = \int_V (\rho_{\text{body}} - \rho_{\text{fluid}})\boldsymbol{g}\, dV, \tag{3.4}$$

explicitly confirming that when the body is made from the same fluid as its surroundings—so that $\rho_{\text{body}} = \rho_{\text{fluid}}$—the resultant force vanishes automatically. In general, however, the distributions of mass in the body and in the displaced fluid will be different.

Karl Friedrich Hieronymus Freiherr von Münchhausen (1720–1797). German soldier, hunter, nobleman, and delightful story-teller. In one of his stories, he lifts himself out of a deep lake by pulling at his bootstraps.

Münchhausen effect: Archimedes' principle is valid even if the gravitational field varies across the body, but fails if the body is so large that its own gravitational field cannot be neglected, such as would be the case if an Earth-sized body fell into Jupiter's atmosphere. The additional gravitational compression of the fluid near the surface of the body generally increases the fluid density and thus the buoyancy force. In semblance with Baron von Münchhausen's adventure, the body in effect lifts itself by its bootstraps (see problems 3.6 and 3.7).

Constant field of gravity

In a constant gravitational field, $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$, everything simplifies. The weight of the body and the buoyancy force become instead,

$$\mathcal{F}_G = M_{\text{body}}\, \boldsymbol{g}_0, \qquad \mathcal{F}_B = -M_{\text{fluid}}\, \boldsymbol{g}_0. \tag{3.5}$$

Since the total force is the sum of these contributions, one might say that buoyancy acts as if the displacement were filled with fluid of negative mass $-M_{\text{fluid}}$. In effect the buoyancy force acts as a kind of antigravity.

The total force on an unrestrained object is now,

$$\mathcal{F} = \mathcal{F}_G + \mathcal{F}_B = (M_{\text{body}} - M_{\text{fluid}})\boldsymbol{g}_0. \tag{3.6}$$

If the body mass is smaller than the mass of the displaced fluid, the total force is directed upwards, and the body will begin to rise, and conversely if the force is directed downwards it will sink. Alternatively, if the body is kept in place, the restraints must deliver a force $-\mathcal{F}$ to prevent the object from moving.

In constant gravity, a body can only hover motionlessly inside a fluid (or on its surface) if its mass equals the mass of the displaced fluid,

$$M_{\text{body}} = M_{\text{fluid}}. \tag{3.7}$$

A fish achieves this balance by adjusting the amount of water it displaces (M_{fluid}) through contraction and expansion of its body by means of an internal air-filled bladder. A submarine, in contrast, adjusts its mass (M_{body}) by pumping water in and out of ballast tanks.

Bermuda Triangle Mystery: It has been proposed that the mysterious disappearance of ships near Bermuda could be due to a sudden release of methane from the vast deposits of methane hydrates known to exist on the continental shelves. What effectively could happen is the same as when you shake a bottle of soda. Suddenly the water is filled with tiny gas bubbles with a density near that of air. This lowers the average density of the frothing water to maybe only a fraction of normal water, such that the mass of the ship's displacement falls well below the normal value. The ship is no more in buoyant equilibrium and drops like a stone, until it reaches normal density water or hits the bottom where it will usually remain forever because it becomes filled with water on the way down. Even if this sounds like a physically plausible explanation for the sudden disappearance of surface vessels, there is no consensus that this is what really happened in the Bermuda Triangle, nor in fact whether there is a mystery at all [4].

The physical phenomenon is real enough. It is, for example, well known to white water sailors that "holes" can form in which highly aerated water decreases the buoyancy, even to the point where it cannot carry any craft. You can yourself do an experiment in your kitchen using a half-filled soda bottle with a piece of wood floating on the surface. When you tap the bottle hard, carbon dioxide bubbles are released, and the "ship" sinks. In this case the "ship" will however quickly reappear on the surface.

Example 3.1 [Gulf stream surface height]: The warm Gulf stream originates in the Gulf of Mexico and crosses the Atlantic to Europe after having followed the North American coast to Newfoundland. It has a width of about 100 km and a depth of about 1000 m. Its temperature is about 10° C above the surrounding ocean, of course warmest close to its origin. Since the density of sea water decreases by about 0.015% per degree, the buoyancy force on the upper 100 m of warmer water will lift the water surface by about 1.5 cm per degree or 15 cm for a difference of 10 degrees, which agrees with the scale of seasonal variations [KSH99]. The absolute height over the surrounding water surface is about one meter.

3.2 The gentle art of ballooning

Joseph Michel Montgolfier (1740–1810). Experimented (together with his younger brother **Jacques Étienne (1745–1799)**) with hot-air balloons. On November 21, 1783, the first human flew in such a balloon for a distance of 9 km at a height of 100 m above Paris. Only one of the brothers ever flew, and then only once!

The first balloon flights are credited to the Montgolfier brothers who on November 21, 1783 flew an untethered manned hot-air balloon, and to Jacques Charles who on December 1 that same year flew a manned hydrogen gas balloon (see fig. 3.1). In the beginning there was an intense rivalry between the advocates of Montgolfier and Charles type balloons, respectively called *la Montgolfière* and *la Charlière*, which presented different advantages and dangers to the courageous fliers. Hot air balloons were easier to make although prone to catch fire, while hydrogen balloons had greater lifting power but could suddenly explode. By 1800 the hydrogen balloon had won the day, culminating in the huge (and dangerous) hydrogen airships of the 1930s. Helium balloons are much safer, but also much more expensive to fill. In the last half of the twentieth century hot-air balloons again came into vogue, especially for sports, because of the availability of modern strong lightweight materials (nylon) and fuel (propane).

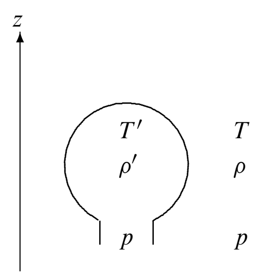
Jacques Alexandre César Charles (1746–1823). French physicist. The first to use hydrogen balloons for manned flight, and made on December 1, 1783, an ascent to about 3 km. Discovered Charles' law, a forerunner of the ideal gas law, stating that the ratio of volume to absolute temperature (V/T) is constant for a given pressure.

The balloon equation

Let M denote the mass of the balloon at height z above the ground. This includes the gondola, the balloon skin, the payload (passengers), but not the gas (be it hot air, hydrogen or helium). The mass of the balloon can diminish if the balloon captain decides to throw out stuff from the gondola to increase its maximal height, also called the *ceiling*, and often sand bags are carried as ballast for this purpose. The condition (3.7) for the balloon to float stably at height z above the ground now takes the form,

$$\boxed{M + \rho' V = \rho V,} \tag{3.8}$$

where ρ' the density of the gas, ρ is the density of the displaced air, and V the volume of the gas at height z. On the right we have left out the tiny buoyancy ρV_M due to the volume V_M of the material of the balloon itself. If the left-hand side of this equation is smaller or larger than the right-hand side, the balloon will rise or fall.



A hot-air balloon has higher temperature $T' > T$ and lower density $\rho' < \rho$ but essentially the same pressure as the surrounding atmosphere because it is open below.

Hot-air balloons

A hot-air balloon is open at the bottom so that the inside pressure is always the same as the atmospheric pressure outside. The air in the balloon is warmer ($T' > T$) than the outside temperature and the density is correspondingly lower ($\rho' < \rho$). Using the ideal gas law (2.27) and the equality of the inside and outside pressures we obtain $\rho' T' = \rho T$, so that the inside density is $\rho' = \rho T/T'$. Up to a height of about 10 km one can use the expressions (2.50) and (2.51) for the homentropic temperature and density of the atmosphere.

Example 3.2 [La Montgolfière]: The first Montgolfier balloon used for human flight on November 21 1783, was about 15 meter in diameter with an oval shape and had a constant volume $V \approx 1700$ m^3. It carried two persons, rose to a ceiling of $z \approx 1000$ m and flew about 9 kilometers in 25 minutes. The machine is reported to have weighed 1600 lbs $\approx$ 725 kg, and adding the two passengers and their stuff the total mass to lift must have been about $M \approx 900$ kg. The November air being fairly cold and dense, we guess that $\rho_0 = 1.2$ kg m^{-3}, which yields a displaced air mass at the ceiling $\rho V = 2039$ kg. Comparing with (3.8) we conclude that the mass $\rho' V$ of the hot air must have been about the same as the mass M of the balloon. More precisely, we find from (3.8) , $\rho'/\rho = T/T' = 1 - M/\rho V = 0.56$. In late November, the temperature a thousand meters above Paris could well have been $T = 0°$ C $= 273$ K, such that the average hot air temperature must have been $T' = T/(1 - M/\rho V) \approx 489K = 216°$ C. This is uncomfortably close to the ignition temperature of paper (451 Fahrenheit = 233 Celsius). The balloon's material did actually get scorched by the burning straw used to heat the air in flight, but the fire was quickly extinguished with water brought along for this eventuality.

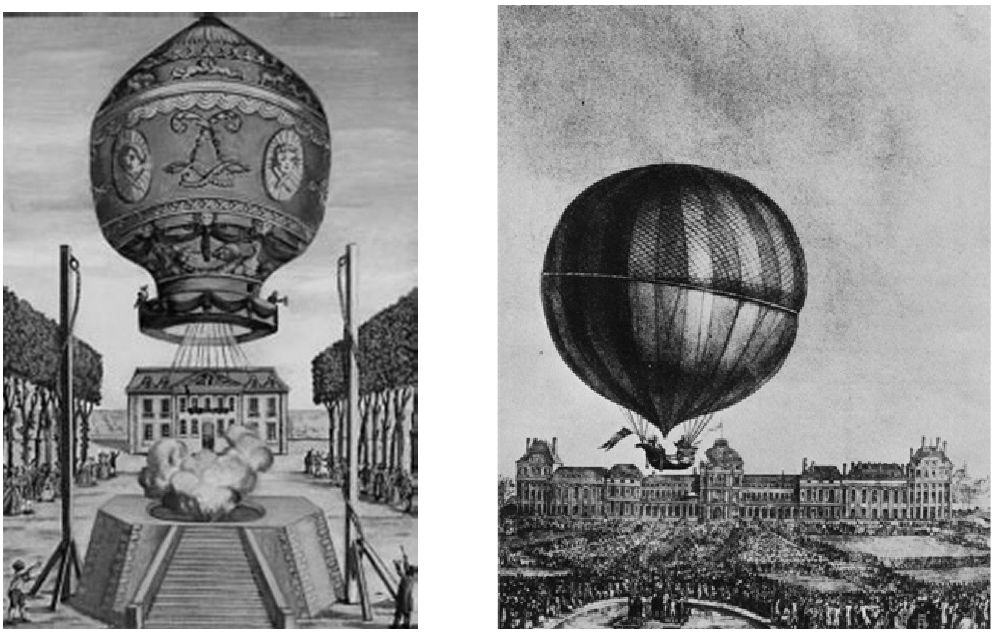


Figure 3.1. Contemporary pictures of the first flights of the Montgolfier hot air balloon (left) and the Charles hydrogen balloon (right). The first ascents were witnessed by huge crowds. Benjamin Franklin, scientist and one of the founding fathers of the US, was present at the first Montgolfier ascent and was deeply interested in the future possibilities of this invention, but did not live to see the first American hot air balloon flight in 1793.

Gas balloons

A modern large hydrogen or helium balloon typically begins its ascent being only partially filled, assuming an inverted tear-drop shape. During the ascent the gas expands because of the fall in ambient air pressure, and eventually the balloon becomes nearly spherical and stops expanding (or bursts) because the "skin" of the balloon cannot stretch further. To avoid bursting the balloon can be fitted with a safety valve. Since the density of the displaced air falls with height, the balloon will eventually reach a ceiling where it would hover permanently if it did not lose gas. In the end no balloon stays aloft forever[1].

Example 3.3 [La Charlière]: The hydrogen balloon used by Charles for the ascent on December 1, 1783, carried two passengers to a height of 600 m. The balloon was made from rubberized silk and nearly spherical with a diameter of 27 ft, giving it a nearly constant volume of $V = 292\ \mathrm{m}^3$. It was open at the bottom to make the pressure the same inside and outside. Taking also the temperatures to be the same inside and outside, it follows from the ideal gas law that the ratio of densities equals the ratio of molar masses, $\rho'/\rho = M_{\mathrm{hydrogen}}/M_{\mathrm{air}} \approx 1/15$, independently of height. Assuming a density, $\rho = 1.2\ \mathrm{kg\ m^{-3}}$ at the ceiling, we obtain from (3.8) $M = (\rho - \rho')V = 327\ \mathrm{kg}$. Assuming furthermore a thickness of 1.5 mm and a density 0.4 g cm^{-3}, the skin mass becomes 127 kg, leaving only 200 kg for the gondola, and passengers. At the end of the first flight, a small accident happened. One passenger jumped out, forgetting to replace his weight with ballast, and the balloon rapidly shot up to about 3000 meters. In problem 3.15 it is shown that this corresponds well with the passenger weighing 76 kilograms, thereby confirming the rough estimates above.

[1] Curiously, no animals seem to have developed balloons for floating in the atmosphere, although both the physics and chemistry of gas ballooning appears to be within reach of biological evolution.

3.3 Stability of floating bodies

Although a body may be in buoyant equilibrium, such that the total force composed of gravity and buoyancy vanishes, $\mathcal{F} = \mathcal{F}_G + \mathcal{F}_B = \mathbf{0}$, it may still not be in complete mechanical equilibrium. The total moment of all the forces acting on the body must also vanish; otherwise an unrestrained body will necessarily start to rotate. In this section we shall discuss the mechanical stability of floating bodies, whether they float on the surface, like ships and ducks, or float completely submerged, like submarines and fish. To find the stable configurations of a floating body, we shall first derive a useful corollary to Archimedes' Principle concerning the moment of force due to buoyancy.

Moments of gravity and buoyancy

The total moment is like the total force a sum of two terms,

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_G + \boldsymbol{\mathcal{M}}_B, \tag{3.9}$$

with one contribution from gravity,

$$\boldsymbol{\mathcal{M}}_G = \int_V \boldsymbol{x} \times \rho_{\text{body}} \boldsymbol{g} \, dV, \tag{3.10}$$

and the other from pressure, the moment of buoyancy,

$$\boldsymbol{\mathcal{M}}_B = \oint_S \boldsymbol{x} \times (-p \, d\boldsymbol{S}). \tag{3.11}$$

If the total force vanishes, $\mathcal{F} = \mathbf{0}$, the total moment will be independent of the origin of the coordinate system, as may be easily shown.

Assuming again that the presence of the body does not change the local hydrostatic balance in the fluid, the moment of buoyancy will be independent of the nature of the material inside V. If the actual body is replaced by an identical volume of the ambient fluid, this fluid volume must be in total mechanical equilibrium, such that both the total force as well as the total moment acting on it have to vanish. Using that $\boldsymbol{\mathcal{M}}_G^{\text{fluid}} + \boldsymbol{\mathcal{M}}_B = \mathbf{0}$, we get

$$\boldsymbol{\mathcal{M}}_B = -\boldsymbol{\mathcal{M}}_G^{\text{fluid}} = -\int_V \boldsymbol{x} \times \rho_{\text{fluid}} \boldsymbol{g} \, dV, \tag{3.12}$$

and we have in other words shown that

- *the moment of buoyancy is equal and opposite to the moment of the weight of the displaced fluid.*

This result is a natural corollary to Archimedes' principle, and of great help in calculating the buoyancy moment. A formal proof of this theorem, starting from the local equation of hydrostatic equilibrium, is found in problem 3.8.

Constant gravity and mechanical equilibrium

In the remainder of this chapter we assume that gravity is constant, $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$, and that the body is in buoyant equilibrium so that it displaces exactly its own mass of fluid, $M_{\text{fluid}} = M_{\text{body}} = M$. The density distributions in the body and the displaced fluid will in general be different, $\rho_{\text{body}}(\boldsymbol{x}) \neq \rho_{\text{fluid}}(\boldsymbol{x})$ for nearly all points.

The moment of gravity (3.10) may be expressed in terms of the center of mass $\boldsymbol{x}_G$ of the body, here called the *center of gravity*,

$$\mathcal{M}_G = \boldsymbol{x}_G \times M\boldsymbol{g}_0, \qquad \boldsymbol{x}_G = \frac{1}{M}\int \boldsymbol{x}\,\rho_{\text{body}}\,dV. \tag{3.13}$$

Similarly the moment of buoyancy (3.12) may be written,

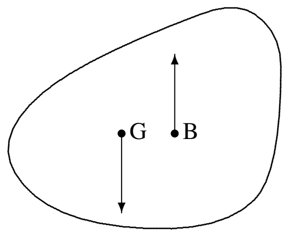
$$\mathcal{M}_B = -\boldsymbol{x}_B \times M\boldsymbol{g}_0, \qquad \boldsymbol{x}_B = \frac{1}{M}\int \boldsymbol{x}\,\rho_{\text{fluid}}\,dV, \tag{3.14}$$

where $\boldsymbol{x}_B$ is the moment of gravity of the displaced fluid, also called the *center of buoyancy*. Although each of these moments depends on the choice of origin of the coordinate system, the total moment,

$$\boxed{\mathcal{M} = (\boldsymbol{x}_G - \boldsymbol{x}_B) \times M\boldsymbol{g}_0,} \tag{3.15}$$

will be independent. This is also evident from the appearance of the difference of the two center positions. A shift of the origin of the coordinate system will affect the centers of gravity and buoyancy in the same way and therefore cancel out.

As long as the total moment is non-vanishing, the unrestrained body is not in complete mechanical equilibrium, but will start to rotate towards an orientation with vanishing moment. Except for the trivial case where the centers of gravity and buoyancy coincide, the above equation tells us that the total moment can only vanish if the centers lie on the same vertical line, $\boldsymbol{x}_G - \boldsymbol{x}_B \propto \boldsymbol{g}_0$. Evidently, there are two possible orientations satisfying this condition: one where the center of gravity lies below the center of buoyancy, and another where the center of gravity is above. At least one of these must be stable, for otherwise the body would never come to rest.

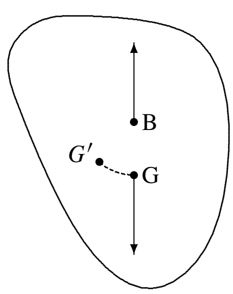


Fully submerged body in buoyant equilibrium with non-vanishing total moment (which here sticks out of the paper). The moment will for a fully submerged body always tend to rotate it (here anti-clockwise) such that the center of gravity is brought below the center of buoyancy.

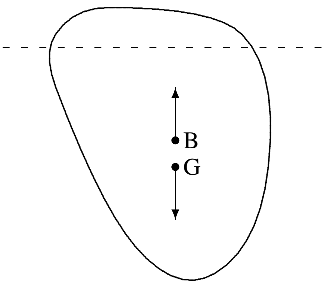
Fully submerged body

In a fully submerged rigid body, for example a submarine, both centers are always in the same place relative to the body, barring possible shifts in the cargo. If the center of gravity does not lie directly below the center of buoyancy, but is displaced horizontally, for example by rotating the body, the direction of the moment will always tend to turn the body so that the center of gravity is lowered with respect to the center of buoyancy. The only stable equilibrium orientation of the body is where the center of gravity lies vertically below the center of buoyancy. Any small perturbation away from this orientation will soon be corrected and the body brought back to the equilibrium orientation, assuming of course that dissipative forces (friction) can seep off the energy of the perturbation, for otherwise it will oscillate. A similar argument shows that the other equilibrium orientation with the center of gravity above the center of buoyancy is unstable and will flip the body over, if perturbed the tiniest amount.

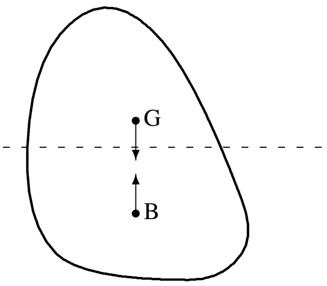
Now we understand better why the gondola hangs below an airship or balloon. If the gondola were on top, its higher average density would raise the center of gravity above the center of buoyancy and thereby destabilize the craft. Similarly, a fish goes belly-up when it dies, because it loses muscular control of the swim bladder which enlarges into the belly and reverses the positions of the centers of gravity and buoyancy. It generally also loses buoyant equilibrium and floats to the surface, and stays there until it becomes completely waterlogged and sinks to the bottom. Interestingly, a submarine always has a conning tower on top to serve as a bridge when sailing on the surface, but its weight will be offset by the weight of heavy machinery at the bottom, so that the boat remains fully stable when submerged. Surfacing, the center of buoyancy of the submarine is obviously lowered, but as we shall see below this needs not destabilize the boat even if it comes to lie below the center of gravity.



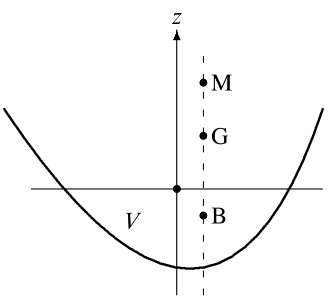
A fully submerged rigid body (a submarine) in stable equilibrium must have the center of gravity situated directly below the center of buoyancy. If G is moved to G', for example by rotating the body, a restoring moment is created which sticks out of the plane of the paper, as shown in the upper figure.



A floating body may have a stable equilibrium with the center of gravity directly *below* the center of buoyancy.



A floating body generally has a stable equilibrium with the center of gravity directly *above* the center of buoyancy.



Ship in an equilibrium orientation with aligned centers of gravity (G) and buoyancy (B). The metacenter (M) lies in this case above the center of gravity, so that the ship is stable against small perturbations. The horizontal line at $z = 0$ indicates the surface of the water. In buoyant equilibrium, the ship always displaces the volume V, independently of its orientation.

Body floating on the surface

At the surface of a liquid, a body such as a ship or an iceberg will according to Archimedes' principle always arrange itself so that the mass of displaced liquid exactly equals the mass of the body. Here we assume that there is vacuum or a very light fluid such as air above the liquid. The center of gravity is always in the same place relative to the body if the cargo is fixed (see however fig. 3.2), but the center of buoyancy depends now on the orientation of the body, because the volume of displaced fluid changes place and shape, while keeping its mass constant, when the body orientation changes.

Stability can – as always – only occur when the two centers lie on the same vertical line, but there may be more than one stable orientation. A sphere made of homogeneous wood floating on water is stable in all orientations. None of them are in fact truly stable, because it takes no force to move from one to the other (disregarding friction). This is however a marginal case.

A floating body may, like a submerged body, possess a stable orientation with the center of gravity directly *below* the center of buoyancy. A heavy keel is, for example, used to lower the center of gravity of a sailing ship so much that this orientation becomes the only stable equilibrium. In that case it becomes virtually impossible to capsize the ship, even in a very strong wind.

The stable orientation for most floating objects, such as ships, will in general have the center of gravity situated directly *above* the center of buoyancy. This happens always when an object of constant mass density floats on top of a liquid of constant mass density, for example an iceberg on water. The part of the iceberg that lies below the waterline must have its center of buoyancy in the same place as its center of gravity. The part of the iceberg lying above the water cannot influence the center of buoyancy whereas it always will shift the center of gravity upwards.

How can that situation ever be stable? Will the moment of force not be of the wrong sign if the ship is perturbed? Why don't ducks and tall ships capsize spontaneously? The qualitative answer is that when the body is rotated away from such an equilibrium orientation, the volume of displaced water will change place and shape in such a way as to shift the center of buoyancy back to the other side of the center of gravity, reversing thereby the direction of the moment of force to restore the equilibrium. We shall now make this argument quantitative.

3.4 Ship stability

Sitting comfortably in a small rowboat, it is fairly obvious that the center of gravity lies above the center of buoyancy, and that the situation is stable with respect to small movements of the body. But many a fisherman has learned that suddenly standing up may compromise the stability and throw him out among the fishes. There is, as we shall see, a strict limit to how high the center of gravity may be above the center of buoyancy. If this limit is violated, the boat becomes unstable and capsizes. As a practical aid to the captain, the limit is indicated by the position of the so-called *metacenter*, a fictive point usually placed on the vertical line through the equilibrium positions of the centers of buoyancy and gravity (the 'mast'). The stability condition then requires the center of gravity to lie below the metacenter (see the margin figure).

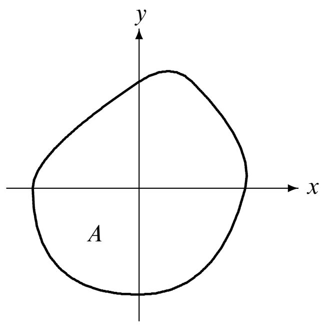
Initially, we shall assume that the ship is in complete mechanical equilibrium with vanishing total force and vanishing total moment of force. The aim is now to calculate the moment of force that arises when the ship is brought slightly out of equilibrium. If the moment tends to turn the ship back into equilibrium, the initial orientation is stable, otherwise it is unstable.

Figure 3.2. The Flying Enterprise (1952). A body can float stably in many orientations, depending on the position of its center of gravity. In this case the list to port was caused by a shift in the cargo which moved the center of gravity to the port side. The ship and its lonely captain Carlsen became famous because he stayed on board during the storm that eventually sent it to the bottom. Photograph courtesy *Politiken*, Denmark, reproduced with permission.

Center of roll

Most ships are mirror symmetric in a plane, but we shall be more general and consider a "ship" of arbitrary shape. In a flat earth coordinate system with vertical z-axis the waterline is naturally taken to lie at $z = 0$. In the waterline the ship covers a horizontal region A of arbitrary shape. The geometric center or *area centroid* of this region is defined by the average of the position,

$$(x_0, y_0) = \frac{1}{A}\int_A (x, y)\, dA, \tag{3.16}$$

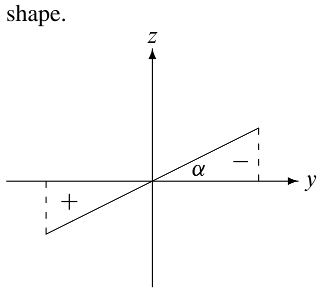


The area A of the ship in the waterline may be of quite arbitrary shape.

where $dA = dx\, dy$ is the area element. Without loss of generality we may always place the coordinate system such that $x_0 = y_0 = 0$. In a ship that is mirror symmetric in a vertical plane the area center will also lie in this plane.

To discover the physical significance of the centroid of the waterline area, the ship is tilted (or "heeled" as it would be in maritime language) through a tiny positive angle α around the x-axis, such that the equilibrium waterline area A comes to lie in the plane $z = \alpha y$. The net change ΔV in the volume of the displaced water is to lowest order in α given by the difference in volumes of the two wedge-shaped regions between new and the old waterline. Since the displaced water is removed from the wedge at $y > 0$ and added to the wedge for $y < 0$, the volume change becomes

$$\Delta V = -\int_A z\, dA = -\alpha \int_A y\, dA = 0. \tag{3.17}$$



Tilt around the x-axis. The change in displacement consists in moving the water from the wedge to the right into the wedge to the left.

In the last step we have used that the origin of the coordinate system coincides with the centroid of the waterline area (i.e. $y_0 = 0$). There will be corrections to this result of order α^2 due to the actual shape of the hull just above and below the waterline, but they are disregarded here. To leading order the two wedges have the same volume.

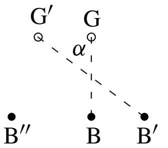
Figure 3.3. The Queen Mary 2 set sail on its maiden voyage on January 2, 2004. It was at that time the world's largest ocean liner with a length of 345 m, a height of 72 m from keel to funnel, and a width of 41 m. Having a draft of only 10 m, its superstructure rises an impressive 62 m over the waterline. The low average density of the superstructure, including 2620 passengers and 1253 crew, combined with the high average density of the 117 megawatt engines and other heavy facilities close to the bottom of the ship nevertheless allow the stability condition (3.28) to be fulfilled. Photograph by Daniel Carneiro.

Since the direction of the x-axis is quite arbitrary, the conclusion is that the ship may be heeled around any line going through the centroid of the waterline area without any first order change in volume of displaced water. This guarantees that the ship will remain in buoyant equilibrium after the tilt. The centroid of the waterline area may thus be called the ship's *center of roll*.

The metacenter

Taking water to have constant density, the center of buoyancy is simply the geometric average of the position over the displacement volume V (below the waterline),

$$(x_B, y_B, z_B) = \frac{1}{V} \int_V (x, y, z)\, dV. \tag{3.18}$$



The tilt rotates the center of gravity from G to G′, and the center of buoyancy from B to B′. In addition, the change in displaced water shifts the center of buoyancy back to B″. In stable equilibrium this point must for $\alpha > 0$ lie to the left of the new center of gravity G′.

In equilibrium the horizontal positions of the centers of buoyancy and gravity must be equal $x_B = x_G$ and $y_B = y_G$. The vertical position z_B of the center of buoyancy will normally be different from the vertical position of the center of gravity z_G, which depends on the actual mass distribution of the ship, determined by its structure and load.

The tilt around the x-axis changes the positions of the centers of gravity and buoyancy. The center of gravity $\boldsymbol{x}_G = (x_G, y_G, z_G)$ is supposed to be fixed with respect to the ship (see however fig. 3.2) and is to first order in α shifted horizontally by a simple rotation through the infinitesimal angle α,

$$\delta y_G = -\alpha z_G. \tag{3.19}$$

There will also be a vertical shift, $\delta z_G = \alpha y_G$, but that is of no importance to the stability because gravity is vertical so that the shift creates no moment.

The center of buoyancy is also shifted by the tilt, at first by the same rule as the center of gravity but because the displacement also changes there will be another contribution Δy_B to the total shift, so that we may write

$$\delta y_B = -\alpha z_B + \Delta y_B. \tag{3.20}$$

As discussed above, the change in the shape of the displacement amounts to moving the water from the wedge at the right ($y > 0$) to the wedge at the left ($y < 0$). The ensuing change in the horizontal position of the center of buoyancy may according to (3.18) be calculated by averaging the position change $y - y_B$ over the volume of the two wedges,

$$\Delta y_B = -\frac{1}{V}\int_A (y - y_B)\, z\, dA = -\frac{\alpha}{V}\int_A y^2\, dA = -\alpha\frac{I}{V}$$

where

$$\boxed{I = \int_A y^2\, dA\,,} \tag{3.21}$$

is the second order moment of the waterline area A around the x-axis. The movement of displaced water will also create a shift in the x-direction, $\Delta x_B = -\alpha J/V$ where $J = \int_A xy\, dA$, which does not destabilize the ship.

The total horizontal shift in the center of buoyancy may thus be written

$$\delta y_B = -\alpha\left(z_B + \frac{I}{V}\right). \tag{3.22}$$

This shows that the complicated shift in the position of the center of buoyancy can be written as a simple rotation of a point M that is *fixed* with respect to the ship with z-coordinate,

$$\boxed{z_M = z_B + \frac{I}{V}.} \tag{3.23}$$

This point, called the *metacenter*, is usually placed on the straight line that goes through the centers of gravity and buoyancy, such that $x_M = x_G = x_B$ and $y_M = y_G = y_B$. The calculation shows that when the ship is heeled through a small angle, the center of buoyancy will always move so that it stays vertically below the metacenter.

The metacenter is a purely geometric quantity (for a liquid with constant density), depending only on the displacement volume V, the center of buoyancy $\boldsymbol{x}_B$, and the second order moment of the shape of the ship in the waterline. The simplest waterline shapes are,

Rectangular waterline area: If the ship has a rectangular waterline area with sides $2a$ and $2b$, the roll center coincides with the center of the rectangle, and the second moment around the x-axis becomes,

$$I = \int_{-a}^{a} dx \int_{-b}^{b} dy\, y^2 = \frac{4}{3}ab^3. \tag{3.24}$$

If $a > b$ this is the smallest moment around any tilt axis because $ab^3 < a^3 b$ (see page 55).

Elliptic waterline area: If the ship has an elliptical waterline area with axes $2a$ and $2b$, the roll center coincides with the center of the ellipse, and the second moment around the x-axis becomes,

$$I = \int_{-a}^{a} dx \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} y^2\, dy = \frac{4}{3}ab^3 \int_0^1 (1-t^2)^{3/2}\, dt = \frac{\pi}{4}ab^3. \tag{3.25}$$

Notice that this is a about half of the value for the rectangle.

Stability condition

The tilt generates a *restoring moment* around the x-axis, which may be calculated from (3.15) ,

$$\mathcal{M}_x = -(y_G - y_B) M g_0. \tag{3.26}$$

Since we have $y_G = y_B$ in the original mechanical equilibrium, the difference in coordinates after the tilt may be written, $y_G - y_B = \delta y_G - \delta y_B$ where δy_G and δy_B are the small horizontal shifts of order α in the centers of gravity and buoyancy, calculated above. The shift $\Delta x_B = -\alpha J/V$ will create a moment $M_y = -\Delta x_B M g_0$ which tends to pitch the ship along x but does not affect its stability.

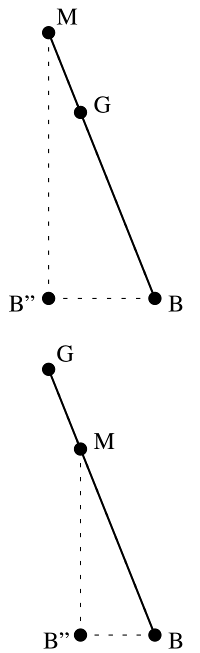
In terms of the height of the metacenter z_M the restoring moment becomes

$$\boxed{\mathcal{M}_x = \alpha(z_G - z_M) M g_0.} \tag{3.27}$$

For the ship to be stable, the restoring moment must counteract the tilt and thus have opposite sign of the tilt angle α. Consequently, the stability condition becomes

$$z_G < z_M. \tag{3.28}$$

Evidently, *the ship is only stable when the center of gravity lies below the metacenter.* For an alternative derivation of the stability condition, see problem 3.14.



The metacenter M lies always directly above the actual center of buoyancy B$''$. **Top:** The ship is stable because the metacenter lies above the center of gravity. **Bottom:** The ship is unstable because the metacenter lies below the center of gravity.

Example 3.4 [Elliptical rowboat]: An elliptical rowboat with vertical sides has major axis $2a = 2$ m and minor axis $2b = 1$ m. The smallest moment of the elliptical area is $I = (\pi/4)ab^3 \approx 0.1$ m^4. If your mass is 75 kg and the boat's is 50 kg, the displacement will be $V = 0.125$ m^3, and the draft $d \approx V/4ab = 6.25$ cm, ignoring the usually curved shape of the boat's hull. The coordinate of the center of buoyancy becomes $z_B = -3.1$ cm and the metacenter $z_M = 75$ cm. Getting up from your seat may indeed raise the center of gravity so much that it gets close to the metacenter and the boat begins to roll violently. Depending on your weight and mass distribution the boat may even become unstable and turn over.

Metacentric height and righting arm

The orientation of the coordinate system with respect to the ship's hull was not specified in the analysis which is therefore valid for a tilt around any direction. For a ship to be fully stable, the stability condition must be fulfilled for all possible tilt axes. Since the displacement V is the same for all choices of tilt axis, the second moment of the area on the right hand side of (3.23) should be chosen to be the *smallest* one. Often it is quite obvious which moment is the smallest. Many modern ships are extremely long with the same cross section along most of their length and with mirror symmetry through a vertical plane. For such ships the smallest moment is clearly obtained with the tilt axis parallel to the longitudinal axis of the ship.

The restoring moment (3.27) is proportional to the vertical distance, $z_M - z_G$, between the metacenter and the center of gravity, also called the *metacentric height*. The closer the center of gravity comes to the metacenter, the smaller will the restoring moment be, and the longer will the period of rolling oscillations be. The actual roll period depends also on the true moment of inertia of the ship around the tilt axis (see problem 3.11). Whereas the metacenter is a purely geometric quantity which depends only on the ship's actual draft, the center of gravity depends on the way the ship is actually loaded. A good captain should always know the positions of the center of gravity and the metacenter of his ship before he sails, or else he may capsize when casting off.

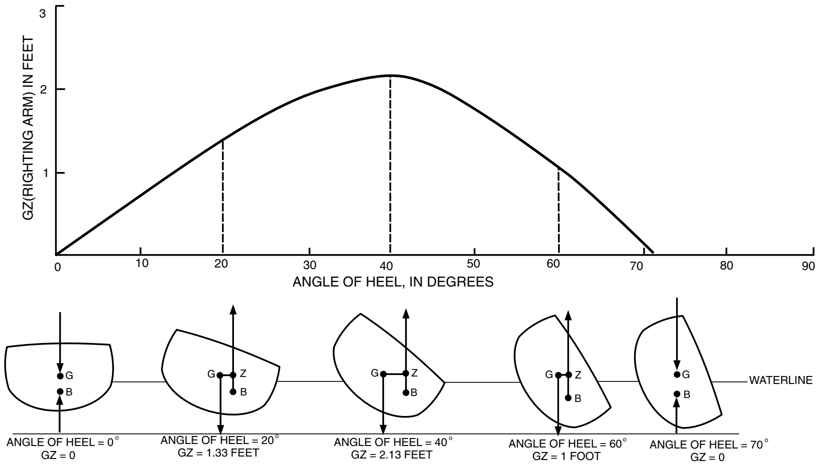
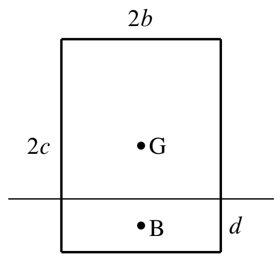


Figure 3.4. Stability curve for large angles of heel. The metacenter is only useful for tiny heel angles where all changes are linear in the angle. For larger angles one uses instead the 'righting arm' which is the distance between the center of gravity and the vertical line through the actual center of buoyancy. Instability sets in when the righting arm reaches zero, in the above plot for about 72° heel. Courtesy John Pike, Global Security.

The metacenter is only useful for tiny heel angles where all changes are linear in the angle. For larger angles one uses instead the *righting arm* which is the horizontal distance $|y_G - y_B|$ between the center of gravity and the vertical line through the actual center of buoyancy. The restoring moment is the product of the righting arm and the weight of the ship, and instability sets in when the righting arm reaches zero for some non-vanishing angle of heel (see figure 3.4).

Case: Floating block

The simplest non-trivial case in which we may apply the stability criterion is that of a rectangular block of dimensions $2a$, $2b$ and $2c$ in the three coordinate directions. Without loss of generality we may assume that $a > b$. The center of the waterline area coincides with the roll center and the origin of the coordinate system with the waterline at $z = 0$. The block is assumed to be made from a uniform material with constant density ρ_1 and floats in a liquid of constant density ρ_0.



Floating block with height h, draft d, width $2b$, and length $2a$ into the paper.

In hydrostatic equilibrium we must have $M = 4abd\rho_0 = 8abc\rho_1$, or

$$\frac{\rho_1}{\rho_0} = \frac{d}{2c}. \tag{3.29}$$

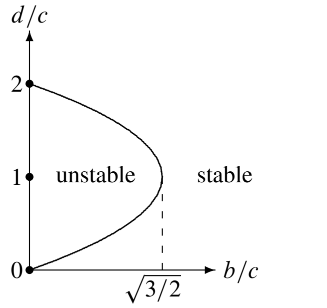
The position of the center of gravity is $z_G = c - d$ and the center of buoyancy $z_B = -d/2$. Using (3.24) and $V = 4abd$, the position of the metacenter is

$$z_M = -\frac{d}{2} + \frac{b^2}{3d}. \tag{3.30}$$

Rearranging the stability condition, $z_M > z_G$, it may be written as

$$\left(\frac{d}{c} - 1\right)^2 > 1 - \frac{2b^2}{3c^2}. \tag{3.31}$$

When the block dimensions obey $a > b$ and $b/c > \sqrt{3/2} = 1.2247\ldots$, the right-hand side becomes negative and the inequality is always fulfilled. On the other hand, if $b/c < \sqrt{3/2}$



Stability diagram for the floating block.

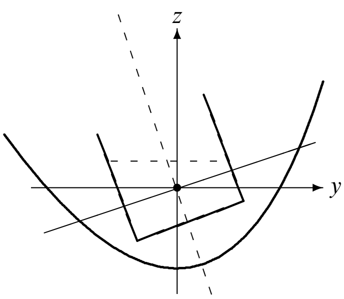
there is a range of draft values around $d = c$ (corresponding to $\rho_1/\rho_0 = 1/2$),

$$1 - \sqrt{1 - \frac{2}{3}\left(\frac{b}{c}\right)^2} < \frac{d}{c} < 1 + \sqrt{1 - \frac{2}{3}\left(\frac{b}{c}\right)^2}, \tag{3.32}$$

for which the block is unstable. If the draft lies in this interval the block will keel over and come to rest in another orientation.

For a cubic block we have $a = b = c$, there is always a range around density $\rho_1/\rho_0 = 1/2$ which cannot be stable. It takes quite a bit of labor to determine which other orientation has smallest metacentric height (see problem 3.13).

Case: Ship with liquid cargo



Tilted ship with an open container filled with liquid.

Many ships carry liquid cargos, oil or water, or nearly liquid cargoes such as grain. When the tanks are not completely filled this kind of cargo may strongly influence the stability of the ship. The main effect of an open liquid surface inside the ship is the movement of real liquid which shifts the center of mass in the same direction as the movement of displaced water shifts the center of buoyancy. This competition between the two centers may directly lead to instability because the movement of the real liquid inside the ship nearly cancels the stabilizing movement of the displaced water. On top that the fairly slow sloshing of the real liquid compared to the instantaneous movement of the displaced water can lead to dangerous oscillations which may also capsize the ship.

For the case of a single open tank the calculation of the restoring moment must now include the liquid cargo. Disregarding sloshing, a similar analysis as before shows that there will be a horizontal change in the center of gravity from the movement of a wedge of real liquid of density ρ_1,

$$\Delta y_G = -\alpha \frac{\rho_1 I_1}{M} = -\alpha \frac{\rho_1}{\rho_0} \frac{I_1}{V} \tag{3.33}$$

where I_1 is the second moment of the open liquid surface. The metacenter position now becomes

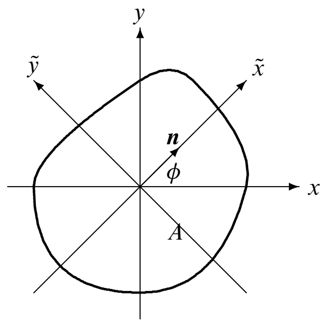
$$\boxed{z_M = z_B + \frac{I}{V} - \frac{\rho_1}{\rho_0} \frac{I_1}{V}.} \tag{3.34}$$

The effect of the moving liquid is to lower the metacentric height (or shorten the righting arm) with possible destabilization as a result. The unavoidable inertial sloshing of the liquid may further compromise the stability. The destabilizing effect of a liquid cargo is often counteracted by dividing the hold into a number of smaller compartments by means of bulkheads along the ship's principal roll axis.

Car ferry instability: In heavy weather or due to accidents, a car ferry may inadvertently get a layer of water on the car deck. Since the car deck of a roll-on-roll-off ferry normally spans the whole ferry, we have $\rho_1 = \rho_0$ and $I_1 \approx I$, implying that $z_M \approx z_B < z_G$, nearly independent of the thickness h of the layer of water (as long as h is not too small). The inequality $z_M < z_G$ spells rapid disaster, as several accidents with car ferries have shown. Waterproof longitudinal bulkheads on the car deck would stabilize the ferry, but are usually avoided because they would hamper efficient loading and unloading of the cars.

* Principal roll axes

It has already been pointed out that the metacenter for absolute stability is determined by the smallest second moment of the waterline area. To determine that we instead tilt the ship around an axis, $\boldsymbol{n} = (\cos\phi, \sin\phi, 0)$, forming an angle ϕ with the x-axis. Since this configuration is obtained by a simple rotation through ϕ around the z-axis, the transverse coordinate to be used in calculating the second moment becomes $\tilde{y} = y\cos\phi - x\sin\phi$ (see eq. (B.29b)), and we find the moment,



Tilt axis $\boldsymbol{n}$ forming an angle ϕ with the x-axis.

$$\tilde{I} = \int_A \tilde{y}^2\, dA = I_{yy}\cos^2\phi + I_{xx}\sin^2\phi - 2I_{xy}\sin\phi\cos\phi, \qquad (3.35)$$

where I_{xx}, I_{yy} and I_{xy} are the elements of the symmetric matrix,

$$\boldsymbol{I} = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{pmatrix} = \int_A \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} dA, \qquad (3.36)$$

which is actually a two-dimensional tensor.

The extrema of $\tilde{I}(\phi)$ are easily found by differentiation with respect to ϕ. They are,

$$\phi_1 = \frac{1}{2}\arctan\frac{2I_{xy}}{I_{xx} - I_{yy}}, \qquad \phi_2 = \phi_1 + \frac{\pi}{2}, \qquad (3.37)$$

with the respective area moments $I_1 = \tilde{I}(\phi_1)$ and $I_2 = \tilde{I}(\phi_2)$. The two angles determine orthogonal *principal directions*, 1 and 2, in the ship's waterline area. The principal direction with the smallest second order moment around the area centroid has the lowest metacentric height. If I_1 is the smallest moment and the actual roll axis forms an angle ϕ with the 1-axis, we can calculate the moment I for any other axis of roll forming an angle ϕ with the x-axis from

$$I = I_1\cos^2\phi + I_2\sin^2\phi. \qquad (3.38)$$

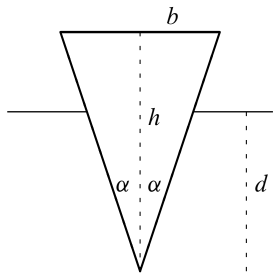
Since $I_1 < I_2$, it follows trivially that if the ship is stable for a tilt around the first principal axis, it will be stable for a tilt around any axis.

Problems

3.1 A stone weighs 1000 N in air and 600 N when submerged in water. Calculate the volume and average density of the stone.

3.2 A hydrometer (an instrument used to measure the density of a liquid) with mass $M = 4$ g consists of a roughly spherical glass container and a long thin cylindrical stem of radius $a = 2$ mm. The sphere is weighed down so that the apparatus will float stably with the stem pointing vertically upwards and crossing the fluid surface at some point. How much deeper will it float in alcohol with mass density $\rho_1 = 0.78$ g cm^{-3} than in oil with mass density $\rho_2 = 0.82$ g cm^{-3}? You may disregard the tiny density of air.

3.3 A cylindrical wooden stick (density $\rho_1 = 0.65$ g cm^{-3}) floats in water (density $\rho_0 = 1$ g cm^{-3}). The stick is loaded down with a lead weight (density $\rho_2 = 11$ g cm^{-3}) at one end such that it floats in a vertical position with a fraction $f = 1/10$ of its length out of the water. **(a)** What is the ratio (M_1/M_2) between the masses of the wooden stick and the lead weight? **(b)** How large a fraction of the stick can be out of the water in hydrostatic equilibrium (disregarding questions of stability)?



Triangular ship of length L (into the paper) floating with its peak vertically downwards.

3.4 A ship of length L has a longitudinally invariant cross section in the shape of an isosceles triangle with half opening angle α and height h (see the margin figure). It is made from homogeneous material of density ρ_1 and floats in a liquid of density $\rho_0 > \rho_1$. **(a)** Determine the stability condition on the mass ratio ρ_1/ρ_0 when the ship floats vertically with the peak downwards. **(b)** Determine the stability condition on the mass ratio when the ship floats vertically with the peak upwards. **(c)** What is the smallest opening angle that permits simultaneous stability in both directions?

3.5 A right rotation cone has half opening angle α and height h. It is made from a homogeneous material of density ρ_1 and floats in a liquid of density $\rho_0 > \rho_1$. **(a)** Determine the stability condition on the mass ratio ρ_1/ρ_0 when the cone floats vertically with the peak downwards. **(b)** Determine the stability condition on the mass ratio when the cone floats vertically with the peak upwards. **(c)** What is the smallest opening angle that permits simultaneous stability in both directions?

3.6 A barotropic compressible fluid is in hydrostatic equilibrium with pressure $p(z)$ and density $\rho(z)$ in a constant external gravitational field with potential $\Phi = g_0 z$. A finite body having a 'small' gravitational field $\Delta\Phi(\boldsymbol{x})$ is submerged in the fluid. **(a)** Show that the change in hydrostatic pressure to lowest order of approximation is $\Delta p(\boldsymbol{x}) = -\rho(z)\Delta\Phi(\boldsymbol{x})$. **(b)** Show that for a spherically symmetric body of radius a and mass M, the extra surface pressure is $\Delta p = g_1 a \rho(z)$ where $g_1 = GM/a^2$ is the magnitude of surface gravity, and that the buoyancy force is increased.

3.7 Two identical homogenous spheres of mass M and radius a are situated a distance $D \gg a$ apart in a barotropic fluid. Due to their field of gravity, the fluid will be denser near the spheres. There is no other gravitational field present, the fluid density is ρ_0 and the pressure is p_0 in the absence of the spheres. One may assume that the pressure corrections due to the spheres are small everywhere in comparison with p_0. **(a)** Show that the spheres will repel each other and calculate the magnitude of the force to leading order in a/D. **(b)** Compare with the gravitational attraction between the spheres. **(c)** Under which conditions will the total force between the spheres vanish?

* **3.8** Prove without assuming constant gravity that the hydrostatic moment of buoyancy equals (minus) the moment of gravity of the displaced fluid (corollary to Archimedes' law).

* **3.9** Assuming constant gravity, show that for a body not in buoyant equilibrium (i.e. for which the total force $\mathcal{F}$ does not vanish), there is always a well-defined point $\boldsymbol{x}_0$ such that the total moment of gravitational plus buoyant forces is given by $\boldsymbol{\mathcal{M}} = \boldsymbol{x}_0 \times \mathcal{F}$.

* **3.10** Let $\boldsymbol{I}$ be a symmetric (2×2) matrix. Show that the extrema of the corresponding quadratic form $\boldsymbol{n} \cdot \boldsymbol{I} \cdot \boldsymbol{n} = I_{xx} n_x^2 + 2 I_{xy} n_x n_y + I_{yy} n_y^2$ where $n_x^2 + n_y^2 = 1$ are determined by the eigenvectors of $\boldsymbol{I}$ satisfying $\boldsymbol{I} \cdot \boldsymbol{n} = \lambda \boldsymbol{n}$.

* **3.11** Show that in a stable orientation the angular frequency of small oscillations around around a principal tilt axis of a ship is

$$\omega = \sqrt{\frac{M g_0}{J}(z_M - z_G)}$$

where J is the moment of inertia of the ship around this axis.

* **3.12** A ship has a waterline area which is a regular polygon with $n \geq 3$ edges. Show that the area moment tensor (3.36) has $I_{xx} = I_{yy}$ and $I_{xy} = 0$.

* **3.13** A homogeneous cubic block has density equal to half that of the liquid it floats on. Determine the stability properties of the cube when it floats **(a)** with a horizontal face below the center, **(b)** with a horizontal edge below the center, and **(c)** with a corner vertically below the center. Hint: problem 3.12 is handy for the last case, which you should be warned is quite difficult.

3.14 **(a)** Show that the work (or potential energy) necessary to tilt a ship through an angle α to second order in α is

$$W = -\frac{1}{2}\alpha^2 M_0 g_0 (z_G - z_B) + \frac{1}{2}\rho_0 g_0 \int_{A_0} z^2 dA \tag{3.39}$$

where $z = \alpha y$, and **(b)** show that this leads to the stability condition (3.28) .

3.15 Estimate the rise of the Charlière when a passenger of 76 kilograms jumps out.