D

Cylindrical coordinates

Cartesian coordinates x, y, z and cylindrical coordinates[1] r, ϕ, z are related by

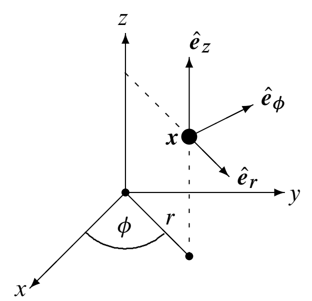
$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z \tag{D.1}$$

with the range of variation $0 \le r < \infty$, $0 \le \phi < 2\pi$, and $-\infty < z < \infty$. The inverse transformation is

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \arctan\frac{y}{x}, \qquad z = z \tag{D.2}$$

The two first equations in both transformations simply define polar coordinates in the xy-plane, whereas the last, $z = z$, is included to emphasize that this is a transformation in three-dimensional space.

D.1 Local basis vectors



Cylindrical coordinates and basis vectors.

The *cylindrical basis vectors* follow from the geometry (see the margin figure),

$$\hat{\boldsymbol{e}}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\cos\phi, \sin\phi, 0), \tag{D.3a}$$

$$\hat{\boldsymbol{e}}_\phi = \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0), \tag{D.3b}$$

$$\hat{\boldsymbol{e}}_z = \frac{\partial \boldsymbol{x}}{\partial z} = (0, 0, 1). \tag{D.3c}$$

They are clearly orthonormal and satisfy $\hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_\phi \cdot \hat{\boldsymbol{e}}_z = 1$.

A Cartesian vector field $\boldsymbol{U}$ may be resolved in this basis

$$\boldsymbol{U} = \hat{\boldsymbol{e}}_r U_r + \hat{\boldsymbol{e}}_\phi U_\phi + \hat{\boldsymbol{e}}_z U_z, \tag{D.4}$$

where

$$U_r = \boldsymbol{U} \cdot \hat{\boldsymbol{e}}_r, \qquad U_\phi = \boldsymbol{U} \cdot \hat{\boldsymbol{e}}_\phi, \qquad U_z = \boldsymbol{U} \cdot \hat{\boldsymbol{e}}_z. \tag{D.5}$$

A tensor field $\boldsymbol{T}$ may similarly be resolved in dyadic products of the local basis vectors.

[1] Some texts use Θ instead of ϕ as the conventional name for the polar angle in the plane. Similarly, the radial variable r is sometimes denoted s to distinguish it from the spherical radial distance.

D.2 Line, surface and volume elements

The differentials along the local coordinate axes,

$$d_r\boldsymbol{x} = \hat{\boldsymbol{e}}_r dr, \qquad d_\phi\boldsymbol{x} = \hat{\boldsymbol{e}}_\phi r d\phi, \qquad d_z\boldsymbol{x} = \hat{\boldsymbol{e}}_z dz, \tag{D.6}$$

allow us to resolve the Cartesian line, surface and volume elements in the local basis,

$$d\boldsymbol{\ell} \equiv d_r\boldsymbol{x} + d_\phi\boldsymbol{x} + d_z\boldsymbol{x} = \hat{\boldsymbol{e}}_r\, dr + \hat{\boldsymbol{e}}_\phi\, r d\phi + \hat{\boldsymbol{e}}_z\, dz, \tag{D.7}$$

$$\begin{aligned} d\boldsymbol{S} &\equiv d_\phi\boldsymbol{x} \times d_z\boldsymbol{x} + d_z\boldsymbol{x} \times d_r\boldsymbol{x} + d_r\boldsymbol{x} \times d_\phi\boldsymbol{x} \\ &= \hat{\boldsymbol{e}}_r\, r\, d\phi dz + \hat{\boldsymbol{e}}_\phi\, dr dz + \hat{\boldsymbol{e}}_z\, r\, d\phi dr, \end{aligned} \tag{D.8}$$

$$dV \equiv d_r\boldsymbol{x} \times d_\phi\boldsymbol{x} \cdot d_z\boldsymbol{x} = r\, d\phi dr dz \tag{D.9}$$

Using these infinitesimals, all integrals can be converted to cylindrical coordinates.

D.3 Resolution of the gradient

The derivatives with respect to the cylindrical coordinates are obtained by differentiation through the Cartesian coordinates,

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial \boldsymbol{x}}{\partial r} \cdot \frac{\partial}{\partial \boldsymbol{x}} = \hat{\boldsymbol{e}}_r \cdot \boldsymbol{\nabla} = \nabla_r, \\ \frac{\partial}{\partial \phi} &= \frac{\partial \boldsymbol{x}}{\partial \phi} \cdot \frac{\partial}{\partial \boldsymbol{x}} = r\hat{\boldsymbol{e}}_\phi \cdot \boldsymbol{\nabla} = r\nabla_\phi. \end{aligned}$$

Nabla may now be resolved on the local basis

$$\boxed{\boldsymbol{\nabla} = \hat{\boldsymbol{e}}_r \nabla_r + \hat{\boldsymbol{e}}_\phi \nabla_\phi + \hat{\boldsymbol{e}}_z \nabla_z = \hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z}.} \tag{D.10}$$

Finally, we note that the only non-vanishing derivatives of the basis vectors are

$$\frac{\partial \hat{\boldsymbol{e}}_r}{\partial \phi} = \hat{\boldsymbol{e}}_\phi, \qquad \frac{\partial \hat{\boldsymbol{e}}_\phi}{\partial \phi} = -\hat{\boldsymbol{e}}_r, \tag{D.11}$$

These are the fundamental tools necessary to convert differential equations from Cartesian to cylindrical coordinates.

D.4 First order expressions

Here follows a list of various combinations of a single nabla and various fields. In writing out the results we refrain from using the nabla projections, ∇_r etc, but express everything in conventional partial derivatives, $\partial/\partial r$ etc.

The three basic first order expressions are the gradient, divergence and curl,

$$\nabla S = \hat{\boldsymbol{e}}_r \frac{\partial S}{\partial r} + \hat{\boldsymbol{e}}_\phi \frac{1}{r}\frac{\partial S}{\partial \phi} + \hat{\boldsymbol{e}}_z \frac{\partial S}{\partial z}, \tag{D.12}$$

$$\nabla \cdot \boldsymbol{U} = \frac{\partial U_r}{\partial r} + \frac{1}{r}\frac{\partial U_\phi}{\partial \phi} + \frac{\partial U_z}{\partial z} + \frac{U_r}{r}, \tag{D.13}$$

$$\nabla \times \boldsymbol{U} = \hat{\boldsymbol{e}}_r \left(\frac{1}{r}\frac{\partial U_z}{\partial \phi} - \frac{\partial U_\phi}{\partial z}\right) + \hat{\boldsymbol{e}}_\phi \left(\frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r}\right) + \hat{\boldsymbol{e}}_z \left(\frac{\partial U_\phi}{\partial r} - \frac{1}{r}\frac{\partial U_r}{\partial \phi} + \frac{U_\phi}{r}\right). \tag{D.14}$$

The tensor gradient is used in solid and fluid mechanics to calculate the stress tensor. In dyadic notation (see appendix B), we have

$$\begin{aligned}
\nabla \boldsymbol{U} &= \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_r \frac{\partial U_r}{\partial r} + \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_\phi \frac{\partial U_\phi}{\partial r} + \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_z \frac{\partial U_z}{\partial r} \\
&+ \hat{\boldsymbol{e}}_\phi\hat{\boldsymbol{e}}_r \left(\frac{1}{r}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r}\right) + \hat{\boldsymbol{e}}_\phi\hat{\boldsymbol{e}}_\phi \left(\frac{1}{r}\frac{\partial U_\phi}{\partial \phi} + \frac{U_r}{r}\right) + \hat{\boldsymbol{e}}_\phi\hat{\boldsymbol{e}}_z \frac{1}{r}\frac{\partial U_z}{\partial \phi} \\
&+ \hat{\boldsymbol{e}}_z\hat{\boldsymbol{e}}_r \frac{\partial U_r}{\partial z} + \hat{\boldsymbol{e}}_z\hat{\boldsymbol{e}}_\phi \frac{\partial U_\phi}{\partial z} + \hat{\boldsymbol{e}}_z\hat{\boldsymbol{e}}_z \frac{\partial U_z}{\partial z}
\end{aligned} \tag{D.15}$$

The dot product with a vector $\boldsymbol{V}$ from the left becomes,

$$\begin{aligned}
(\boldsymbol{V}\cdot\nabla)\boldsymbol{U} &= \hat{\boldsymbol{e}}_r \left(V_r \frac{\partial U_r}{\partial r} + \frac{V_\phi}{r}\frac{\partial U_r}{\partial \phi} + V_z \frac{\partial U_r}{\partial z} - \frac{V_\phi U_\phi}{r}\right) \\
&+ \hat{\boldsymbol{e}}_\phi \left(V_r \frac{\partial U_\phi}{\partial r} + \frac{V_\phi}{r}\frac{\partial U_\phi}{\partial \phi} + V_z \frac{\partial U_\phi}{\partial z} + \frac{V_\phi U_r}{r}\right) \\
&+ \hat{\boldsymbol{e}}_z \left(V_r \frac{\partial U_z}{\partial r} + \frac{V_\phi}{r}\frac{\partial U_z}{\partial \phi} + V_z \frac{\partial U_z}{\partial z}\right)
\end{aligned} \tag{D.16}$$

It is used for calculating the advective terms in equations of motion.

Finally, the divergence of a full-fledged tensor field becomes (also in dyadic notation)

$$\begin{aligned}
\nabla \cdot \boldsymbol{T} &= \hat{\boldsymbol{e}}_r \left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{\partial T_{zr}}{\partial z} + \frac{T_{rr}}{r} - \frac{T_{\phi\phi}}{r}\right) \\
&+ \hat{\boldsymbol{e}}_\phi \left(\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{z\phi}}{\partial z} + \frac{T_{r\phi}}{r} + \frac{T_{\phi r}}{r}\right) \\
&+ \hat{\boldsymbol{e}}_z \left(\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi z}}{\partial \phi} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r}\right)
\end{aligned} \tag{D.17}$$

This may be used in formulating the equations of motion for continuum physics, although it is normally not necessary.

D.5 Second order expressions

Expressions involving two nabla factors also turn up everywhere in continuum physics. They can of course be derived by combinations of first order expressions, but it is nevertheless useful to list them separately.

The Laplacian of a scalar field is calculated from the divergence of the gradient, $\nabla^2 S = \nabla \cdot (\nabla S)$, and becomes after the dust has settled,

$$\nabla^2 S = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \phi^2} + \frac{\partial^2 S}{\partial z^2} + \frac{1}{r}\frac{\partial S}{\partial r}. \tag{D.18}$$

The Laplacian can also be applied to a vector field, and may be obtained from the divergence of the gradient of the vector field, $\nabla^2 S = \nabla \cdot (\nabla \boldsymbol{U})$. It is somewhat more complicated,

$$\begin{aligned}
\nabla^2 \boldsymbol{U} = \hat{\boldsymbol{e}}_r & \left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_r}{\partial \phi^2} + \frac{\partial^2 U_r}{\partial z^2} + \frac{1}{r}\frac{\partial U_r}{\partial r} - \frac{2}{r^2}\frac{\partial U_\phi}{\partial \phi} - \frac{U_r}{r^2} \right) \\
+ \hat{\boldsymbol{e}}_\phi & \left(\frac{\partial^2 U_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial \phi^2} + \frac{\partial^2 U_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial U_\phi}{\partial r} + \frac{2}{r^2}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r^2} \right) \\
+ \hat{\boldsymbol{e}}_z & \left(\frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_z}{\partial \phi^2} + \frac{\partial^2 U_z}{\partial z^2} + \frac{1}{r}\frac{\partial U_z}{\partial r} \right)
\end{aligned} \tag{D.19}$$

Another second order expression is the gradient of a divergence,

$$\begin{aligned}
\nabla(\nabla \cdot \boldsymbol{U}) = \hat{\boldsymbol{e}}_r & \left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}\frac{\partial^2 U_\phi}{\partial r \partial \phi} + \frac{\partial^2 U_z}{\partial r \partial z} + \frac{1}{r}\frac{\partial U_r}{\partial r} - \frac{1}{r^2}\frac{\partial U_\phi}{\partial \phi} - \frac{U_r}{r^2} \right) \\
+ \hat{\boldsymbol{e}}_\phi & \left(\frac{1}{r}\frac{\partial^2 U_r}{\partial \phi \partial r} + \frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial \phi^2} + \frac{1}{r}\frac{\partial^2 U_z}{\partial \phi \partial z} + \frac{1}{r^2}\frac{\partial U_r}{\partial \phi} \right) \\
+ \hat{\boldsymbol{e}}_z & \left(\frac{\partial^2 U_r}{\partial z \partial r} + \frac{1}{r}\frac{\partial^2 U_\phi}{\partial z \partial \phi} + \frac{\partial^2 U_z}{\partial z^2} + \frac{1}{r}\frac{\partial U_r}{\partial z} \right)
\end{aligned} \tag{D.20}$$

In practice we do not need more complex expressions than these.