E

Spherical coordinates

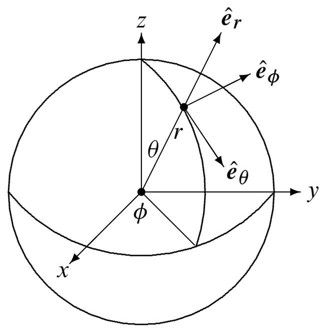
Cartesian coordinates x, y, z and spherical (or polar) coordinates r, θ and ϕ are related by

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r \cos\theta, \tag{E.1}$$

The domain of variation is $0 \le r < \infty$, $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. The inverse transformation is,

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \qquad \phi = \arctan \frac{y}{x} \tag{E.2}$$

The choice of the z-axis as polar axis and the symbols for the angles are nearly universal.



Spherical coordinates and their basis vectors.

E.1 Curvilinear basis

The normalized tangent vectors along the directions of the spherical coordinate are,

$$\hat{\boldsymbol{e}}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{E.3a}$$

$$\hat{\boldsymbol{e}}_\theta = \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \tag{E.3b}$$

$$\hat{\boldsymbol{e}}_\phi = \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0). \tag{E.3c}$$

They are orthogonal, such that an arbitrary vector field may be resolved in these directions,

$$\boldsymbol{U} = \hat{\boldsymbol{e}}_r U_r + \hat{\boldsymbol{e}}_\theta U_\theta + \hat{\boldsymbol{e}}_\phi U_\phi \tag{E.4}$$

with spherical components

$$U_r = \hat{\boldsymbol{e}}_r \cdot \boldsymbol{U}, \qquad U_\theta = \hat{\boldsymbol{e}}_\theta \cdot \boldsymbol{U}, \qquad U_\phi = \hat{\boldsymbol{e}}_\phi \cdot \boldsymbol{U} \tag{E.5}$$

A tensor field $\boldsymbol{T}$ may similarly be resolved in dyadic products of the local basis vectors.

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E.2 Line, surface, and volume element

The differentials along the local coordinate axes,

$$d_r\boldsymbol{x} = \hat{\boldsymbol{e}}_r dr, \qquad d_\theta\boldsymbol{x} = \hat{\boldsymbol{e}}_\theta r d\theta, \qquad d_\phi\boldsymbol{x} = \hat{\boldsymbol{e}}_\phi r r \sin\theta d\phi, \tag{E.6}$$

allow us to resolve the Cartesian line, surface and volume elements in the local basis,

$$d\boldsymbol{\ell} \equiv d_r\boldsymbol{x} + d_\theta\boldsymbol{x} + d_\phi\boldsymbol{x} = \hat{\boldsymbol{e}}_r\, dr + \hat{\boldsymbol{e}}_\theta\, r d\theta + \hat{\boldsymbol{e}}_\phi\, r\sin\theta d\phi, \tag{E.7}$$

$$\begin{aligned} d\boldsymbol{S} &\equiv d_\theta\boldsymbol{x} \times d_\phi\boldsymbol{x} + d_\phi\boldsymbol{x} \times d_r\boldsymbol{x} + d_r\boldsymbol{x} \times d_\theta\boldsymbol{x} \\ &= \hat{\boldsymbol{e}}_r\, r^2 \sin\theta\, d\theta\, d\phi + \hat{\boldsymbol{e}}_\theta r \sin\theta d\phi dr + \hat{\boldsymbol{e}}_\phi\, r dr d\theta, \end{aligned} \tag{E.8}$$

$$dV \equiv d_r\boldsymbol{x} \times d_\theta\boldsymbol{x} \cdot d_\phi\boldsymbol{x} = r^2 \sin\theta\, dr d\theta d\phi. \tag{E.9}$$

Using these infinitesimals, all integrals can be converted to spherical coordinates.

E.3 Resolution of the gradient

The derivatives with respect to the spherical coordinates are obtained by differentiation through the Cartesian coordinates

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial\boldsymbol{x}}{\partial r} \cdot \frac{\partial}{\partial\boldsymbol{x}} = \hat{\boldsymbol{e}}_r \cdot \boldsymbol{\nabla} = \nabla_r, \\ \frac{\partial}{\partial\theta} &= \frac{\partial\boldsymbol{x}}{\partial\theta} \cdot \boldsymbol{\nabla} = r\hat{\boldsymbol{e}}_\theta \cdot \boldsymbol{\nabla} = r\nabla_\theta, \\ \frac{\partial}{\partial\phi} &= \frac{\partial\boldsymbol{x}}{\partial\phi} \cdot \boldsymbol{\nabla} = r\sin\theta\hat{\boldsymbol{e}}_\phi \cdot \boldsymbol{\nabla} = r\sin\theta\nabla_\phi. \end{aligned}$$

This allows us to resolve the nabla operator in the curvilinear basis

$$\boxed{\boldsymbol{\nabla} = \hat{\boldsymbol{e}}_r\nabla_r + \hat{\boldsymbol{e}}_\theta\nabla_\theta + \hat{\boldsymbol{e}}_\phi\nabla_\phi = \hat{\boldsymbol{e}}_r\frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{e}}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi},} \tag{E.10}$$

Finally, the non-vanishing derivatives of the basis vectors are

$$\frac{\partial\hat{\boldsymbol{e}}_r}{\partial\theta} = \hat{\boldsymbol{e}}_\theta, \qquad \frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial\theta} = -\hat{\boldsymbol{e}}_r, \tag{E.11a}$$

$$\frac{\partial\hat{\boldsymbol{e}}_\theta}{\partial\phi} = \cos\theta\hat{\boldsymbol{e}}_\phi, \qquad \frac{\partial\hat{\boldsymbol{e}}_r}{\partial\phi} = \sin\theta\,\hat{\boldsymbol{e}}_\phi, \qquad \frac{\partial\hat{\boldsymbol{e}}_\phi}{\partial\phi} = -\sin\theta\,\hat{\boldsymbol{e}}_r - \cos\theta\,\hat{\boldsymbol{e}}_\theta. \tag{E.11b}$$

These are all the relations necessary to convert differential equations from Cartesian to spherical coordinates.

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E.4 First order expressions

Here follows a list of various combinations of a single nabla and various fields. In writing out the results we refrain from using the nabla projections, ∇_r etc, but express everything in conventional partial derivatives, $\partial/\partial r$ etc.

The three basic first order expressions are the gradient, divergence and curl,

$$\nabla S = \hat{\boldsymbol{e}}_r \frac{\partial S}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial S}{\partial \theta} + \hat{\boldsymbol{e}}_z \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}, \tag{E.12}$$

$$\nabla \cdot \boldsymbol{U} = \frac{\partial U_r}{\partial r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial U_\phi}{\partial \phi} + \frac{2U_r}{r} + \frac{U_\theta}{r\tan\theta}, \tag{E.13}$$

$$\begin{aligned}\nabla \times \boldsymbol{U} &= \hat{\boldsymbol{e}}_r \left(\frac{1}{r}\frac{\partial U_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial U_\theta}{\partial \phi} + \frac{U_\phi}{r\tan\theta}\right) \\ &+ \hat{\boldsymbol{e}}_\theta \left(\frac{1}{r\sin\theta}\frac{\partial U_r}{\partial \phi} - \frac{\partial U_\phi}{\partial r} - \frac{U_\phi}{r}\right) \\ &+ \hat{\boldsymbol{e}}_\phi \left(\frac{\partial U_\theta}{\partial r} - \frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{U_\theta}{r}\right).\end{aligned} \tag{E.14}$$

The tensor gradient becomes

$$\begin{aligned}\nabla \boldsymbol{U} &= \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_r \frac{\partial U_r}{\partial r} + \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_\theta \frac{\partial U_\theta}{\partial r} + \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_\phi \frac{\partial U_\phi}{\partial r} \\ &+ \hat{\boldsymbol{e}}_\theta\hat{\boldsymbol{e}}_r \left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} - \frac{U_\theta}{r}\right) + \hat{\boldsymbol{e}}_\theta\hat{\boldsymbol{e}}_\theta \left(\frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right) + \hat{\boldsymbol{e}}_\theta\hat{\boldsymbol{e}}_\phi \frac{1}{r}\frac{\partial U_\phi}{\partial \theta} \\ &+ \hat{\boldsymbol{e}}_\phi\hat{\boldsymbol{e}}_r \left(\frac{1}{r\sin\theta}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r}\right) + \hat{\boldsymbol{e}}_\phi\hat{\boldsymbol{e}}_\theta \left(\frac{1}{r\sin\theta}\frac{\partial U_\theta}{\partial \phi} - \frac{U_\phi}{r\tan\theta}\right) \\ &+ \hat{\boldsymbol{e}}_\phi\hat{\boldsymbol{e}}_\phi \left(\frac{1}{r\sin\theta}\frac{\partial U_\phi}{\partial \phi} + \frac{U_\theta}{r\tan\theta} + \frac{U_r}{r}\right).\end{aligned} \tag{E.15}$$

Dotting from the left with $\boldsymbol{U}$ we get

$$\begin{aligned}(\boldsymbol{V}\cdot\nabla)\boldsymbol{U} &= \hat{\boldsymbol{e}}_r \left(V_r \frac{\partial U_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial U_r}{\partial \theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial U_r}{\partial \phi} - \frac{V_\theta U_\theta}{r} - \frac{V_\phi U_\phi}{r}\right) \\ &+ \hat{\boldsymbol{e}}_\theta \left(V_r \frac{\partial U_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial U_\theta}{\partial \phi} + \frac{V_\theta U_r}{r} - \frac{V_\phi U_\phi}{r\tan\theta}\right) \\ &+ \hat{\boldsymbol{e}}_\phi \left(V_r \frac{\partial U_\phi}{\partial r} + \frac{V_\theta}{r}\frac{\partial U_\phi}{\partial \theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial U_\phi}{\partial \phi} + \frac{V_\phi U_r}{r} + \frac{V_\phi U_\theta}{r\tan\theta}\right)\end{aligned} \tag{E.16}$$

Finally, the left divergence of a tensor field becomes,

$$\begin{aligned}\nabla\cdot\boldsymbol{T} &= \hat{\boldsymbol{e}}_r \left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{2T_{rr}}{r} + \frac{T_{\theta r}}{r\tan\theta} - \frac{T_{\theta\theta}}{r} - \frac{T_{\phi\phi}}{r}\right) \\ &+ \hat{\boldsymbol{e}}_\theta \left(\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial \phi} + \frac{2T_{r\theta}}{r} + \frac{T_{\theta r}}{r} + \frac{T_{\theta\theta}}{r\tan\theta} - \frac{T_{\phi\phi}}{r\tan\theta}\right) \\ &+ \hat{\boldsymbol{e}}_\phi \left(\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{2T_{r\phi}}{r} + \frac{T_{\phi r}}{r} + \frac{T_{\theta\phi}}{r\tan\theta} + \frac{T_{\phi\theta}}{r\tan\theta}\right)\end{aligned} \tag{E.17}$$

This may be used in formulating the equations of motion for continuum physics, although it is normally not necessary.

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E.5 Second order expressions

The Laplacian of a scalar field becomes

$$\nabla^2 S = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 S}{\partial \phi^2} + \frac{2}{r}\frac{\partial S}{\partial r} + \frac{1}{r^2 \tan\theta}\frac{\partial S}{\partial \theta}. \tag{E.18}$$

The Laplacian of a vector field becomes

$$\begin{aligned}
\nabla^2 \boldsymbol{U} = \hat{\boldsymbol{e}}_r & \left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_r}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_r}{\partial \phi^2} \right. \\
& \left. + \frac{2}{r}\frac{\partial U_r}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial U_r}{\partial \theta} - \frac{2}{r^2}\frac{\partial U_\theta}{\partial \theta} - \frac{2}{r^2 \sin\theta}\frac{\partial U_\phi}{\partial \phi} - \frac{2U_r}{r^2} - \frac{2U_\theta}{r^2 \tan\theta} \right) \\
+ \hat{\boldsymbol{e}}_\theta & \left(\frac{\partial^2 U_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_\theta}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_\theta}{\partial \phi^2} \right. \\
& \left. + \frac{2}{r}\frac{\partial U_\theta}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial U_\theta}{\partial \theta} + \frac{2}{r^2}\frac{\partial U_r}{\partial \theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial U_\phi}{\partial \phi} - \frac{U_\theta}{r^2 \sin^2\theta} \right) \\
+ \hat{\boldsymbol{e}}_\phi & \left(\frac{\partial^2 U_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_\phi}{\partial \phi^2} \right. \\
& \left. + \frac{2}{r}\frac{\partial U_\phi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial U_\phi}{\partial \theta} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial U_\theta}{\partial \phi} + \frac{2}{r^2 \sin\theta}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r^2 \sin^2\theta} \right).
\end{aligned} \tag{E.19}$$

Finally, the gradient of the divergence

$$\begin{aligned}
\nabla(\nabla \cdot \boldsymbol{U}) = \hat{\boldsymbol{e}}_r & \left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}\frac{\partial^2 U_\theta}{\partial r \partial \theta} + \frac{1}{r \sin\theta}\frac{\partial^2 U_\phi}{\partial r \partial \phi} \right. \\
& \left. + \frac{2}{r}\frac{\partial U_r}{\partial r} + \frac{\cot\theta}{r}\frac{\partial U_\theta}{\partial r} - \frac{1}{r^2}\frac{\partial U_\theta}{\partial \theta} - \frac{1}{r^2 \sin\theta}\frac{\partial U_\phi}{\partial \phi} - \frac{2U_r}{r^2} - \frac{U_\theta}{r^2 \tan\theta} \right) \\
+ \hat{\boldsymbol{e}}_\theta & \left(\frac{1}{r}\frac{\partial^2 U_r}{\partial \theta \partial r} + \frac{1}{r^2}\frac{\partial^2 U_\theta}{\partial \theta^2} + \frac{1}{r^2 \sin\theta}\frac{\partial^2 U_\phi}{\partial \theta \partial \phi} \right. \\
& \left. + \frac{2}{r^2}\frac{\partial U_r}{\partial \theta} + \frac{\cot\theta}{r^2}\frac{\partial U_\theta}{\partial \theta} - \frac{\cos\theta}{r^2 \sin^2\theta}\frac{\partial U_\phi}{\partial \phi} - \frac{U_\theta}{r^2 \sin^2\theta} \right) \\
+ \hat{\boldsymbol{e}}_\phi & \left(\frac{1}{r \sin\theta}\frac{\partial^2 U_r}{\partial \phi \partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial^2 U_\theta}{\partial \phi \partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_\phi}{\partial \phi^2} \right. \\
& \left. + \frac{2}{r^2 \sin\theta}\frac{\partial U_r}{\partial \phi} + \frac{\cos\theta}{r^2 \sin^2\theta}\frac{\partial U_\theta}{\partial \phi} \right)
\end{aligned} \tag{E.20}$$

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