0. The Calculus of Variations

0.1 *Introduction to the calculus of variations*: what the calculus of variations is good for; calculus of variations with one dependent and one independent variable, variations with fixed end-points, Euler's equation; readily integrable systems, conservation laws

0.2 *Several dependent and several independent variables*: classical mechanics; the scalar wave equation; Maxwell's equations; the Schrödinger equation

0.3 *Variations at boundaries*: natural boundary conditions; superconductivity

0.4 *The second variation*: stationarity versus extremality; the criteria of Legendre, Euler and Jacobi

0.4 *Constraints*: implementing constraints via Lagrange multipliers; isoperimetric problems; connection with eigenproblems; variational approximation schemes

1. Partial differential equations of mathematical physics

1.1 *A selection of important partial differential equations (or PDEs)*: the diffusion equation (heat conduction, stochastic processes, polymer statistics); the wave equation (motion of transverse displacements of a stretched string); Maxwell's equations (*in vacuo*) and potentials in electromagnetism; the time-dependent Schrödinger equation; the Navier-Stokes equation (motion of simple fluids); time-independent Ginzburg-Landau equation (equilibrium states of superconductors); time-independent Landau-Lifshitz equation (equilibrium states of hard magnets)

1.2 *Classification of partial differential equations and boundary conditions*: how to think about boundary conditions; cases of interest to us; general and special PDEs; the Cauchy-Kovalevska theorem; classification of quasi-linear second order PDEs; characteristic curves; qualitative features of characteristic curves

2. Separation of variables

2.1 *Introduction*

2.2 *Separating the time dependence*

2.3 *Separable coordinates*: rectangular, circular cylindrical and spherical polar

2.4 *Series solutions of ordinary differential equations*

2.5 *Sturm-Liouville eigenproblems*: matrix eigenproblems; linear operators, adjoint operators; boundary conditions, adjoint and self-adjoint; Sturm-Liouville form of the general eigenproblem; examples of Sturm-Liouville forms of eigenproblems; properties of Sturm-Liouville operators; Fourier series; Fourier transforms and Fourier integrals

3. Spherical harmonics and their applications

3.1 *Construction of spherical harmonics*: series solution of the associated Legendre equation; Legendre polynomials and their properties; associated Legendre functions and their properties

3.2 *Spherical harmonic functions and their properties*: addition theorem for spherical harmonics; multipole expansions

3.3 *Laplace's equation in spherical polar coordinates*: uniqueness of solutions of the Laplace equation; interior Laplace problem; exterior Laplace problem; region between concentric spheres; neutral conducting sphere in a uniform external field

4. Bessel functions and their applications

4.1 *Series solution of Bessel's equation*

4.2 *Neumann functions*

4.3 *Some properties of solutions of Bessel's equation*

4.4 *Bessel functions of imaginary argument*

4.5 *Laplace's equation in cylindrical polar coordinates*: application to fluid mechanics

4.6 *Bessel series as analogues of Fourier series*

4.7 *Solution of Laplace's equation inside an infinitely long cylinder*

4.8 *Bessel transforms as analogues of Fourier transforms*

5. Normal mode eigenproblems

5.1 *Separating the time-dependence*

5.2 *Diffusion equation in a closed region of space*

5.3 *Normal mode treatment of the drumhead*

5.4 *Normal mode treatment of the time dependent Schrödinger equation*

5.5 *Acoustic wave guides*

6. Inhomogeneous ordinary differential equations

6.1 *Introduction*

6.2 *Inhomogeneous ordinary differential equations*: variation of parameters; Green functions (GFs) for inhomogeneous ordinary differential equations (ODEs); the issue of boundary conditions for GFs for ODEs

6.3 *Equivalence of inverse matrices and GFs*: eigenvector expansion for matrix inverses; reciprocity and its physical origin

6.4 *Example – GF for inhomogeneous ODE*: the bowed stretched string; inhomogeneity

6.5 *Eigenfunction expansion for GF*

7. Inhomogeneous partial differential equations and Green functions

7.1 *Poisson's equation*: electrostatics in the presence of charge; solution using Green's theorem; the basic GF – Coulomb's law; Poisson's equation when there is no boundary; Green function for Poisson's equation inside a sphere; expansion of Green function in spherical polar coordinates; example – electrostatic potential inside a grounded conducting sphere with charges; how to compute a Green function when no images trick is apparent; Green function for Poisson's equation for the interior of an infinite cylinder

7.2 *Green functions and conversion of differential equations into integral equations*

7.3 *Green functions in the time-dependent domain*: wave equation; boundary and initial conditions for the wave equation; how we would use the Green function if we knew it; source-varying Green function; Green function for wave equation in an infinite spatial region; use of the causal Green function; Liénard-Wiechert potential; computation and use of a source-varying Green function – example

8. Integral equations

8.1 *Introduction*: why we study integral equations; classification of linear integral equations

8.2 *Integral transforms*: review of some familiar integral transforms

8.3 *Integral equations with degenerate kernels*

8.4 *The Fredholm alternative*

8.5 *Neumann series solution to integral equations*

8.6 *Fredholm's formula*: some properties of gaussian integrals; Wick's theorem from gaussian integrals; complex gaussian integrals; Grassmann gaussian integrals

8.7 *Fredholm's method from Grassmann functional integrals*: example of Fredholm's method; Fredholm eigenproblem

9. Boundary integral methods

9.1 *Single and double layer potentials*

9.2 *Jump conditions and boundary integral equations*

9.3 *Applications to spectral geometry: "Can one hear the shape of a drum?"*