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[redacted]

FIELD THEORY

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(lecture notes prepared by Ejnar Gyldenkerne)

NORDITA LECTURE NOTES
January 1983

PREFACE

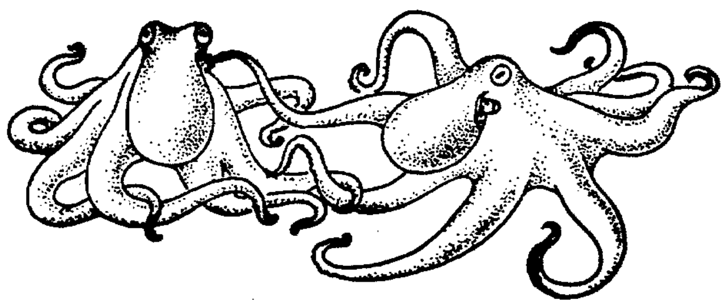
In the fall of 1979, Benny Lautrup and I set out to write the ultimate Quantum Chromodynamics review. The report was going to consist of four parts, one for each line of

From Ghoulies and Ghosties
and Long-leggety Beasties
and Things that go bump in the Night
Good Lord, deliver us!

Ghoulies are body-snatchers and grave robbers; they are those revel in that which is revolting.

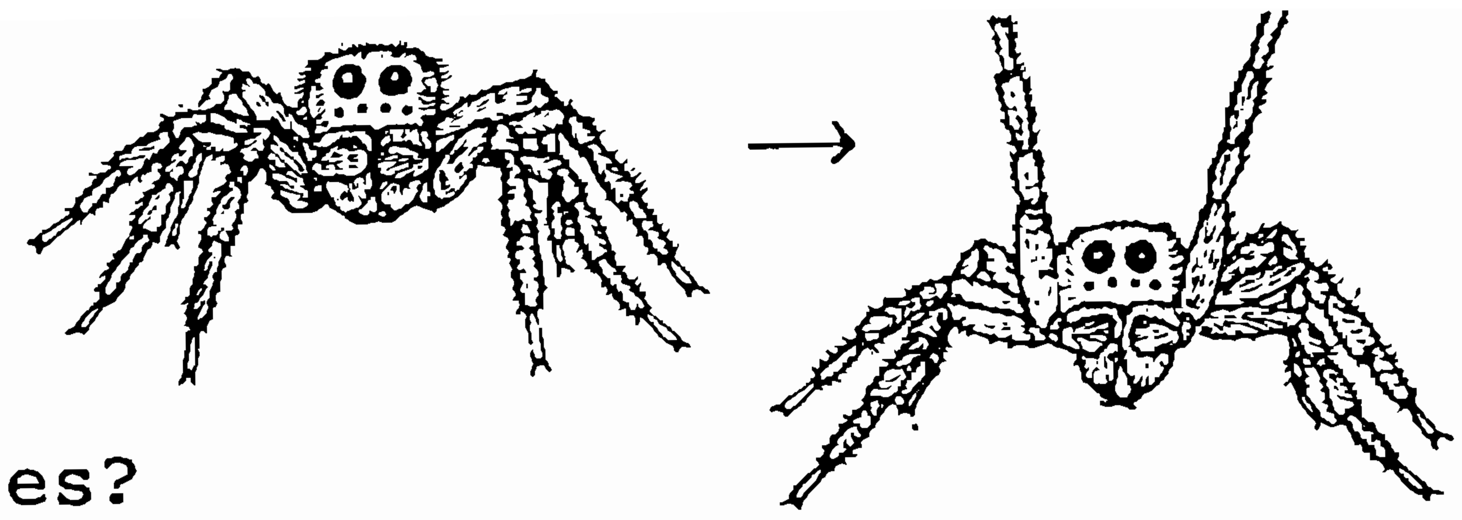
Benny had previously described ghosties in a very nice set of QCD lectures† which we were going to use as the first part. Green functions resemble long-leggety beasties; they, and the general formalism of field theory, were to be developed in the second part. The things that go bump in the night are clearly the many unpleasant surprises of field theory; divergences, together with the regularization and computation techniques, were to be covered in the third part. Finally, good Lord deliver us, we were going to actually calculate a few basic QCD integrals.

Well, while I was lecturing about the long-leggety beasties, Benny deserted me for lattice, and the ultimate QCD review was never written. That these lectures appear at all is largely due to tireless work by Ejnar Gyldenkerne and to the criticisms of the QCD study group at the Niels Bohr Institute. In writing these lectures I have profited much from discussions with Benny Lautrup, to whom I direct my thanks.



† B. Lautrup, "Of ghoulies and ghosties - an introduction to QCD", Basko Polje 1976 lectures, available as Niels Bohr Institute preprint NBI-HE-76-14.

1. INTRODUCTION



What are long-leggety beasties?

Long-leggety beasties are to be seen in any field theory or statistical mechanics textbook; they are Feynman diagrams, Green functions, S-matrix elements, correlation functions, and so on. They represent sums of probabilities (statistical mechanics) or probability amplitudes (quantum mechanics).

There are two ways of visualizing long-leggety beasties†.

In the first picture the transition probabilty (amplitude) is the sum of all ways in which particles can propagate, disintegrate and recombine before reaching a detector. Each possibility is represented by a Feynman diagram, and the penalty associated with each choice is given by a Feynman integral.

In the second picture the transition probability (amplitude) is a sum over all "paths" which the system can take between the initial and the final state. The penalty to be paid for a particular path is assessed by a Boltzmann factor (phase factor). A process is dominated by the classical paths, and the fluctuation (quantum) effects arise from the heavily penalized deviations away from the beaten path.

The two pictures are equivalent. The second (path integrals) is a "Fourier" transform of the first (generating functionals). In some contexts, such as in perturbative calculations, generating functionals are the practical choice. In others, such as in identifying the dominant classical configurations, or in exploiting symmetries of a theory, the path integral formulation might be more suggestive.

In these notes we put the usual logic of field theory textbooks on its head; we start with the Feynman rules and end with Lagrangians. We find it easier to understand field theory this way: for many particle physicists, diagrams are an important tool for developing field-theoretic intuition.

Our attitude will be eclectic. We shall start by building up generating functionals using vertices and propagators as

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† R. Herrick has in his poem "On Julia's Legs" suggested a third way: "Fain would I kiss my Julia's dainty leg, which is as white and hairless as an egg".

simple building blocks. Then we shall rewrite the results in terms of path integrals, and from then on use either formalism, whichever may be more expedient. Each particular physical theory brings in its own set of ailments (ultraviolet divergences, ill-defined path integrals, etc.), but the general formalism should be good enough to describe anything under the sun, from statistical mechanics to lattice gauge theories to continuum theories to gravity and cosmology. The general formalism is straightforward and intuitive. The real work starts only with specialization to a particular theory; the dominant classical configurations have to be identified, divergent sums (integrals) regularized, etc.

We will apply the general formalism to QCD. Chapter 6 is a rehash of Benny Lautrup's "Ghoulies and Ghosties". This construction yields QCD Feynman rules and bare Ward identities. In chapter 7 we feed these into the general formalism to obtain the Ward identities for full Green functions. At this point our patience runs out, and the proof of renormalizability of QCD and the evaluation of the running coupling constants, scaling violations and hadron masses are left as exercises for the reader.

I have included much graphic gore in these notes. The reason is that I fear that the perturbation theory is here to stay; it will not go away even if the gauge theories do. At least, if I ever have to do a perturbative calculation again, I will know where to look up the diagrams. The reader is advised to skip over lengthy perturbative expansions - most particle physicists reach tenure without doing anything more strenuous than one-loop Feynman integrals. The exercises are another matter - we have relegated much of the conceptually dull but technically important material to the exercises. They are of three kinds: trivial, undoable, and wrong.

There is nothing in these lectures that is not well-known and has not been published many other places. The only excuse for writing them up is that they seem to resemble no other field theory text on the market. It cannot be precluded that that might be considered a virtue.

A. Land of Quefithe

Once (and it was not yesterday) there lived a very young mole and a very young crow who, having heard of the fabulous land called Quefithe, decided to visit it. Before starting out, they went to the wise owl and asked what Quefithe was like.

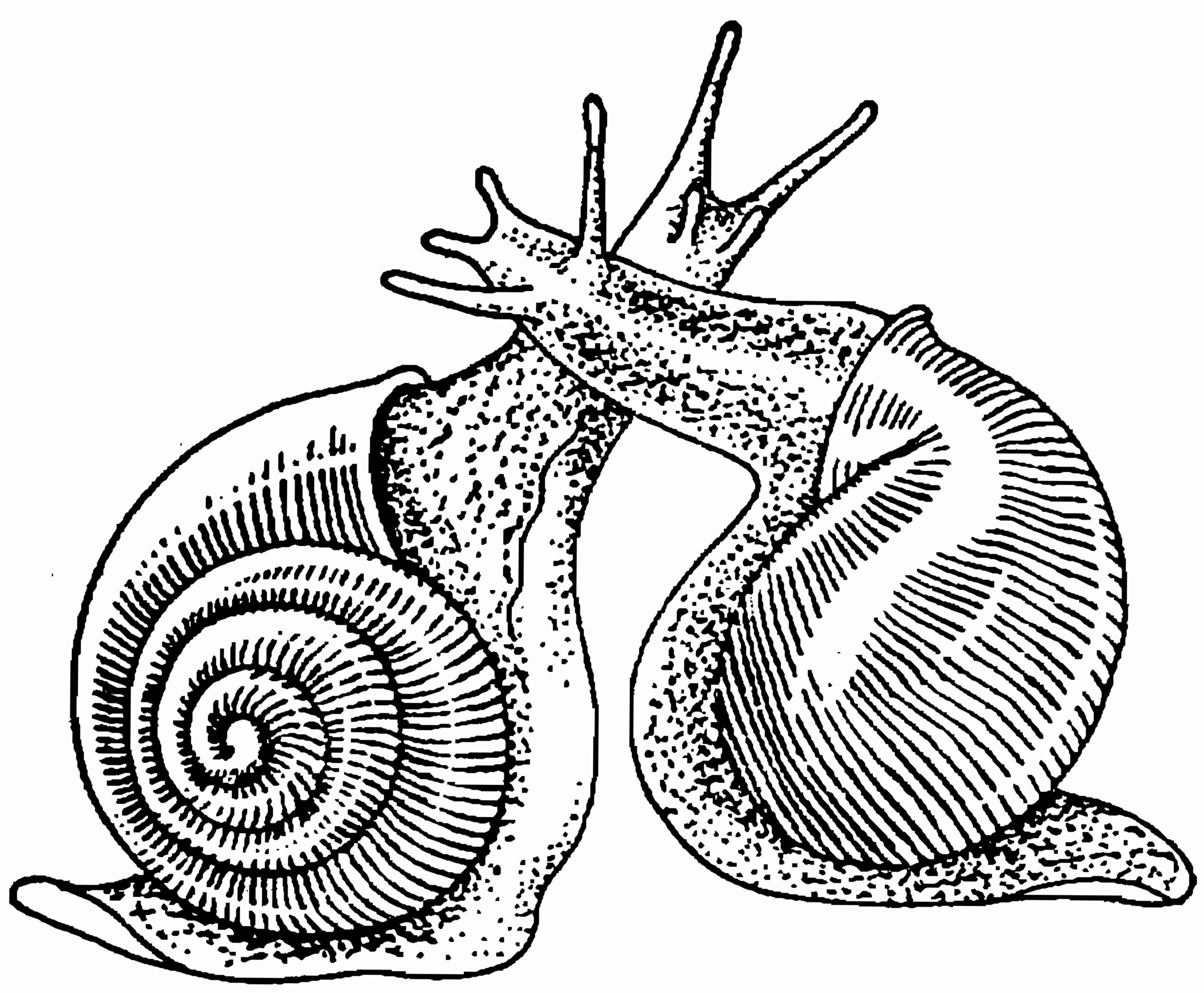
Owl's description of Quefithe was quite confusing. He said that in Quefithe everything was both up and down. If you knew where you were, there was no way of knowing where you were going, and conversely, if you knew where you were going, there was no way of knowing where you were. The young mole and the young crow did not understand much, so they went instead to the old eagle and asked him what Quefithe was like. The eagle shook his white-feathered head, sized them up with his fierce eyes, and said: "Action gives automatically invariant description of Quefithe. You must study the unitary representations of the Lorentz group". The mole and the crow waited for more, but the eagle remained silent, his gaze fixed on an unfathomable string in the sky.

Clearly, if they were ever going to learn anything about Quefithe, they had to see it for themselves. And that is what they did.

After a few years had passed, the mole came back. He said that Quefithe consisted of lots of tunnels. One entered a hole and wandered through a maze, tunnels splitting and rejoining, until one found the next hole and got out. Quefithe sounded like a place only a mole would like, and nobody wanted to hear more about it.

Not much later the crow landed, flapping its wings and crowing excitedly. Quefithe was amazing, it said. The most beautiful landscape with high mountains, perilous passes and deep valleys. The valley floors were teeming with little moles who were scurrying down rutted paths. The crow sounded like he had taken too many bubble baths, and many who heard him shook their heads. The frogs kept on croaking "it is not rigorous, it is not rigorous!" The eagle said: "It is frightful nonsense. One must study the unitary representations of the Lorentz group". But there was something about crow's enthusiasm that was infectious.

The most puzzling thing about it all was that the mole's description of Quefithe sounded nothing like the crow's description. Some even doubted that the mole and the crow had ever gotten to the mythical land. Only the fox, who was by nature very curious, kept running back and forth between the mole and the crow and asking questions, until he was sure that he understood them both. Nowadays, anybody can get to Quefithe - even snails.



two hermaphroditic snails.

2. GENERATING FUNCTIONALS

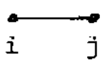
A. Propagators and vertices

A particle (an elementary excitation of a theory) is specified by a list of attributes; its name, its state (spin up, ingoing, ...), its spacetime location, etc. To develop the formalism of field theory, one does not need any specific part of this information, so we hide it in a single collective index:

$$i = \{q, a, \alpha, \mu, x_\mu, \; .... \}$$

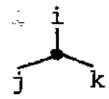
q : particle type
a : colour
α : spin
μ : Minkowski indices
x_μ : spacetime coordinates (2.1)

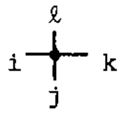
A particle is an interesting particle only if it can do something. The simplest thing it can do is to change its position, its spin or some other attribute. The probability (amplitude) that this happens is described by the (bare) <u>propagators</u>:

$\Delta_{ij} =$ 

. (2.2)

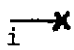
Beyond this, many things can happen; a particle can split into two, or three, or many other particles. The probability (amplitude) that this happens is described by (bare) <u>vertices</u>:

$\gamma_{ijk} =$ 

$\gamma_{ijk\ell} =$ 

$\gamma_{ijk\ell m} =$ $\vdots$ (2.3)

A particle can also be created (or removed from the system). This is described by a <u>source</u> (or a <u>sink</u>):

$J_i =$ 

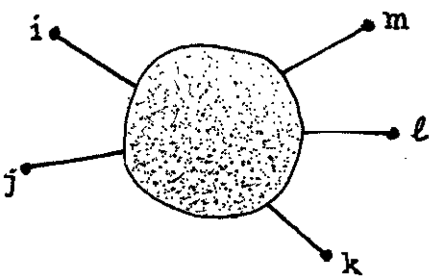
. (2.4)

The concept of a particle makes sense only if its persistence probability (2.2) is appreciable, i.e. if (2.3), the probability of its disintegration, is relatively small. In that case the interactions (2.3) may be treated as small corrections, and the perturbation theory applies. If the "particle" described by attributes (2.1) has a negligible persistence probability, the theory should be reformulated in terms of another set of "elementary excitations" which are a better approximation to the physical spectrum of the theory (an easy thing to say).

How many identical particles (particles with all the same labels) can coexist? We shall consider two extremes: infinity (bosons) or at most one (fermions). Other more perverse possibilities cannot be excluded. Assumption of additivity of probabilities/amplitudes then implies that the bosonic propagators and vertices must be symmetric under interchange of indices $\Delta_{ij} = \Delta_{ji}$, $\gamma_{ijk} = \gamma_{jik} = \gamma_{ikj} = \ldots$. (The argument is similar to the one we shall use for fermions in chapter 4). For the time being, we assume that the vertices (2.3) are symmetric.

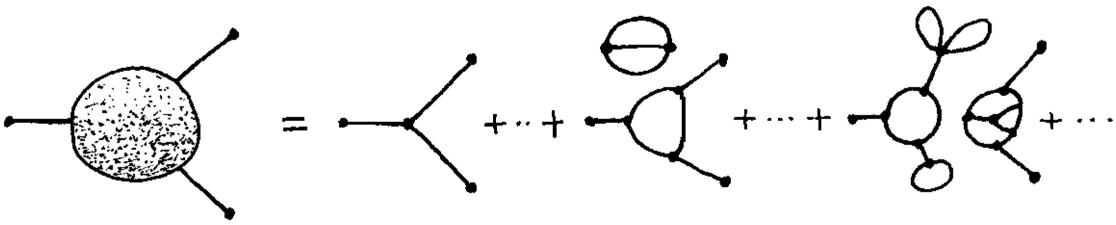
B. Green functions

A typical experiment consists of a setup of the initial particle configuration, followed by a measurement of the final configuration. The theoretical prediction is expressed in terms of the Green functions. For example, if we are considering an experiment in which particles i and j interact, and the outcome is particles k, ℓ, and m, we draw the corresponding Green functions

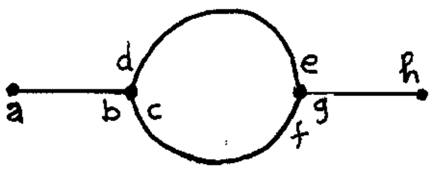
$G_{ijk\ell m} =$  (2.5)

(remember that labels i, j, ... stand for all variables and indices which specify a particle.)

A Green function is a sum of the probabilities (amplitudes) associated with all possible ways in which the final state can be reached. This is represented by an infinite sum of Feynman diagrams:



Each Feynman diagram corresponds to a sum (or an integral). For example, diagram



represents the probability that 1) a particle whose type, location, etc. is described by the collective index a reached any state labeled b; 2) that b splits into any two particles labeled c and d, and so forth. The intermediate states are summed over the entire range of possible index values

$$= \sum_{b,c,d,e,f,g} \Delta_{ab} \gamma_{bcd} \Delta_{cf} \Delta_{de} \gamma_{efg} \Delta_{gh} \; .$$

Here the summation signs imply sums over discrete indices (such as spin) and integrals over continuous indices (such as position). In the future we shall drop the explicit summation signs, and use instead Einstein's repeated index convention; if an index appears twice in a term, it is summed (integrated) over.

Exercise 2.B.1 Continuous indices. For QCD the collective index i stands for:

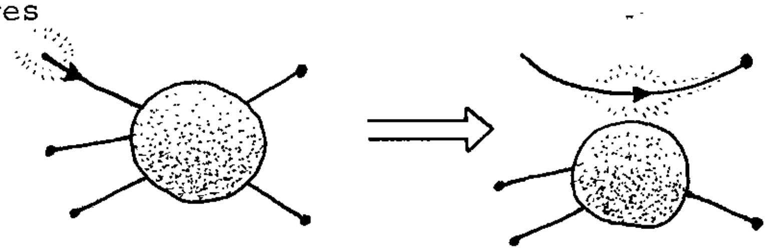
x^μ	spacetime coordinates,
$\mu = 1,2,\ldots,d$	Minkowski indices,
$j = 1,2,\ldots,N$	gluon colours.

If the propagator is denoted by $D^{ij}_{\mu\nu}(x,y)$ and the three-gluon vertex by $\gamma^{ijk}_{\mu\nu\sigma}(x,y,z)$, write down the complete expression for the above self-energy diagram.

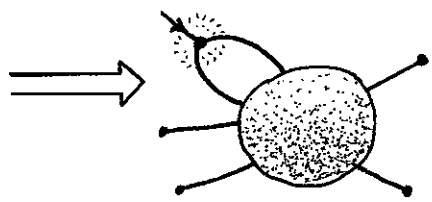
C. Dyson-Schwinger equations

A Green function consists of an infinity of Feynman diagrams. For a theory to be manageable, it is essential that these diagrams can be generated systematically, in order of their relative importance.

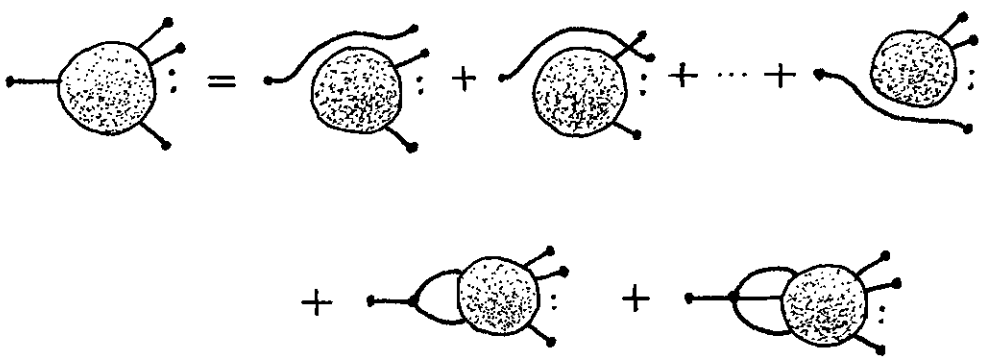
Consider (for simplicity) a theory with only cubic and quartic vertices†. Take a Green function and follow a particle into the blob. Two things can happen; either the particle survives



or it interacts at least once:



More precisely, entering the diagram via leg 1, we either reach leg 2, or leg 3, ... , or hit a three-vertex, or a four-vertex, etc. Adding up all the possibilities, we end up with the Dyson-Schwinger equations:



(2.6)

†Remember that the different particle types are covered by a single collective index, so QCD is also this type.

Iteration of the Dyson-Schwinger (DS) equations yields all Feynman diagrams contributing to a given process, ordered by the number of vertices (the order in perturbation theory).

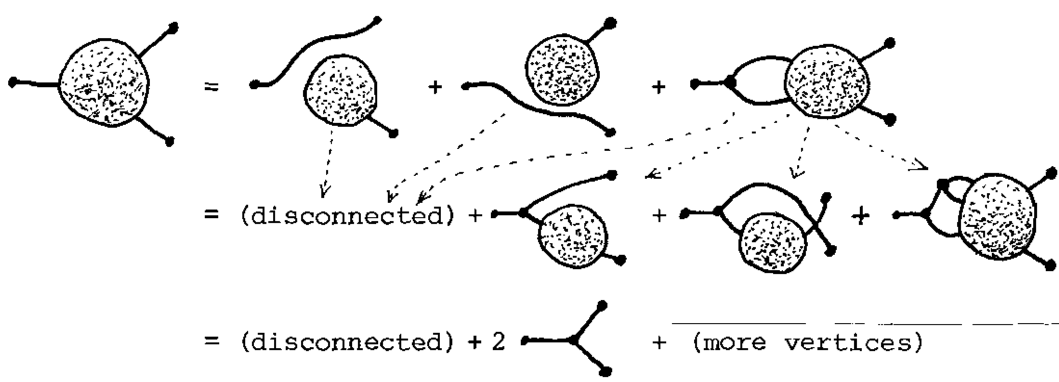
A few words about the diagrammatic notation; a diagrammatic equation like (2.6) contains precisely the same information as its algebraic transcription

$$G_{ij..k\ell} = \Delta_{i\ell}G_{j..k} + \Delta_{ik}G_{j..\ell} + \dots + \Delta_{ij}G_{..k\ell}$$
$$+ \Delta_{ir}\gamma_{rst}G_{tsj..k\ell} + \Delta_{ir}\gamma_{rstu}G_{utsj..k\ell} \,.$$

Indices can always be omitted. An internal line implies a summation/integration over the corresponding indices, and for external lines the equivalent points on each diagram represent the same index in all terms of a diagrammatic equation. The advantages of the diagrammatic notation are obvious to all those who prefer the comic strip editions of "The greatest story ever told" to the unwieldy, fully indexed version†. Two of the principal benefits are that it eliminates "dummy indices" and that it does not force Feynman integrals into one-dimensional format (both being means whereby identical integrals can be made to look totally different).

D. Combinatoric factors

For a three-leg Green function the DS equations yield



It is rather unnatural that an expansion of a three-leg Green function does not start with the bare three-vertex, but twice

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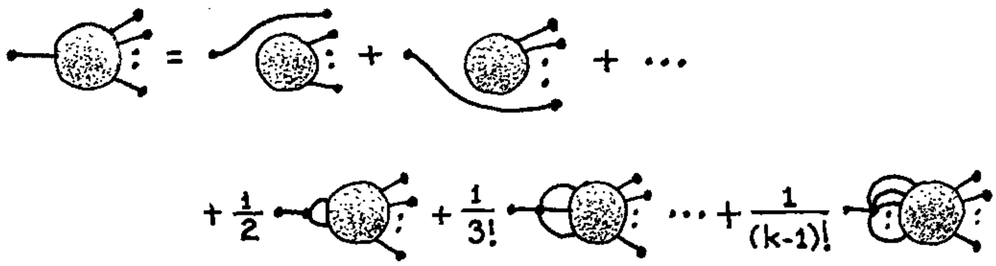
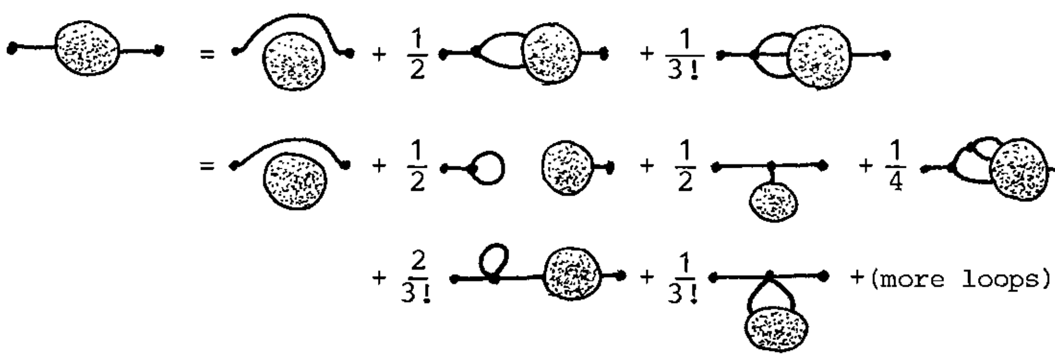
†C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, N.Y., 1980).

the bare three-vertex. This is easily fixed-up by including compensating combinatorial factors into DS equations:

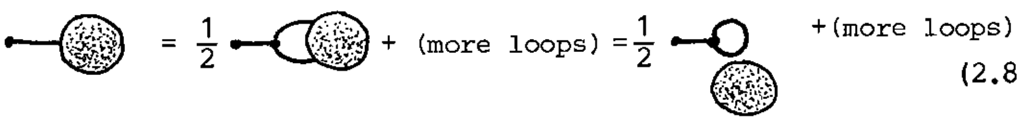


(2.7)

To illustrate how the DS equations generate the perturbation expansion, we expand a two-leg Green function up to one loop:

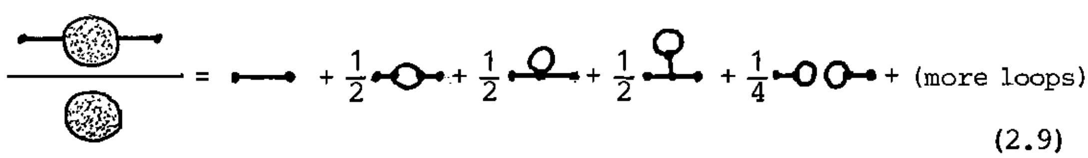


The one-loop tadpole is given by



(2.8)

Substituting the tadpole into the above, we finally obtain the self-energy expansion up to two vertices with all the correct combinatoric factors:

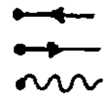


(2.9)

This expansion looks like the usual $\phi^3 + \phi^4$ theory, but it is not only that: the combinatoric factors are correct for any theory with cubic and quartic vertices, such as QCD with its full particle content.

Exercise 2.D.1 Feynman diagrams in the collective index notation look like diagrams for scalar field theories. Nevertheless, they do contain the perturbative expansion for theories with arbitrary particle content. As an example, consider a QED-type theory with an "in" particle (electron), and "out" particle (positron) and a scalar particle (photon). The collective index (2.1) now ranges over an array of three sub-collective indices

$$i = \begin{bmatrix} a,\ \text{in} \\ a,\ \text{out} \\ \mu \end{bmatrix} =$$



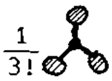
Index a stands for the charged particle's position and spin, and index μ stands for all labels characterizing the neutral particle. The "in" - "out" labels can be eliminated by taking a to be an upper index for "in" particles, and a lower index for "out" particles. Diagrammatically they are distinguished by drawing arrows pointing away from upper indices and down into lower indices:

$$\Delta^a_b = $$

$$\gamma_{\mu a}^{\ \ b} =$$

Show that if the sources and fields are replaced by $J = (\eta^a, \eta_b, J_\mu)$, $\phi = (\psi_a, \psi^b, A^\mu)$, the combinatoric factors in (2.9) cancel, and the vertices such as the electron-positron-photon vertex have no combinatoric weight:

$\frac{1}{2}$  $= \frac{1}{2} J_i \Delta_{ij} J_j =$  $+ \frac{1}{2}$ 

$\frac{1}{3!}$  $= \frac{1}{3!} \gamma_{ijk} \phi_i \phi_j \phi_k = \gamma_{\mu b}^{\ \ a} \psi^b A_\mu \psi_a$.

Exercise 2.D.2 Write the Dyson-Schwinger equations for QED-like theories. (We say "QED-like" because electrons are fermions. We shall return to the fermion DS equations later.)

Exercise 2.D.3 Determine the one-loop self-energy diagrams (2.9) for QED-like theories.

E. Generating functionals

The structure of the DS equations is very general; still, at present we have to write them separately for two-leg Green function, three-leg Green function, and so on. To state relations between Green functions in a more compact way we introduce generating functionals. A generating functional is the vacuum (legless) Green function for a theory with sources (2.4):

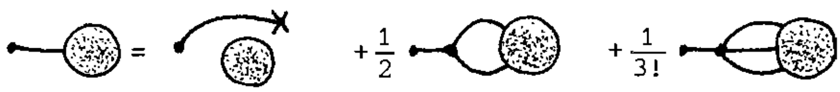
$$Z[J] = \sum_{m=0}^{\infty} \frac{1}{m!} G_{i_1 i_2 \ldots i_m} J_{i_1} J_{i_2} \ldots J_{i_m}$$

 (2.10)

(as J_i is a function which depends on both discrete and continuous indices, $Z[J]$ is a functional). The coefficients in this expansion are the usual Green functions. They can be retrieved from the generating functional by differentiation:

$$G_{ijk} = \frac{d}{dJ_i}\frac{d}{dJ_j}\frac{d}{dJ_k} Z[J]\Big|_{J=0} \text{, etc.} \tag{2.11}$$

The DS equations (2.7) can be written as



$$\frac{d}{dJ_i} Z[J] = \Delta_{ij}\left\{J_j + \frac{1}{2}\gamma_{jk\ell}\frac{d}{dJ_\ell}\frac{d}{dJ_k} + \frac{1}{3!}\gamma_{jk\ell m}\frac{d}{dJ_m}\frac{d}{dJ_\ell}\frac{d}{dJ_k}\right\}Z[J] \,. \tag{2.12}$$

The bare propagators and vertices can themselves be collected in a functional called the <u>action</u>:

$$S[\phi] = -\frac{1}{2}\phi_i\Delta^{-1}_{ij}\phi_j + S_I[\phi] \,, \tag{2.13}$$

$$S_I[\phi] = \sum_m \underbrace{\gamma_{ijk..\ell}}_{m\text{ legs}} \frac{\phi_i\phi_j\ldots\phi_\ell}{m!} \,. \tag{2.14}$$

Now the Dyson-Schwinger equations can be stated in an even more elegant way:

$$0 = \left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] + J_i\right)Z[J] \,, \tag{2.15}$$

where

$$\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] \equiv \frac{dS[\phi]}{d\phi_i}\Big|_{\phi=\frac{d}{dJ}} \,.$$

The action (or the Lagrangian) is just another way of defining the propagators and vertices for a given theory. Giving the Lagrangian or listing the Feynman rules is one and the same thing.

Exercise 2.E.1 Functional derivatives. For continuous indices the Kronecker deltas are replaced by Dirac deltas. For example, check that in d-dimensions

$$\frac{dJ(x)}{dJ(y)} = \delta^d(x-y) \quad ,$$

is the correct definition of the derivative in (2.11).

Exercise 2.E.2 Feynman rules. Consider ϕ^3 theory given by the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}\,m^2\phi(x)^2 - \frac{\mu}{3!}\,\phi(x)^3$$

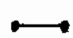
$$S = \int d^dx\mathcal{L}(x) .$$

Read off the bare propagators and vertices (the Feynman rules) from the Lagrangian.

Hint: $$\gamma_{ij..k} = \frac{d}{d\phi_i}\frac{d}{d\phi_j}\cdots\frac{d}{d\phi_k}\,S[\phi]\Big|_{\phi=0} \quad ,$$

and the derivatives are in this case functional derivatives.

Exercise 2.E.3. Zero-dimensional field theory. Consider a ϕ^3 theory defined by trivial Feynman rules

 $= 1$,  $= g$.

The value of a graph with k vertices is g^k, and k-th order contribution to Green function is basically the number of contributing diagrams. More precisely, if

$$Z[J] = \sum_{k,m} G_k^{(m)} g^k \frac{J^m}{m!}$$

the Green function

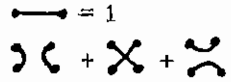
$$G_k^{(m)} = \sum_G C_G$$

is the sum of combinatoric factors of all diagrams with m legs and k vertices. Use the Dyson-Schwinger equation (2.7) to show that for a free field theory

$$G_0^{(m)} = (m-1)!! \qquad m \text{ even}$$
$$= 0 \qquad m \text{ odd} \quad .$$

Diagrammatically

$G^{(2)} =$ $= 1$

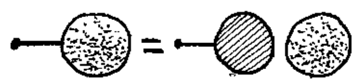
$G^{(4)} =$  $= 3$, etc.

The zero-dimensional field theory is about the only field theory which is easily computable to all orders. We shall use it often to illustrate in a concrete way various field-theoretic relations.

F. Connected Green functions

Generating functionals are a powerful tool for stating relations between Green functions. For example, we can use them to derive relations between the full and the connected Green functions:

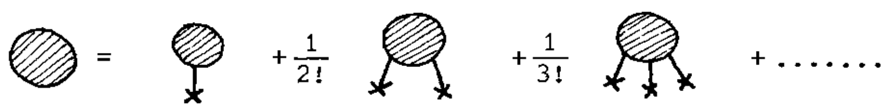
Pick out a leg and follow it into a full Green function. This separates all associated Feynman diagrams into two parts - the part that is connected to the initial leg, and the remainder:



$$\frac{d}{dJ_i} Z[J] = \frac{dW[J]}{dJ_i} Z[J] \qquad (2.16)$$

The generating functional for the connected Green functions is defined in the same way as (2.10), the generating functional for the full Green functions:

$$W[J] = \sum_{m=1}^{\infty} \frac{1}{m!} G^{(c)}_{i_1 i_2 \cdots i_m} J_{i_1} J_{i_2} \cdots J_{i_m} \quad ,$$

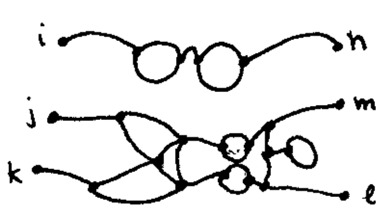


(2.17)

The differential equation (2.16) is easily solved

$$Z[J] = e^{W[J]} \qquad (2.18)$$

A disconnected Feynman diagram such as



represents a product of two independent processes; one could take place on the moon, and the other in Aarhus. The physically interesting processes are described by the connected Green functions. To obtain a systematic perturbation series which includes only the connected Feynman diagrams, we use the identity[†]

$$\frac{1}{Z[J]}\frac{d}{dJ_i}Z[J] = \frac{dW[J]}{dJ_i} + \frac{d}{dJ_i} \tag{2.19}$$

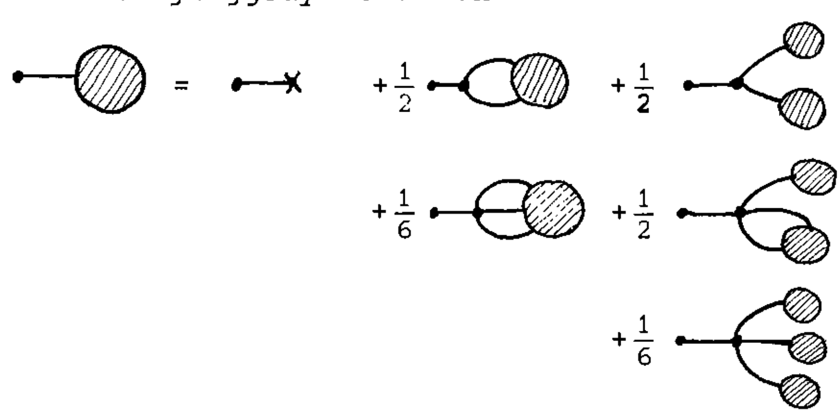
to rewrite the DS equations (2.15) in terms of the connected Green functions:

$$0 = \frac{dS}{d\phi_i}\left[\frac{dW[J]}{dJ} + \frac{d}{dJ}\right] + J_i \ . \tag{2.20}$$

This is very elegant, but possibly not too transparent. To get a feeling for these equations, take the $\phi^3 + \phi^4$ DS equations (2.12) and substitute $Z[J] = \exp(W[J])$. The result is, in the functional notation

$$\frac{dW[J]}{dJ_i} = \Delta_{ij}\left\{ J_j + \frac{1}{2}\gamma_{jk\ell}\left(\frac{d^2W[J]}{dJ_\ell dJ_k} + \frac{dW[J]}{dJ_k}\frac{dW[J]}{dJ_\ell}\right) + \frac{1}{6}\gamma_{jk\ell m}\left(\frac{d^3W[J]}{dJ_m dJ_\ell dJ_k} + 3\frac{dW[J]}{dJ_k}\frac{d^2W[J]}{dJ_m dJ_\ell} + \frac{dW[J]}{dJ_k}\frac{dW[J]}{dJ_\ell}\frac{dW[J]}{dJ_m}\right)\right\}, \tag{2.21}$$

and in the longleggedy notation



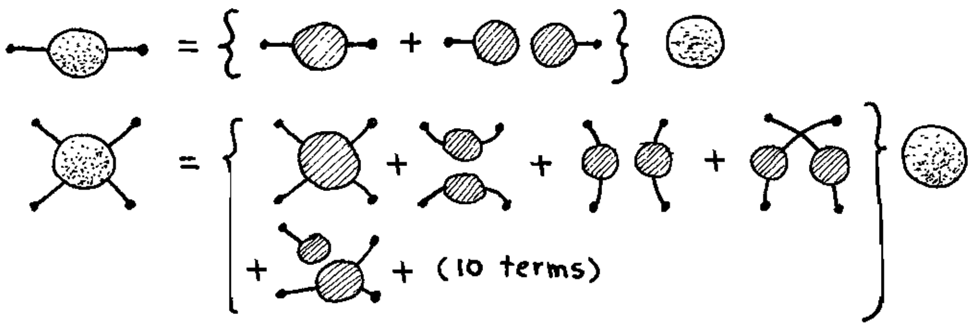
After reaching a vertex, one continues into diagrams that are either mutually disconnected, or connected - that is the reason that there are extra terms in the connected DS equations, compared with the full Green functions equations (2.12).

[†] more explicitly

$$\frac{1}{Z[J]}\frac{d}{dJ}(Z[J]f[J]) = \left(\frac{dW[J]}{dJ} + \frac{d}{dJ}\right)f[J].$$

Exercise 2.F.1 Use DS equations (2.21) to compute self-energy to one loop. How does the result differ from (2.9)?

Exercise 2.F.2 Expand some full Green functions in terms of the connected ones:

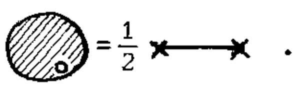


Hint: iterating (2.19) is probably the fastest way.

G. Free field theory

The connected generating functional for a free field theory is trivial: there are no interactions, so the only connected Feynman diagram is the propagator:

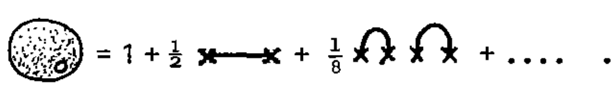
$$W_0[J] = \frac{1}{2} J_i \Delta_{ij} J_j$$



(2.22)

For the free field theory (2.18) gives an explicit expression for the generating functional:

$$Z_0[J] = e^{\frac{1}{2} J_i \Delta_{ij} J_j}$$

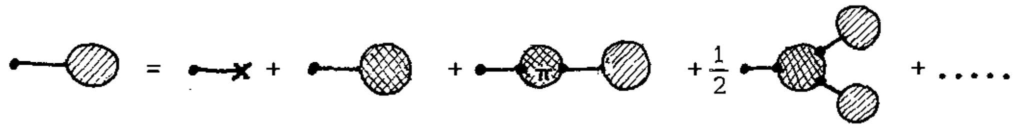


(2.23)

H. One-particle irreducible Green functions

A one-particle irreducible (1PI) diagram cannot be cut into two disconnected parts by cutting a single internal line. An arbitrary connected diagram has in general a number of such lines. The connected and the 1PI Green functions can be related by our usual diagrammatic trick:

Pick out a leg of a connected diagram. This pulls out a 1PI piece, which ends in 0, 1, 2, ... lines whose cutting would disconnect the diagram. Those lines continue into further connected pieces:



$$\phi_i = \Delta_{ij}(J_j + \Gamma_j + \pi_{jk}\phi_k + \tfrac{1}{2}\,\Gamma_{jk\ell}\phi_k\phi_\ell + \ldots.) \quad (2.24)$$

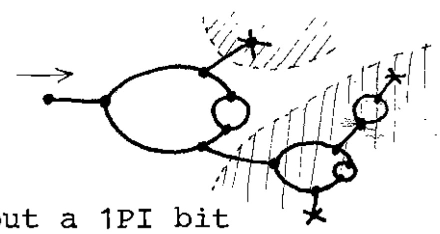
Here the "field" ϕ is defined by

$$= \phi_i = \frac{dW[J]}{dJ_i} \quad (2.25)$$

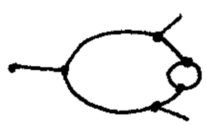
We draw the 1PI Green functions as cross-hatched blobs

$\Gamma_{ij..k} =$ 

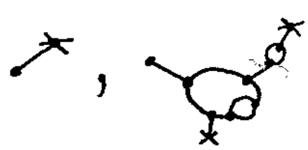
Unlike the full and the connected Green functions, the 1PI ones do not have propagators on external legs - the external indices always belong to a vertex of an 1PI diagram. This is indicated by drawing dots on the edges of 1PI Green functions. Any connected diagram belongs to one and only one term in the expansion (2.24). For example, going into connected diagram



we pull out a 1PI bit



followed by connected bits



Multiplying both sides of (2.24) by the inverse of the bare propagator we obtain

$$0 = J_i + \Gamma_i + (-\Delta^{-1} + \pi)_{ij}\phi_j + \tfrac{1}{2}\,\Gamma_{ijk}\phi_k\phi_j + \ldots. \ .$$

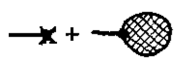
(For reasons which should soon be clear, it is convenient to define the two-leg Γ as $\Gamma_{ij} = -\Delta^{-1}_{ij} + \pi_{ij}$, where π_{ij} is the 1PI two-leg Green function, or the proper self-energy.)

Collecting all 1PI Green functions into the <u>effective action</u> functional

$$\Gamma[\phi] = \sum_{m=1} \Gamma_{ij..k} \frac{\phi_k \cdot\cdot \phi_j \phi_i}{m!} \quad , \tag{2.26}$$

we can write (2.24), the relation between the connected and the 1PI Green functions, as:

$$0 = J_i + \frac{d\Gamma[\phi]}{d\phi_i} \; ,$$

0 =  . (2.27)

This, together with (2.25), can be summarized by a <u>Legendre transformation</u>

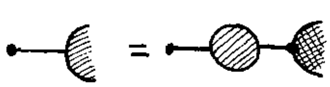
$$W[J] = \Gamma[\phi] + \phi_i J_i \quad . \tag{2.28}$$

(2.27) guarantees that W is independent of ϕ, and (2.25) guarantees that Γ is independent of J:

$$\frac{dW[J]}{d\phi} = 0, \qquad \frac{d\Gamma[\phi]}{dJ} = 0 \; .$$

This is elegant, but how does it help us to get 1PI Green functions? The point is that we are not interested in extracting 1PI Green functions from the connected ones; what we need are the 1PI Dyson-Schwinger equations, i.e. the systematics of generating 1PI diagrams (and only 1PI diagrams). To achieve this, we must first eliminate J-derivatives in favour of ϕ-derivatives (cf. (2.25)):

$$\frac{d}{dJ_i} = \frac{d\phi_j}{dJ_i}\frac{d}{d\phi_j} = \frac{d^2W[J]}{dJ_i dJ_j}\frac{d}{d\phi_j}$$

 (2.29)

This accounts for all self-energy insertions. The right-hand side can be expressed in terms of 1PI Green functions by taking a derivative of (2.27):

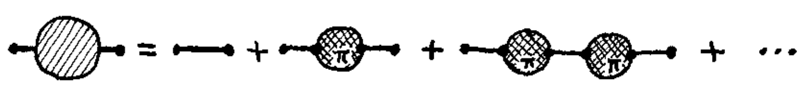
$$0 = \delta_{ij} + \frac{d}{dJ_j}\frac{d\Gamma[\phi]}{d\phi_i} = \delta_{ij} + \frac{d^2W[J]}{dJ_j dJ_k}\frac{d^2\Gamma[\phi]}{d\phi_k d\phi_i} \ . \tag{2.30}$$

In order to understand this relation diagrammatically, we separate out the bare propagator in (2.26) by defining the "interaction" part of Γ:

$$\Gamma[\phi] = -\tfrac{1}{2}\phi_i \Delta^{-1}_{ij}\phi_j + \Gamma_I[\phi] \ . \tag{2.31}$$

Now (2.30) can be written as

$$\frac{d^2W[J]}{dJ_i dJ_j} = \Delta_{ij} + \Delta_{ik}\frac{d^2\Gamma_I[\phi]}{d\phi_k d\phi_\ell}\Delta_{\ell j} + \ldots .$$



$$W[J]'' = \frac{1}{\Delta^{-1} - \Gamma_I[\phi]''} \ . \tag{2.32}$$

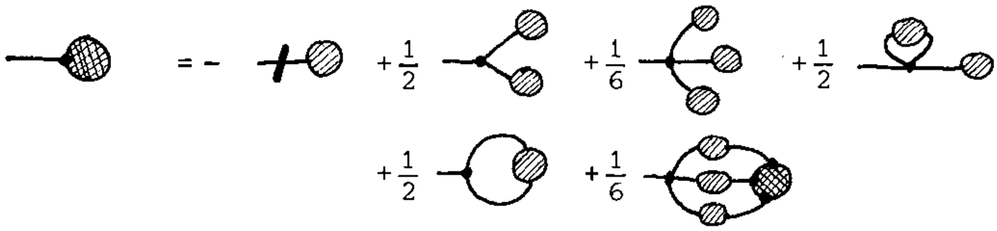
Diagrammatically W'' is a complete propagator which sums up all proper self-energies.

We can use (2.25) and (2.27) to eliminate source-dependent functionals in favour of field-dependent functionals, and (2.29) to replace J-derivatives by ϕ-derivatives, in order to rewrite (2.20) as the <u>1PI Dyson-Schwinger equation</u>:

$$\frac{d\Gamma[\phi]}{d\phi_i} = \frac{dS}{d\phi_i}\left[\phi + W''[J]\frac{d}{d\phi}\right] . \tag{2.33}$$

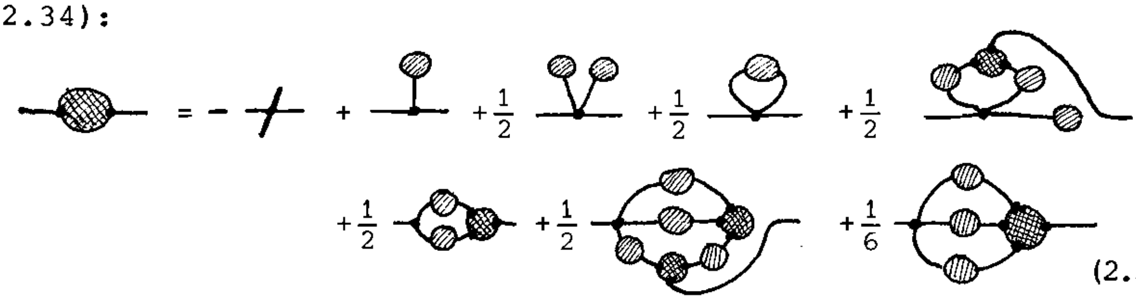
The form of this equation is one of the reasons why the generating functional for 1PI Green functions is called the <u>effective action</u>. If the derivatives are dropped, the effective action reduces to the classical action. The role of the derivatives is to generate loops, i.e. quantum corrections (or statistical fluctuations). We shall return to this in our discussion of path integrals.

DS equations (2.33) are again so elegant that one is probably at a loss as to what to do with them. To get a feeling for their utility, we write them out for the $\phi^3 + \phi^4$ example (2.21):



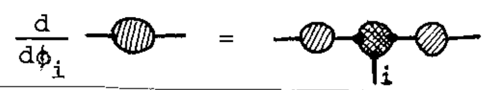
$$\frac{d\Gamma[\phi]}{d\phi_i} = -\Delta^{-1}_{ij}\phi_j + \tfrac{1}{2}\gamma_{ijk}\phi_k\phi_j + \tfrac{1}{6}\gamma_{ijk\ell}\phi_\ell\phi_k\phi_j + \tfrac{1}{2}\gamma_{ijk\ell}\phi_\ell \frac{d^2W[J]}{dJ_k dJ_j}$$
$$+ \tfrac{1}{2}\gamma_{ijk}\frac{d^2W[J]}{dJ_k dJ_j} + \tfrac{1}{6}\gamma_{ijk\ell}\frac{d^2W[J]}{dJ_j dJ_m}\frac{d^2W[J]}{dJ_k dJ_n}\frac{d^2W[J]}{dJ_\ell dJ_\sigma}\frac{d^3\Gamma[\phi]}{d\phi_m d\phi_n d\phi_\sigma} \quad . \tag{2.34}$$

Such equations are used iteratively. For example, to obtain the DS equation for the proper self-energy†, take a derivative of (2.34):



(2.35)

Exercise 2.H.1 Use (2.32) to show that

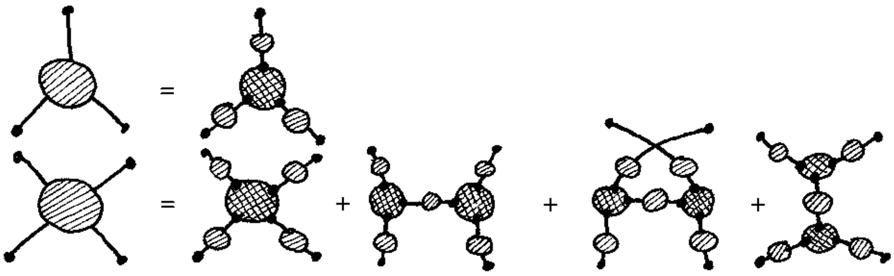


(2.36)

This is a useful identity for deriving relations such as (2.34) and (2.35).

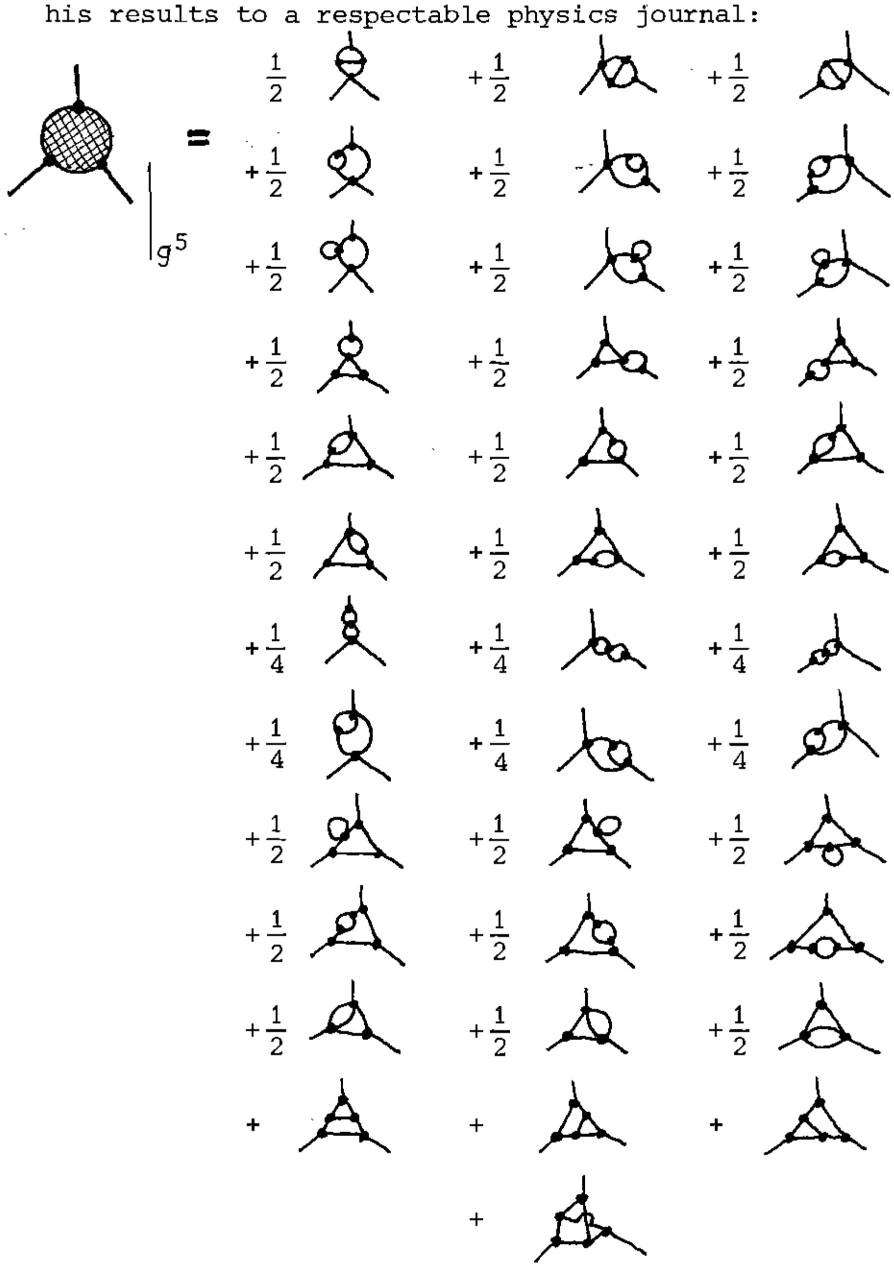
Exercise 2.H.2 Take successive derivatives of (2.30) to show that the connected Green functions can be expanded in terms of 1PI Green functions as

† Here the slash stands for inverse propagator; diagrammatically it is a two-leg vertex. Other vertices are denoted by dots, and a line connecting two vertices is always a propagator, so that $\Delta^{-1}_{ij}\Delta_{jk} = i$—┼•—$k = i$——$k = \delta_{ik}$.



(2.37)

Exercise 2.H.3 Jens J. Jensen, a serious young student of field theory, is getting set to compute the two-loop QCD beta-function. He has drawn up a list of gluon corrections to the three-gluon vertex. Use the 1PI Dyson-Schwinger equations to check this list and make Jens aware of 7 (seven) errors before he rushes his results to a respectable physics journal:



I. Vacuum bubbles

The Green function formalism we have developed so far is tailored to scattering problems; all the Green functions we have considered had external legs. Processes without external particles (the corresponding legless diagrams are called vacuum bubbles) are also physically interesting. For example, if a particle is propagating through a hot, dense soup†, a particle-particle scattering experiment would be a hopeless and messy undertaking. Such systems are probed by varying bulk parameters, such as temperature. Indeed, the generating functionals do not depend only on the single-particle sources J_i, but on all interaction parameters

$$Z[J] \equiv Z[J,\gamma_{ij},\gamma_{ijk},\gamma_{ijk\ell},\dots] \quad . \tag{2.38}$$

Any of these, or any combination of these, can be varied. Diagrammatically we view an n-vertex as an n-particle source. For example, if we rescale $\gamma_{ij..k} \to g\gamma_{ij..k}$ and vary infinitesimally the coupling constant g, we "touch" each $\gamma_{ij..k}$ vertex in a Green function:

$$g\frac{d}{dg}Z[J] = \frac{1}{k!}$$  $$= \frac{g}{k!}\gamma_{ij..k}\frac{d}{dJ_k}\cdots\frac{d}{dJ_j}\frac{d}{dJ_i}Z[J] \quad . \tag{2.39}$$

We can use such generalizations of the Dyson-Schwinger equations (from varying single-particle sources J_i to varying many-particle sources $\gamma_{ijk..\ell}$) to compute hosts of physically significant quantities. One such quantity is the expectation value of the action. We rescale the entire action (2.13)

$$\frac{1}{\hbar}S[\phi] = -\frac{1}{2\hbar}\phi_i\Delta^{-1}_{ij}\phi_j + \frac{1}{3!\hbar}\gamma_{ijk}\phi_k\phi_j\phi_k + \dots , \tag{2.40}$$

and vary $\hbar$ (depending on the context, $\hbar$ could be the Planck constant, coupling constant, inverse temperature or something else):

$$\hbar\frac{d}{d\hbar}Z[J] = -\frac{1}{\hbar}\Big(-\frac{1}{2}$$  $$+\frac{1}{3!}$$  $$+\frac{1}{4!}$$  $$+\dots\Big)$$

$$= -\frac{1}{\hbar}S\left[\frac{d}{dJ}\right]Z[J] \quad . \tag{2.41}$$

---

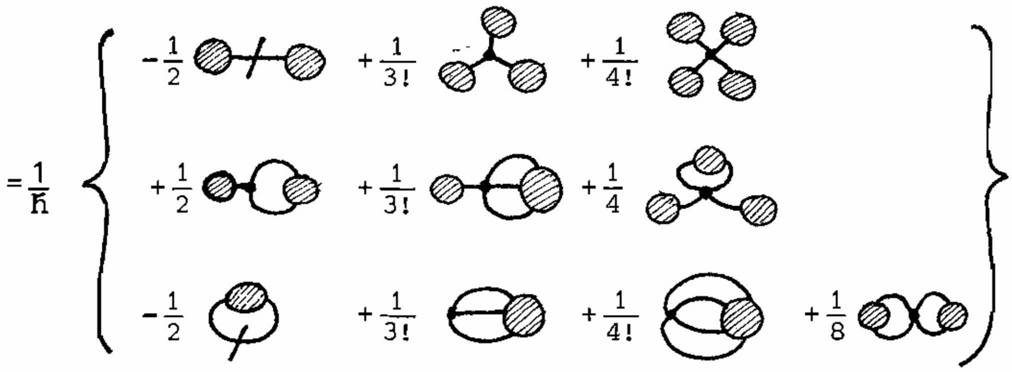
† minestrone, to be specific.

To normalize the expectation value properly, we divide by Z[J]:

$$\langle S[\phi]\rangle = \frac{1}{Z[J]}\, S\left[\frac{d}{dJ}\right] Z[J] \quad . \tag{2.42}$$

That this is really an expectation value will perhaps be easier to grasp in the path-integral formalism, cf. (3.11) in the next chapter. Anyway, we can use (2.19) to rewrite the above in terms of connected Green functions:

$$\langle \frac{1}{\hbar} S[\phi]\rangle = -\hbar \frac{dW[J]}{d\hbar} = \frac{1}{\hbar}\, S\left[\frac{dW[J]}{dJ_i} + \frac{d}{dJ_i}\right]$$



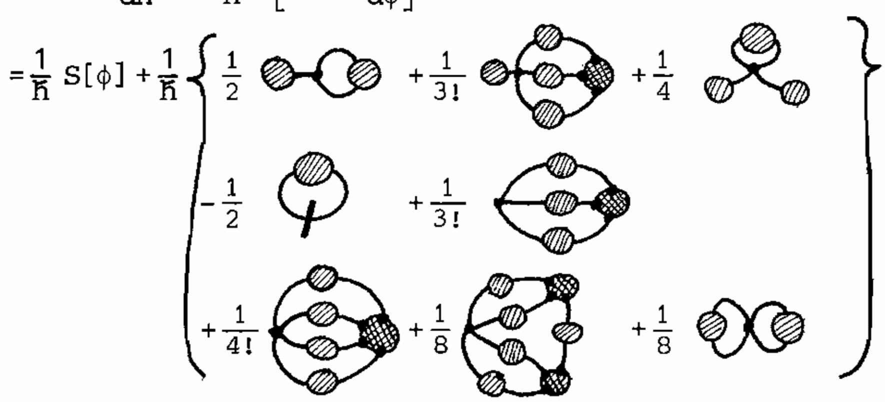
(2.43)

(the diagrammatic expansion is for the $\phi^3 + \phi^4$ theories). Even better, we can use (2.25) and (2.29) together with the identity (follows from (2.28))

$$\frac{dW[J]}{d\hbar} = \frac{d\Gamma[\phi]}{d\hbar} \tag{2.44}$$

to relate the $\langle S[\phi]\rangle$ to the effective action:

$$\frac{1}{\hbar}\langle S[\phi]\rangle = -\hbar\, \frac{d\Gamma[\phi]}{d\hbar} = \frac{1}{\hbar}\, S\left[\phi + W''\frac{d}{d\phi}\right]$$



(2.45)

The above expansions can be used to compute the perturbative expansions for the connected and 1PI vacuum bubbles (see exercises). Their physical significance will become clearer in the next chapter.

Exercise 2.I.1 Loop expansion. Show that with action (2.40) the expansion in powers of $\hbar$ is the loop expansion, i.e. that each loop in a Feynman diagram carries a factor $\hbar$. Hence the loop expansion offers a systematic way of computing quantum corrections (or thermal fluctuations in statistical mechanics). Hint: each propagator carries a factor $\hbar$, while each vertex carries $\hbar^{-1}$.

Exercise 2.I.2 Free energy W[0]. Compute

$\frac{1}{\hbar} W[0] = \frac{\delta_{ii}}{2} \frac{\ln\hbar}{\hbar} + \frac{1}{12}$  $+ \frac{1}{8}$  $+ \frac{1}{8}$  $+ \ldots$

for $\phi^3 + \phi^4$ theory. Hint: use (2.43) and the DS equations (2.21).

Exercise 2.I.3 Gibbs free energy Γ[0]. Compute

$\frac{1}{\hbar} \Gamma[0] = \frac{\delta_{ii}}{2} \frac{\ln\hbar}{\hbar} + \{\frac{1}{12}$  $+ \frac{1}{8}$  $\}$

$+ \{\frac{1}{24}$  $+ \frac{1}{16}$  $+ \frac{1}{8}$  $+ \frac{1}{8}$  $+ \frac{1}{16}$  $+ \frac{1}{48}$  $\}\hbar$

$+ \ldots$ (2.46)

for $\phi^3 + \phi^4$ theory. Hint: use (2.45) and the DS equations (2.34). Note that the one-particle reducible diagrams from W[0] are indeed missing. The vacuum-bubble combinatoric weights are not always obvious - equation (2.45) provides the fastest way of computing them, as far as I know.

Exercise 2.I.4 Show that for the zero-dimensional ϕ^3 theory (continuation of exercise 2.E.3)

$$G^{(m)} = (m - 1 + 3g \frac{d}{dg}) G^{(m-2)} .$$

Hint: use (2.39) together with the Dyson-Schwinger equations (2.12).
Show also that

$$G^{(1)} = \frac{g}{2} G^{(2)} .$$

Hence all Green functions can be computed from $Z \equiv G^{(0)}$, the vacuum bubbles. Show that these satisfy

$$g\frac{d}{dg}Z = g^2\left(\frac{5}{12} + \frac{9}{4} g \frac{d}{dg} + \frac{3}{4} g^2 \frac{d^2}{dg^2}\right)Z .$$

Compute the first few terms of the expansion in powers of g. The complete solution is given in exercise 3.C.1.

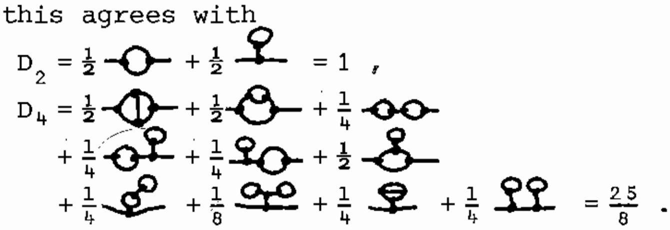
Exercise 2.I.5 Zero-dimensional field theory. Show that the connected vacuum bubbles $W \equiv W[0]$ satisfy

$$g\frac{d}{dg}W = g^2\left[\frac{5}{12} + \frac{9}{4}\,g\frac{dW}{dg} + \frac{3}{4}\,g^2\left(\frac{d^2W}{dg^2} + \left(\frac{dW}{dg}\right)^2\right)\right] .$$

Use this equation to derive recursion relations for connected m-leg Green functions. Compute the exact propagator $D = G^{c(2)}$

$$D = 1 + g^2 + \frac{25}{8}\,g^4 + \frac{390}{32}\,g^6 + \ldots\ldots$$

and check that this agrees with



Hint: establish first that

$$\frac{d}{dJ}W[J] = \frac{g}{2} + J + \frac{g}{2}\left(J\frac{d}{dJ} + 3g\frac{d}{dg}\right)W[J] .$$

That relates $G^{c(m)}$ to the vacuum bubbles W.

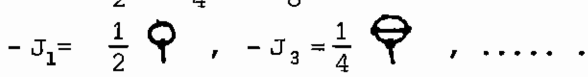
Exercise 2.I.6 Zero-dimensional ϕ^3 theory. Combine the DS equation (2.34) and the previous results to relate 1PI Green functions with different numbers of legs:

$$\frac{d}{d\phi}\Gamma[\phi] = \frac{g}{2} - \phi + \frac{g}{2}\left(-\phi\frac{d}{d\phi} + 3g\frac{d}{dg}\right)\Gamma[\phi] ,$$

and show that the proper tadpoles $J = -\Gamma^{(1)}$ satisfy

$$J = -\frac{g}{2} + \frac{g}{2}\left(1 - \frac{3}{2}g\frac{d}{dg}\right)J^2$$

$$= -\frac{1}{2}\,g - \frac{1}{4}\,g^3 - \frac{5}{8}\,g^5 - \ldots .$$

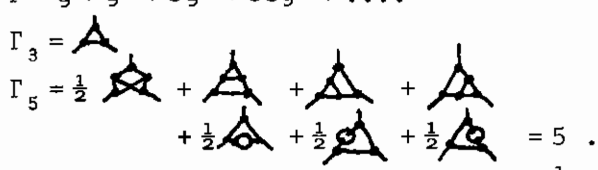


Compute the proper self-energy

$$\pi = \frac{1}{2}\,g^2 + g^4 + \frac{35}{8}\,g^6 + \ldots\ldots$$

and the proper three-vertex $\Gamma \equiv \Gamma^{(3)}$

$$\Gamma = g + g^3 + 5g^5 + 35g^7 + \ldots .$$



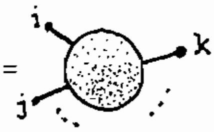
Compare π with the preceeding exercise, $D = (1 - \pi)^{-1}$.

Exercise 2.I.7 Check (2.44).

J. Summary of the generating functional formalism

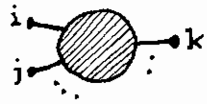
generating functionals

Full Green functions: $G_{ij..k}$ =



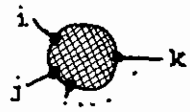
$Z[J]=$

Connected Green functions: $G^{(c)}_{ij..k}$ =

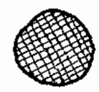


$W[J]=$

1PI Green functions: $\Gamma_{ij..k}$ =

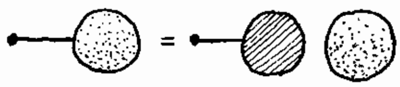


$\Gamma[\phi]=$



Full ↔ connected relation:

$$\frac{1}{Z[J]}\frac{d}{dJ_i}Z[J]=\frac{dW[J]}{dJ_i}+\frac{d}{dJ_i}$$



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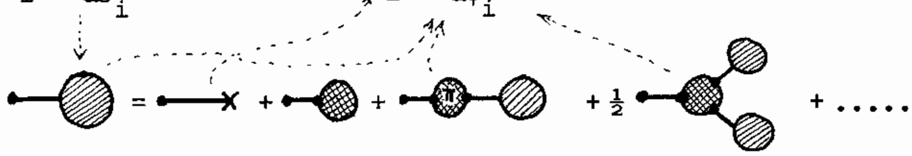
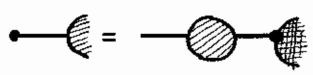
Connected ↔ 1PI relations:

$$\phi_i=\frac{dW[J]}{dJ_i} \qquad\qquad 0=J_i+\frac{d\Gamma[\phi]}{d\phi_i}$$



$$\frac{d}{dJ_i}=\frac{d^2W[J]}{dJ_i dJ_k}\frac{d}{d\phi_k} \quad \leftrightarrow$$



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Dyson-Schwinger equations:

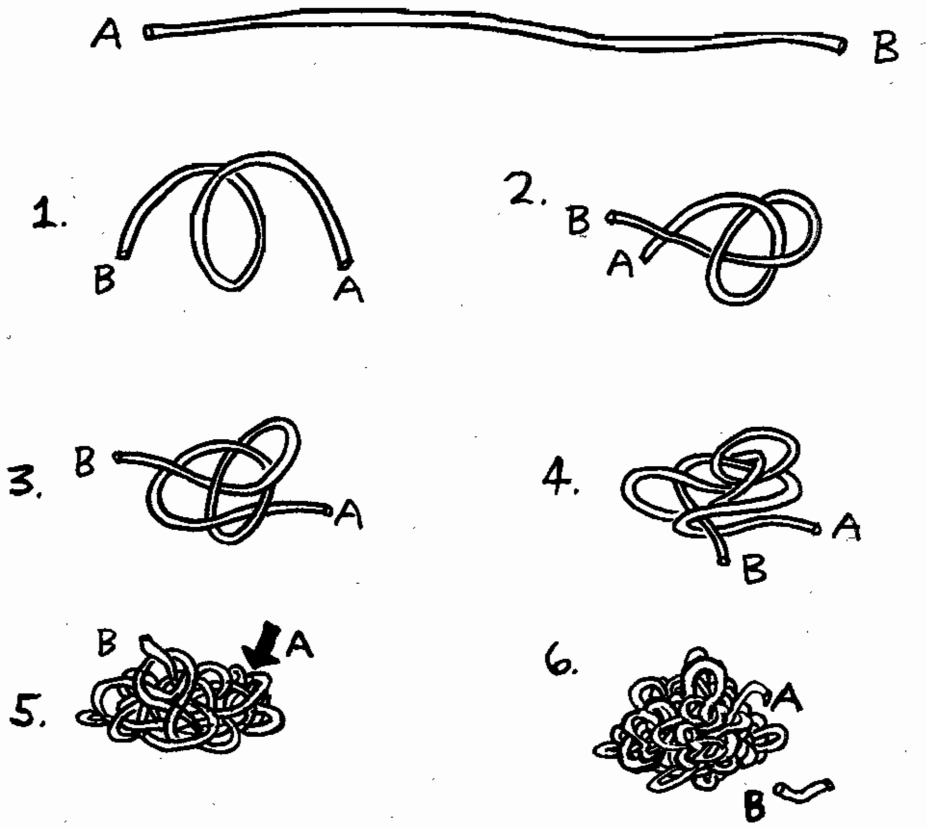
full $$\left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right]+J_i\right)Z[J]=0 \quad ,$$

connected $$\frac{dS}{d\phi_i}\left[\frac{dW[J]}{dJ}+\frac{d}{dJ}\right]+J_i=0 \quad ,$$

1PI $$\frac{d\Gamma[\phi]}{d\phi_i}=\frac{dS}{d\phi_i}\left[\phi+W''[J]\frac{d}{d\phi}\right] .$$

By now we are thoroughly fed up with longleggedy beasties, and the diagrammatic manipulations:

TYING THE NUDO DEL DIABLO OR DEVIL'S KNOT



(CONTINUED NEXT WEEK)

Let us now see whether the crow's vision of Quefithe is any more fun than the mole's version.

Critics say:

... Seen in [Cvitanović's] framework, field theory books are like every other form in the universe: they are generated by changing intervals of tension between a dominant system and a competing system in a space-time continuum that is dependent on the process of competition between these two stabilities and not on any General Concept of Space and Time ... [Cvitanović's] method thus valorizes the microcosm which illuminates macrocosmic form by the high tendency of microcosmic patterns to repeat themselves and so greatly limit structural variation in the macrocosm ... But on another level, as in the sagas, the Song of Rolland, the Illiad, the Odyssey, the Nibelungenlied, the Aeneid and Beowulf, the real dynamic focus of the book is the power of anger.

Patricia Harris Stablein

A distinguished reviewer says: