Lie Groups and Lie Algebras
(see H. Georgi, Lie Algebras in Particle Physics)

So far, we have focused primarily on discrete groups (ie, groups having a finite, or discretely infinite number of elements) and have seen that they certainly have their uses in physics.

We now turn our attention to continuous groups (ie, groups having a continuously infinite number of elements).

Examples include:

$SO(3)$	the special orthogonal group of rotations in 3D — preserve lengths and relative angles of vectors; no inversions
$O(3)$	like $SO(3)$ but also has inversions
$SU(2)$	the special unitary group of transformations of complex valued 2-component entities via complex unitary matrices of determinant +1
$SO(3,1)$	the proper Lorentz group of special relativity

By continuous we mean that the group elements $g \in G$ depend smoothly on a set of continuous parameters α. Smoothly means there is a notion of closeness: nearby group elements correspond to nearby parameters

We shall focus on compact Lie groups.

compact: the "volume" of the parameter space for α is finite — this rules out $SO(3,1)$ but keeps $SU(2)$ (for example)

Lie: all elements can be continuously deformed into the identity element — this rules out $O(3)$ but keeps $SO(3)$ (for example)

Generators

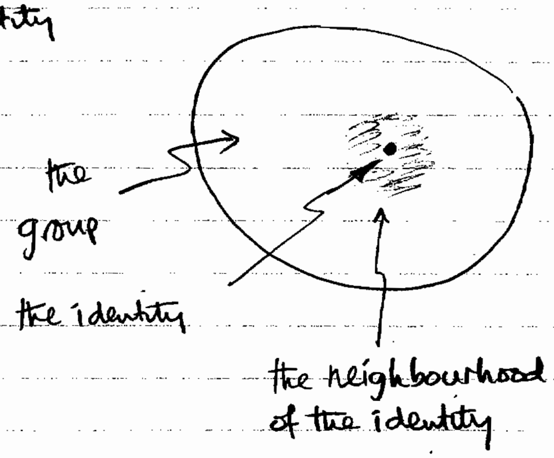
The identity element is important – let us choose to parametrise G so that $g(0) = e$ ← the identity

↑ all parameters zero

Suppose that to move around the group requires N real parameters

$$\{\alpha_a\}_{a=1}^{N}$$

← Collectively denote α



Then $g(\alpha)\big|_{\alpha=0} = e$.

As usual, we shall not only be concerned with groups but also with their representations (by matrices or differential operators), and we shall parametrise reps in the same way that we parametrise the group itself

$$\underset{\text{group element}}{g(\alpha)} \rightarrow \underset{\text{rep (matrix or differential operator)}}{D(\alpha)}$$

Then at $\alpha = 0$ we have $D(\alpha)\big|_{\alpha=0} = I$ ← identity operator

And near the identity we may Taylor expand

$$D(\delta\alpha) = D(0 + \delta\alpha) \approx I + i\sum_{a=1}^{N} \delta\alpha_a X_a$$

- $D(\delta\alpha)$: operator
- I: identity operator
- i: inserted for convenience – as D will be unitary, X will be hermitian
- $\delta\alpha_a$: (small) parameters
- X_a: Set of operators, one for each parameter, called generators

So the (representation) of the generator X_a can be constructed from the representation of the group via

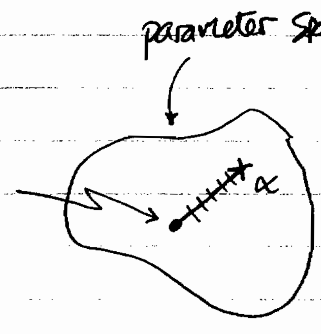
$$X_a = \frac{1}{i} \frac{d}{d\alpha_a} D(\alpha) \Big|_{\alpha=0}$$

We shall assume that the parametrisation is parsimonious – all parameters are needed, and therefore all generators are independent.

Aside: Marius Sophus Lie defined generators without recourse to representations, and showed that the entire local structure of (now called) Lie groups is determined by their behaviour near the identity. We shall be primarily concerned with group representations, so we shall not pursue this (beautiful) development.

The key point is this: because all group elements in Lie groups are continuously connected to the identity, we can build them up by the repetition of elements close to the identity. So the structure of the group (by which we mean the combination rule for group elements) is determined by the properties of the group in the neighbourhood of the identity – hence the focus on generators.

Constructing $D(\alpha)$ for non-infinitesimal α:



Define this to be the element we get to $\alpha = 0$ after k applications of the element $D(\frac{\alpha}{k})$.

$$\text{Then } D(\alpha) \equiv D\left(\frac{\alpha}{k}\right)^k = \lim_{k\to\infty} D\left(\frac{\alpha}{k}\right)^k$$

same for any $k = 1, 2, 3, \ldots$

but now $D(\frac{\alpha}{k})$ is near the identity

$$= \lim_{k\to\infty} \left(I + i\frac{1}{k}\, \alpha \cdot X \right)^k$$

means $\alpha_a X_a \equiv \sum_{a=1}^{N} \alpha_a X_a$

But X is hermitian, so there is always a basis in which $\alpha \cdot X$ is diagonal with real eigenvalues. Temporarily working in this basis, and using

$$\lim_{k \to \infty} \left(1 + \frac{z}{k}\right)^k = e^z \qquad (\text{for } z \in \mathbb{C})$$

we find
$$\boxed{D(\alpha) = \exp(i\alpha \cdot X)}$$
— called the exponential parametrisation of the rep.

the group rep is expressed in terms of the generators (ie, in terms of properties near the identity)

Remark: We often refer to linear combinations of generators X_a also as generators.

Nice feature: the generators X_a form a vector space; we can multiply them by real numbers and we can add them.

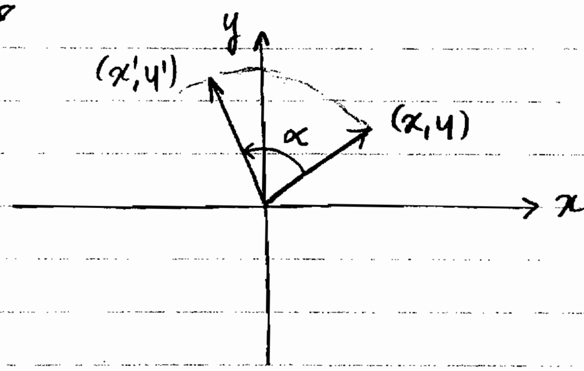
Example: The compact Lie group SO(2) (see Jones, pp. 96–101)

This is the group of proper rotations in 2 dimensions.

no reflections

All rotations take place about the same axis, through an angle α (with $0 \le \alpha < 2\pi$)

Compact



Inversions $(x, y) \to (-x, -y)$ are allowed – they correspond to rotations by π.

By plane geometry we have
$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{\underline{R}(\alpha)} \begin{pmatrix} x \\ y \end{pmatrix}$$

Strictly speaking, the 2x2 matrices $\underline{R}(\alpha)$ provide a rep of SO(2), but the kernel of the mapping is $\underline{I}$ so the rep. is faithful (ie isomorphic to the group itself).

Aside: over the reals this rep is irreducible; but over the field $\mathbb{C}$ we can write
$$\begin{pmatrix} x' + iy' \\ x' - iy' \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} x + iy \\ x - iy \end{pmatrix}$$
and hence we see that the rep $\underline{R}(\alpha)$ is reducible to two 1-dimensional representations – faithful reps via (1x1) unitary matrices – so SO(2) can also be designated $U(1)$. End aside.

Composition:
$$\underline{R}(\alpha_1)\,\underline{R}(\alpha_2) \underset{\substack{\text{by trigonometric} \\ \text{addition formulae}}}{=} \underline{R}(\alpha_1 + \alpha_2) = \underline{R}(\alpha_2 + \alpha_1) = \underline{R}(\alpha_2)\,\underline{R}(\alpha_1)$$

$\Rightarrow$ the group is Abelian, just as we would expect for rotations about a fixed axis.

The matrices are orthogonal:

$$\underline{\underline{R}}(\alpha)^T \cdot \underline{\underline{R}}(\alpha) = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\underline{I}}$$

Their determinant has modulus 1 (by orthogonality) but in fact it is +1:

$$\det \underline{\underline{R}} = c^2 + s^2 = +1. \quad \text{Cf } \det \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}^{\text{refl.}}\overbrace{\begin{pmatrix} c & -s \\ s & c \end{pmatrix}}^{\text{rot.}} = -1.$$

Generators? Note that $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then we see that

$$\underline{\underline{R}}(\alpha) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\alpha + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} i \sin\alpha$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sum_{n \text{ even}} \frac{(i\alpha)^n}{n!} + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \sum_{n \text{ odd}} \frac{(i\alpha)^n}{n!}$$

$$= \sum_{n \text{ even}} \frac{(i\alpha)^n}{n!} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^n + \sum_{n \text{ odd}} \frac{(i\alpha)^n}{n!} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ i\alpha \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \right\}^n = \exp i\alpha \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

So we see that there is

- one parameter, α, and
- one generator $\underline{\underline{X}} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

whose relationship with Pauli matrices should not go unnoticed.

Realisation of the group SO(2) via its action on complex functions

Under a rotation α we have $f(\theta) \to f(\theta - \alpha)$.

In terms of the Fourier expansion $f(\theta) = \sum_{m=-\infty}^{\infty} f_m e^{+im\theta}$

this becomes $\sum_m f_m e^{+im\theta} \to \sum_m f_m e^{+im(\theta-\alpha)}$

$$= \sum_m (f_m e^{-im\alpha}) e^{+im\theta}$$

Thus, in the basis for functions $f(\theta)$ spanned by the Fourier amplitudes $\{f_m\}$ the representation is fully reduced to one dimensional components

$$f_m \to \underbrace{D^{(m)}(\alpha)}_{\text{irrep labelled by } m} f_m = e^{-im\alpha} f_m$$

So, we have irreps $D^{(m)}(\alpha) = e^{-im\alpha}$ $\quad (m = 0, \pm 1, \pm 2, \ldots)$

The corresponding irreps of the generator are

$$X^{(m)} = \frac{1}{i} \frac{d}{d\alpha} D^{(m)}(\alpha) \Big|_{\alpha=0} = -m$$

And for the grand orthogonality theorem we have

$$\underbrace{\frac{1}{2\pi} \int_0^{2\pi} d\alpha}_{\text{the analogue of}} \; D^{(m)}(\alpha) \; D^{(m')}(\alpha)^* = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \; e^{-im\alpha} e^{im'\alpha} = \delta_{m,m'}$$

the analogue of

$\frac{1}{[g]} \sum_{g \in G}$ (ie a suitable invariant average over the group)

The generator as an operator:

We have $X^{(m)} f_m = -m f_m$ ← how the rep m of the generator acts

But we can also enquire about the action of the generator as an operator in the full space:

$$\hat{X} f(\theta) = \hat{X} \textstyle\sum_m f_m e^{im\theta} = \sum_m (X^{(m)} f_m) e^{im\theta}$$

$$= \sum_m (-m f_m) e^{im\theta} = i \frac{d}{d\theta} \sum_m f_m e^{im\theta} = i \frac{d}{d\theta} f(\theta).$$

Thus we can identify $\hat{X} = i \frac{d}{d\theta}$

Note the relationship with the z-component of angular momentum

$$\hat{L}_z = -i\hbar \frac{d}{d\theta}$$

From Lie groups to Lie algebras

In any "direction" α, group multiplication is simple:

$$\exp(i\lambda\,\alpha\cdot X)\,\exp(i\mu\,\alpha\cdot X) = \exp(i(\lambda+\mu)\,\alpha\cdot X)$$

because $\lambda\,\underset{\sim}{\alpha}\cdot X$ and $\mu\,\alpha\cdot X$ commute with one another.

real number

linear combination of generators

But for arbitrary "directions" α and β we do not have

$$\exp(i\alpha\cdot X)\,\exp(i\beta\cdot X) = \exp(i(\alpha+\beta)\cdot X) \quad \leftarrow \underline{\text{false}}$$

because, in general, $\alpha\cdot X$ and $\beta\cdot X$ do not commute.

However, by the group property, the left hand side is some element, say

$$\exp(i\gamma\cdot X) \quad \text{where} \quad \gamma = \gamma(\alpha,\beta).$$

At this stage we find something quite beautiful: the l.h.s is a group element (ie, we have closure) only if the generators form what is called a commutator algebra, i.e., the commutator is equal to a linear combination of generators

$$\underbrace{[X_a, X_b]}_{X_a X_b - X_b X_a} = i\,\underbrace{f_{abc}}_{}\,X_c \quad \text{(summation implied over } c\text{)}$$

Called: structure constants — they are real, and evidently obey $f_{abc} = -f_{bac}$.

If it were not for this commutator algebra property, terms arising from the product of the expansions of the l.h.s. exponentials would not organise as they need to.

In principle, we can compute δ as follows

$$i\delta\cdot X = \ln(\exp i\delta\cdot X) = \ln(\exp i\alpha\cdot X \; \exp i\beta\cdot X)$$

$$= \ln\Big[1 + \underbrace{\big(e^{i\alpha\cdot X}e^{i\beta\cdot X} - 1\big)}_{K}\Big] \approx K - \tfrac{1}{2}K^2 + \cdots$$

$$= \Big(1 + i\alpha\cdot X - \tfrac{1}{2}(\alpha\cdot X)^2 + \cdots\Big)\Big(1 + i\beta\cdot X - \tfrac{1}{2}(\beta\cdot X)^2 + \cdots\Big) - 1$$

$$+ \tfrac{1}{2}(\alpha\cdot X + \beta\cdot X)^2 + \cdots$$

Careful with order when expanding

$$= + i(\alpha\cdot X) + i(\beta\cdot X)$$

$$- (\alpha\cdot X)(\beta\cdot X) - \cancel{\tfrac{1}{2}(\alpha\cdot X)^2} - \cancel{\tfrac{1}{2}(\beta\cdot X)^2}$$

$$+ \cancel{\tfrac{1}{2}(\alpha\cdot X)^2} + \cancel{\tfrac{1}{2}(\beta\cdot X)^2} + \tfrac{1}{2}(\alpha\cdot X)(\beta\cdot X) + \tfrac{1}{2}(\beta\cdot X)(\alpha\cdot X) + \cdots$$

$$= i(\alpha+\beta)\cdot X - \tfrac{1}{2}(\alpha\cdot X)(\beta\cdot X) + \tfrac{1}{2}(\beta\cdot X)(\alpha\cdot X) + \cdots$$

$$= i(\alpha+\beta)\cdot X - \tfrac{1}{2}\alpha_a\beta_b[X_a, X_b] + \cdots$$

$$= i(\alpha+\beta)\cdot X - \tfrac{1}{2}\alpha_a\beta_b\, i f_{abc} X_c + \cdots$$

$$\boxed{\Rightarrow \quad \delta_c = \alpha_c + \beta_c - \tfrac{1}{2}\alpha_a\beta_b f_{abc} + \text{cubic order} + \text{higher}}$$

- the commutation relations play the role of the group multiplication table
- the commutation relations are enough to obtain closure to all orders; hence one can compute δ as accurately as one wishes
- the structure constants embody the combination law
- instead of facing the infinity of group elements we only face the finite number of generators.

- reps of the group imply reps of the algebra

  Matrices, differential operators that obey the group multiplication table

  ones that instead obey the commutation relations

- the structure constants are determined by the properties of the abstract group in the neighbourhood of the identity.

- the notions of equivalence and reducibility / irreducibility can be transferred from the group setting to the algebra setting.

- unitarity of $e^{i\alpha\cdot X}$ $\Rightarrow$ hermiticity of X

$$1 = (e^{i\alpha\cdot X})^\dagger (e^{i\alpha\cdot X}) = (1 + i\alpha\cdot X + \cdots)^\dagger (1 + i\alpha\cdot X + \cdots)$$

$$= (1 - i\alpha\cdot X^\dagger + \cdots)(1 + i\alpha\cdot X + \cdots)$$

$$= 1 + i\alpha\cdot(X - X^\dagger) + \cdots \quad \Rightarrow \quad X^\dagger = X$$

- hermiticity of $X \Rightarrow$ reality of f_{abc}

$$[X_a, X_b] = i f_{abc} X_c$$

so $$[X_a, X_b]^\dagger = -i f^*_{abc} X_c^\dagger$$

or $$[X_b, X_a] = -i f^*_{abc} X_c$$, or using $[X_b, X_a] = -[X_a, X_b]$

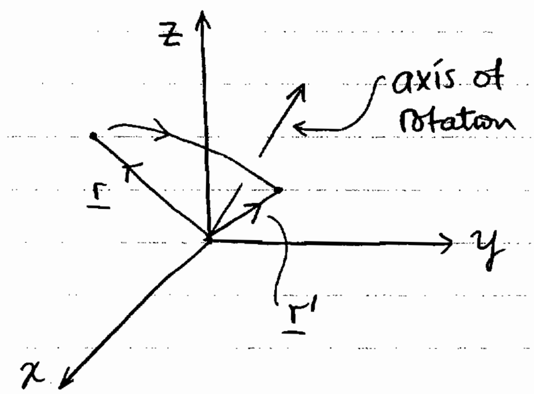
$$[X_a, X_b] = i f^*_{abc} X_c \quad \Rightarrow \quad f^*_{abc} = f_{abc}.$$

- Jacobi identity : $[X_a, [X_b, X_c]] + \text{cyclic perms} = 0$

check this by expanding out $\Rightarrow$ 12 cancelling terms.

Example : The compact Lie group $SO(3)$

This is the group of proper rotations in 3 dimensions.
no inversions $\Leftrightarrow$ all elements continuously connected to the identity



a unit vector

Each rotation can be specified by an axis of rotation $\underline{n}$ and an angle of rotation ω, which can conveniently be bundled together into a vector $\underline{\omega} \equiv \omega \underline{n}$

The parameter space is finite – the ball $\underline{\omega}$ such that $\omega \leq \pi$; longer vectors $\underline{\omega}$ repeat rotations contained in the ball.

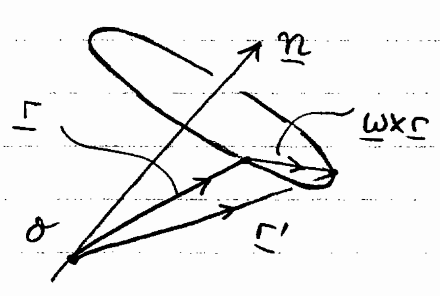
The defining representation follows from the geometry of rotations in 3D and is a faithful representation (ie isomorphic to the group itself).

Under an infinitesimal rotation $\underline{\omega}$ (with $|\underline{\omega}| \ll 1$) we have

$$\underline{r} \rightarrow \underline{r}' = \underline{r} + \underline{\omega} \times \underline{r}$$

perpendicular to $\underline{n}$ and $\underline{r}$

usual vector product



Let us explore the infinitesimal rotations further, in order to identify the (representation of the) generators :

In Cartesian coordinates (and with summation convention) we have

$$
\begin{aligned}
r_a \to r'_a &\simeq r_a + (\underline{\omega}\times\underline{r})_a \\
&= r_a + \varepsilon_{acb}\,\omega_c r_b \\
&= \{ \delta_{ab} + i\,\omega_c (i\,\varepsilon_{cab}) \}\, r_b \\
&= \{ \underline{\underline{I}}|_{ab} + i\alpha_c\, \underline{\underline{X}}_c|_{ab} \}\, r_b
\end{aligned}
$$

$\Rightarrow$ parameters $\boxed{\alpha_c = \omega_c}$ $(c=1,2,3)$

generators $\underline{\underline{X}}_c$ $(c=1,2,3)$, where $\boxed{\underline{\underline{X}}_c|_{ab} = i\,\varepsilon_{cab}}$.

Armed with the parameters and generators, we can write down the matrix $\underline{\underline{R}}(\underline{\alpha})$ representing a non-infinitesimal rotation :

$$\underline{\underline{R}}(\underline{\alpha}) = \operatorname{Exp}(i\,\alpha_c\,\underline{\underline{X}}_c).$$

linear combination of matrices (summation convention implied)

defined by its series expansion

We may also construct the commutator algebra and identify the structure constants :

$$
\begin{aligned}
[\underline{\underline{X}}_a, \underline{\underline{X}}_b]_{cd} &= \underline{\underline{X}}_a|_{ce}\,\underline{\underline{X}}_b|_{ed} - \underline{\underline{X}}_b|_{ce}\,\underline{\underline{X}}_a|_{ed} \\
&= i\varepsilon_{ace}\, i\varepsilon_{bed} - i\varepsilon_{bce}\, i\varepsilon_{aed} \\
&= \varepsilon_{ebc}\,\varepsilon_{eda} - \varepsilon_{eac}\,\varepsilon_{edb} \\
&= (\delta_{bd}\delta_{ca} - \cancel{\delta_{ba}\delta_{cd}}) - (\delta_{ad}\delta_{cb} - \cancel{\delta_{ab}\delta_{cd}}) \\
&= \varepsilon_{bag}\,\varepsilon_{dcg} \\
&= i\underbrace{(-\varepsilon_{abg})}_{f_{abg}}\underbrace{(i\varepsilon_{gcd})}_{\underline{\underline{X}}_g|_{cd}} = i f_{abg}\,\underline{\underline{X}}_g|_{cd}.
\end{aligned}
$$

So we see that the structure constants f_{abc} are given by

$$f_{abc} = -\varepsilon_{abc} \quad \leftarrow \text{real and antisymmetric}$$

It is reassuring to see the structure constants turn out to be something we know well from rotations and angular momentum.

Now let us realise the group $SO(3)$ in terms of its action on functions. We introduce the rotation operator (cf quantum mechanics)

$$\hat{R}(\underline{\alpha}) \equiv \exp i\underline{\alpha}\cdot\underline{\hat{\Lambda}}$$

3 generator operators

which acts as follows:

$$\hat{R}(\underline{\alpha})\,\psi(\underline{r}) = \psi\big(\underbrace{\underline{\underline{R}}(\underline{\alpha})^{-1}}_{\exp(-i\underline{\alpha}\cdot\underline{\underline{X}})}\cdot\underline{r}\big)$$

Expanding for small rotations we find:

$$(1 + i\underline{\alpha}\cdot\underline{\hat{\Lambda}})\,\psi(\underline{r}) \simeq \psi(\underline{r} - i\underline{\alpha}\cdot\underline{\underline{X}}\cdot\underline{r})$$

$$\simeq \psi(\underline{r}) - i(\underline{\alpha}\cdot\underline{\underline{X}})\cdot\underline{r}\,\big|_a\;\partial\psi(\underline{r})/\partial r_a$$

$$\Rightarrow \quad i\alpha_c\hat{\Lambda}_c\psi = -i\alpha_c\,\underline{\underline{X}}_c\big|_{ab}\,r_b\,\partial\psi(\underline{r})/\partial r_a\,.$$

Now, this holds for arbitrary infinitesimal $\underline{\alpha}$, so we have

$$\hat{\Lambda}_c = -i\,\varepsilon_{cab}\,r_b\frac{\partial}{\partial r_a} = +i\,\varepsilon_{cba}\,r_b\frac{\partial}{\partial r_a}$$

or, in vector notation, $\underline{\hat{\Lambda}} = -\underline{r}\times(-i\underline{\nabla})$. Compare this with the quantum mechanical orbital angular momentum operator

$$\underline{\hat{L}} = \underline{\hat{r}}\times\underline{\hat{p}} = \underline{r}\times(-i\hbar\underline{\nabla}) = -\hbar\,\underline{\hat{\Lambda}}\,.$$

We see that the angular momentum operators are the generators of rotations.

Back to generalities – the adjoint representation

Let us introduce N matrices $\underline{\underline{I}}_a$, each N×N, defined by

$$\underset{\text{imaginary}}{\underline{\underline{I}}_a\big|_{bc}} = -i\underset{\text{real}}{f_{abc}}$$

What commutator algebra do the $\underline{\underline{I}}$'s obey?

$$[\underline{\underline{I}}_a, \underline{\underline{I}}_b]_{ce} = -f_{acd}f_{bde} + f_{bcd}f_{ade}. \qquad \circledast$$

But, from the Jacobi identity we have

$$\sum_{\substack{\text{cyclic}\\\text{perms}}} [\underline{X}_a, [\underline{X}_b, \underline{X}_c]] = \underline{0},$$ which becomes, in terms of the structure constants

$$\sum_{\substack{\text{cyclic}\\\text{perms of}\\abc}} if_{ade}\,\underline{X}_e\, if_{bcd} = \underline{0},$$ and because the generators are linearly independent, this gives

$$\sum_{\substack{\text{cyclic perms}\\\text{of } abc}} f_{ade}f_{bcd} = 0 \qquad (e \text{ is free}),$$ ie,

$$\underset{\text{on rhs of } \circledast}{+f_{ade}f_{bcd}} + \underset{\substack{-f_{bde}f_{acd} \text{ (by antisymmetry of commutator)}\\ \text{on rhs of } \circledast}}{f_{bde}f_{cad}} + f_{cde}f_{abd} = 0,$$

$$\Rightarrow [\underline{\underline{I}}_a, \underline{\underline{I}}_b]_{ce} = -f_{cde}f_{abd} = +f_{abd}\underset{i\underline{\underline{I}}_d|_{ce}}{f_{dce}}$$

$$= +if_{abd}\,\underline{\underline{I}}_d\big|_{ce}$$

$$\Rightarrow [\underline{\underline{I}}_a, \underline{\underline{I}}_b] = +if_{abd}\,\underline{\underline{I}}_d$$

So, we see that the structure constants furnish an N-dimensional representation of the algebra – the <u>adjoint</u> representation.

Normalisation and orthogonality in the adjoint rep.

Suppose we make a linear transformation of our abstract generators

$$X_a \to X'_a = L_{ab} X_b$$

This induces a transformation on our structure constants, because we now have

$$[X'_a, X'_b] = L_{ac} L_{bd} [X_c, X_d] = L_{ac} L_{bd}\, i f_{cde} X_e$$

$$= i \underbrace{\left(L_{ac} L_{bd} (L^{-1})_{eg} f_{cde} \right)}_{f'_{abg}} X'_g$$

and so we have $f'_{abg} = L_{ae} L_{bd} (L^{-1})_{eg} f_{cde}$.

This induces a similarity transformation (and more) on the adjoint rep:

$$\underline{\underline{I}}_a \to \underline{\underline{I}}'_a = ?$$

Well, $\underline{I}_a|_{bc} = -i f_{abc}$

$$\to \underline{I}'_a|_{bc} = -i f'_{abc}$$

$$= -i L_{ag} L_{bd} (L^{-1})_{ec} f_{gde}$$

$$= L_{ag} L_{bd} (L^{-1})_{ec} \underline{I}_g|_{de}$$

and a linear combination of generators

similarity transform on matrix indices

Let us define a scalar product on the linear vector space spanned by the matrices $\underline{\underline{T}}_a$:

matrix product

$$\mathrm{Tr}\ \underline{\underline{T}}_a \cdot \underline{\underline{T}}_b \qquad \left(= \sum_{cd} \underline{\underline{T}}_a|_{cd}\ \underline{\underline{T}}_b|_{dc}\right)$$

trace on the matrix labels

- this quantity is symmetric under $a \leftrightarrow b$ and is real

Under our linear transformation $\underline{\underline{L}}$ we have

$$\mathrm{Tr}\ \underline{\underline{T}}_a \underline{\underline{T}}_b \longrightarrow \mathrm{Tr}\ \underline{\underline{T}}'_a \cdot \underline{\underline{T}}'_b = L_{ac} L_{bd}\ \mathrm{Tr}\ \underline{\underline{T}}_c \underline{\underline{T}}_d$$

(due to the cyclic property of the trace, only the L's acting on the labels (not the components) of the $\underline{\underline{T}}$'s remain)

By choosing suitable $\underline{\underline{L}}$ we can arrange for

$$\boxed{\mathrm{Tr}\ \underline{\underline{T}}'_a\ \underline{\underline{T}}'_b = \lambda_a\ \delta_{ab} \qquad \text{(no sum)}}$$

signs (ie + or −)

but we cannot eliminate the signs. If all are positive then we have a compact Lie algebra (e.g. not the Poincaré group).

In this basis we have that f is completely antisymmetric :

$$f_{abc} = f_{cab} = f_{bca} = -f_{acb} = -f_{bac} = -f_{cba}.$$

Proof: $\mathrm{Tr}\,[\underline{\underline{T}}_a, \underline{\underline{T}}_b]\,\underline{\underline{T}}_d = i f_{abc}\ \mathrm{Tr}\ \underline{\underline{T}}_c \underline{\underline{T}}_d = i f_{abd}\ \lambda_d$ (no sum)

$$\Rightarrow f_{abd} = \frac{1}{i\lambda_d}\ \mathrm{Tr}\left(\underline{\underline{T}}_a \underline{\underline{T}}_b \underline{\underline{T}}_d - \underline{\underline{T}}_b \cdot \underline{\underline{T}}_a \cdot \underline{\underline{T}}_d\right)$$

all +1 $\qquad dab = bda \qquad dba = adb \qquad$ by cyclic invariance of the trace

Hence we see explicitly the stated complete antisymmetry of f.
(Then $\underline{\underline{T}}$'s are a/s and imag (ie hermitian) so $\underline{\underline{D}}$'s are unitary.)

Subalgebras, invariant subalgebras; simple and semisimple algebras

Recall that groups can have subgroups (subsets that themselves form groups with the original combination law), and that the subgroups can be invariant (aka normal) subgroups:

If H is a subgroup of G and $gHg^{-1} = H \quad \forall g \in G$ (ie H consists of whole conjugacy classes of G) then H is an invariant subgroup of G.

These notions also apply to algebras (linear combinations of group generators that can be "combined" via commutators).

Suppose that we have a commutator algebra in which a generic generator Y is a linear combination of basis generators Y_a, such that $[Y_a, Y_b] = i f_{abc} Y_c$.

Then a subalgebra of generators X is a subset of the Y's that close under commutation (ie produce i times an element of the subset when any pair's commutator is calculated using the original commutation properties)

And a subalgebra is invariant (aka normal) if any of its elements, when put into a commutator with any element of the original algebra, returns (i times) an element of the subalgebra

$$[X, Y] = i \cdot \text{element } X'$$

any element of the subalgebra

any element of the original algebra

is in the subalgebra

– note the connection between conjugation (for group elements) and commutator formation (for algebra elements)

Invariant subalgebras generate invariant subgroups

Consider $h = e^{iX}$ and $g = e^{iY}$

h: in the corresponding subgroup; X: in an invariant subalgebra; g: in the original group; Y: in the original algebra

Question: Is $g^{-1} h g = e^{iX'}$ with X' in the invariant subalgebra

(ie is h an element of an invariant subgroup)?

Well,
$$e^{iX'} = e^{-iY} e^{iX} e^{iY}$$
$$= e^{-iY} \sum_{n=0}^{\infty} \frac{(iX)^n}{n!} e^{iY}$$
$$= \sum_{n=0}^{\infty} (i\, e^{-iY} X e^{iY})^n / n!$$

So that
$$X' = e^{-iY} X e^{iY}$$

Now
$$e^{-iY} X e^{iY} = e^{-i\varepsilon Y} X e^{i\varepsilon Y}\Big|_{\varepsilon=1}$$
$$= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{d\varepsilon^m} \left(e^{-i\varepsilon Y} X e^{i\varepsilon Y} \right)\Big|_{\varepsilon=0}$$

Taylor series in $\varepsilon\ (=1)$ about $\varepsilon = 0$

Every derivative adds a commutator by pulling down $-iY$ before the X and a $+iY$ after.

So
$$X' = X - i[Y, X] - \tfrac{1}{2}[Y, [Y, X]] + \cdots\cdots .$$

All contributions to X' are in the invariant subalgebra; so X' is, and so $\exp iX$ forms an invariant subgroup.

All algebras have 0 and the full algebra as invariant subalgebras, but these are trivial invariant subalgebras.

If an algebra has only these invariant subalgebras and no others, it is called a simple algebra; simple algebras generate simple groups.

One can show that the adjoint representation of a simple Lie algebra is an irreducible representation of that algebra

Cannot simultaneously block-diagonalise all generators.

a set of matrices realizing the commutation relations

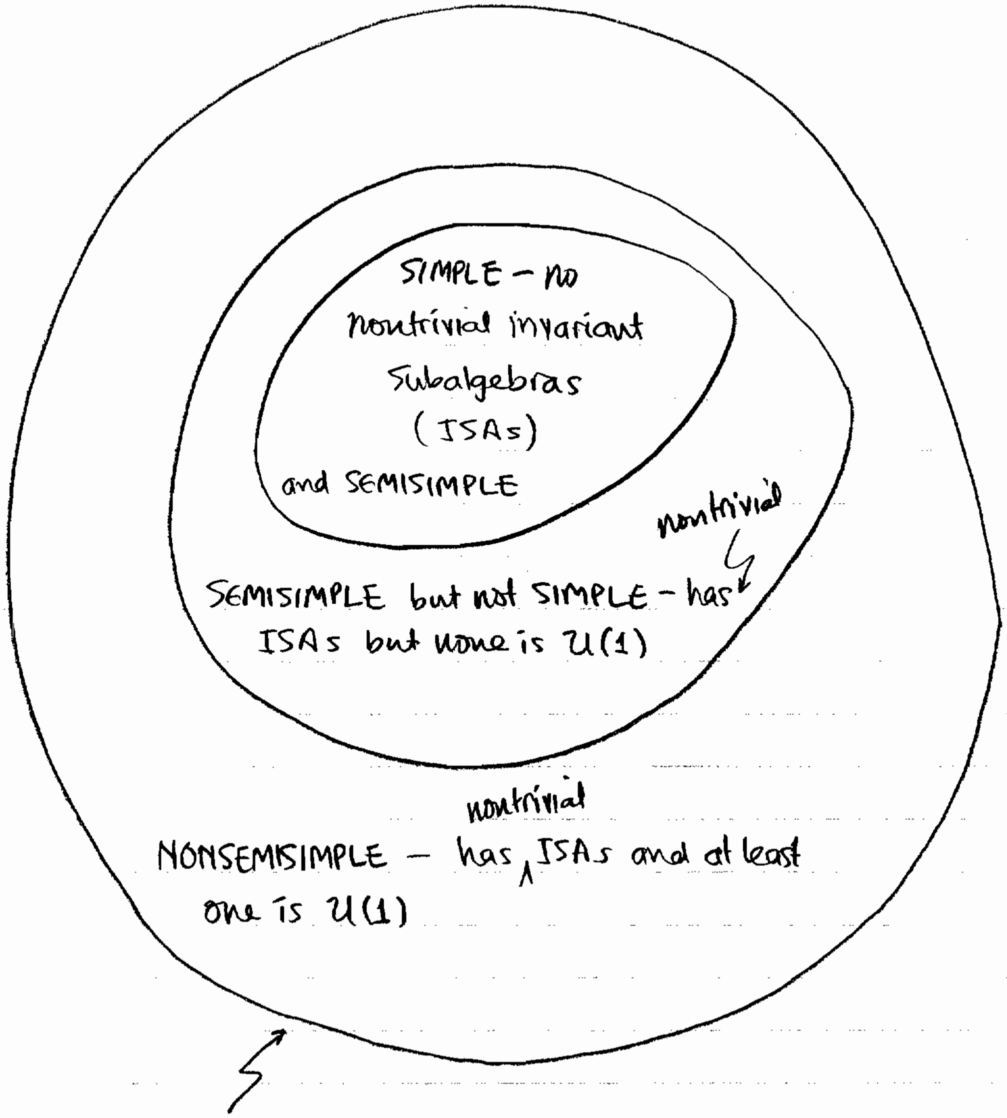
Suppose we can identify a single generator that commutes with all generators of a group. Then we say that this generator generates a special Abelian invariant subalgebra and that the corresponding group contains a $U(1)$ factor

[Aside: the group $U(1)$ is the group of phase transformations $\exp i\alpha$ (α real); ie unitary 1×1 matrices.]

$U(1)$ factors don't show up in the structure constants;
their generators yield directions in which $K_a = 0$ (in the norm);
if X_a is a $U(1)$ generator then $f_{abc} = 0$ (for all b, c);
the structure constants tell us nothing about $U(1)$ subalgebras;
algebras that have no $U(1)$ subalgebras are called semisimple.
(All simple algebras are evidently semisimple.)

We can build semisimple algebras by putting simple algebras together so that every generator has a nonzero commutator with some generator.

For semisimple algebras the structure constants tell us a great deal.



- Classifying Lie algebras – this is the collection of Lie algebras

- note that simple algebras are (by the definitions) also semisimple

Generators, operators and states in Dirac notation

Our representations of generators can be in terms of

- linear differential operators
- matrices

(think of orbital angular momentum and spin angular momentum).

In Dirac notation we have for the action of generators,

$$\underbrace{\hat{X}_a |i\rangle}_{\text{generator } X_a \text{ acting on basis state } |i\rangle} = \underbrace{\sum_j |j\rangle\langle j|}_{\text{resolution of the identity}} \hat{X}_a |i\rangle = \underbrace{\sum_j |j\rangle \, \underline{\underline{X}}_a|_{ji}}_{\text{linear combination of basis states}} .$$

For the action of group elements we have

$$\left.\begin{aligned} |i\rangle &\to e^{i\alpha\cdot\hat{X}}|i\rangle \\ \langle i| &\to \langle i| e^{-i\alpha\cdot\hat{X}} \end{aligned}\right\} \text{ which describe how states transform}$$

In order to preserve generic matrix elements we have that operators transform as follows

$$O \to e^{i\alpha\cdot X} \, O \, e^{-i\alpha\cdot X}$$

(then for example $O|i\rangle \to e^{i\alpha\cdot X} \, O|i\rangle$)

The action of the algebra determines changes under infinitesimal transformations:

$$\delta|i\rangle = e^{i\alpha\cdot\hat{X}}|i\rangle - |i\rangle \approx i\alpha\cdot\hat{X}|i\rangle$$

$$\delta\hat{O} = e^{i\alpha\cdot\hat{X}}\,\hat{O}\,e^{-i\alpha\cdot\hat{X}} \approx [i\alpha\cdot\hat{X}, \hat{O}]$$

The Special Unitary Groups SU(N) $(N = 2, 3, 4, \ldots)$

Fix a value of N. Consider the collection of unitary matrices M.

complex elements, $M^{+} = M^{-1}$

Recall $(M^{+})_{jk} \equiv (M_{kj})^{*}$

Because $M^{+} = M^{-1}$
we certainly have

$$1 = \det I = \det M M^{-1} = \det M M^{+} = (\det M)(\det M^{+})$$
$$= (\det M)(\det M)^{*}$$

and thus $\det M =$ a pure phase, $\exp i\chi$ (χ real).

The additional constraint <u>Special</u> further restricts the collection to those matrices having $\chi = 0$, ie $\det M = +1$.

It is straightforward to check that the collection of special unitary NxN matrices form a group.

In quantum mechanics, the group SU(2) is especially important, being very closely related to SO(3) and providing the transformation properties of Pauli spinors.

In the phenomenology of high energy physics, the groups SU(2) and SU(3) play a vital role in organizing particles and their interactions

Let us return to the algebra obeyed by the generators of $SO(3)$, namely

(*) Can change sign here by re-signing all X's

$$[X_a, X_b] = \underset{\uparrow (*)}{i}\,\varepsilon_{abc} X_c \qquad (a,b,c = 1,\dots,3).$$

When considering transformations acting on scalar wave functions, which are single-valued functions of the spatial coordinate $\underline{r}$, we find irreps labelled $l = 0,1,2,\dots$ corresponding to distinct values of total orbital angular momentum.

However, from the algebraic/operator treatment of the generator eigenproblem we find that there are also irreps corresponding to distinct half-integral values of angular momentum

$$l = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \dots$$

which can only be realised by actions of matrices on (even component) entities of complex numbers – the spinors.

Focusing on the $l = 1/2$ case, we have that the generators X_a are represented by ($\frac{1}{2}\times$) the Pauli matrices (up to equivalence), ie,

$$X_1 = \tfrac{1}{2}\sigma_x = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},$$

$$X_2 = \tfrac{1}{2}\sigma_y = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, \qquad X_3 = \tfrac{1}{2}\sigma_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

These X's are 2×2 Hermitian, traceless matrices and, in fact, form a basis for such matrices:

most general such matrix

$$2\,\underset{\text{real}}{\alpha}\cdot X = \begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}.$$

Exponentiating the generator, we obtain

$$U(\alpha) = \exp(-i\alpha\cdot X) = \exp(-\tfrac{i}{2}\alpha\cdot\sigma)$$

2×2 special unitary matrix; complex entries

- 2x2 complex matrices — well, we have exponentiated 2x2 complex matrices
- unitary — yes, because we have exponentiated (i times) a Hermitian matrix, a basis for which makes it real and diagonal
- special — yes, because of the tracelessness — the sum of the eigenvalues of $\alpha \cdot X$ vanishes and, hence, so does the sum of the phases of the eigenvalues of u

So we have the group SU(2) of matrices $u(\alpha) = e^{-\frac{i}{2}\alpha\cdot\sigma}$.

This u can be computed explicitly by expanding the exponential and using $\sigma_i \sigma_j = I\delta_{ij} + i\varepsilon_{ijk}\sigma_k$ so that $(\alpha\cdot\sigma)^2 = \alpha^2 I$.

I: 2x2 identity — sometimes written σ_0

α^2 means $\sum_{a=1}^{3} \alpha_a^2$

Hence we find $u(\omega \underline{n}) = I \cos\tfrac{1}{2}\omega - i\,\underline{\sigma}\cdot\underline{n} \sin\tfrac{1}{2}\omega$

α; ω is magnitude, $\underline{n}$ is unit vector

note half factors

Suppose we combine two transformations to make a third:

$$u(\omega\underline{n}) = u(\omega_1\underline{n}_1)\, u(\omega_2\underline{n}_2).$$

Then, at least for infinitesimal transformations the combination law $\omega\underline{n} = \omega\underline{n}(\omega_1\underline{n}_1, \omega_2\underline{n}_2)$ is the same as it is for 3D rotations because the commutator algebra is identical to that for 3D rotations. We say that the local properties of the groups SU(2) and SO(3) are identical.

But, as we shall now discuss, the global properties of the two groups are distinct. (The discussion in K. Gottfried, Quantum Mechanics, Sec. 34.3 is highly recommended.)

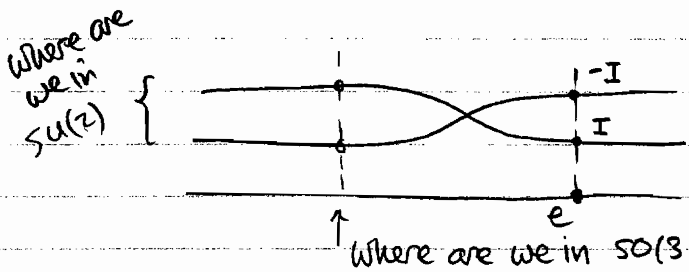
In $SO(3)$, transformations with $\omega \underline{n} = \underline{0}$ and $\omega \underline{n} = 2\pi \underline{n}$ both parametrise the identity element of $SO(3)$.

But in $SU(2)$ we have, from our 2 (complex) component rep, that $\omega \underline{n} = \underline{0}$ parametrises the identity element but that $\omega \underline{n} = 2\pi \underline{n}$ parametrises minus the identity element.

More generally, for every element of our abstract group of rotations there correspond two elements of the group $SU(2)$; there is a 2:1 homomorphism from $SU(2)$ into $SO(3)$, the kernel being

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

the elements of $SU(2)$ that map to the identity of $SO(3)$.



Caricature of the homomorphism

So the matrices $\mathcal{U} = e^{-\frac{i}{2}\omega \underline{n} \cdot \underline{\sigma}}$ form

- a single-valued representation of the group $SU(2)$ but
- a double-valued representation of the group $SO(3)$

not, strictly speaking, a rep*; but important because in quantum mechanics we must allow for the possibility that states belong to multivalued representations
(* in the sense that – in complex variables – a multifunction is not a function)

Globally, we have: $SO(3) \cong SU(2) / \mathbb{Z}_2$.

Topology of the groups SO(3) and SU(2)

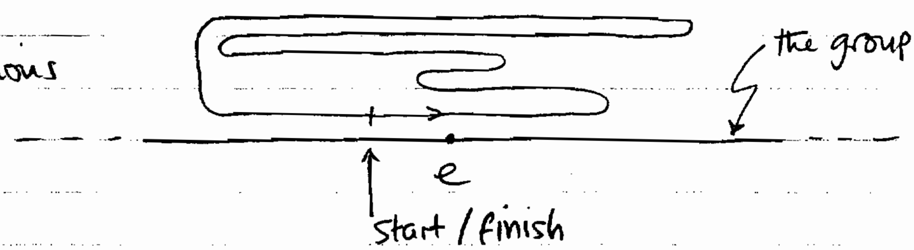
Think of the following 2 groups
- translations on the (infinite) line
- rotations in the plane

Locally they are very similar (in fact identical) – there is one group parameter in each case and it is additive.

But globally the groups differ – they have different topologies. Roughly speaking, this means that the structure of their parameter spaces is different in a way that cannot be accommodated by (smooth) changes of variables

This difference shows up when we consider continuous paths through the groups, especially the closed paths.

- translations

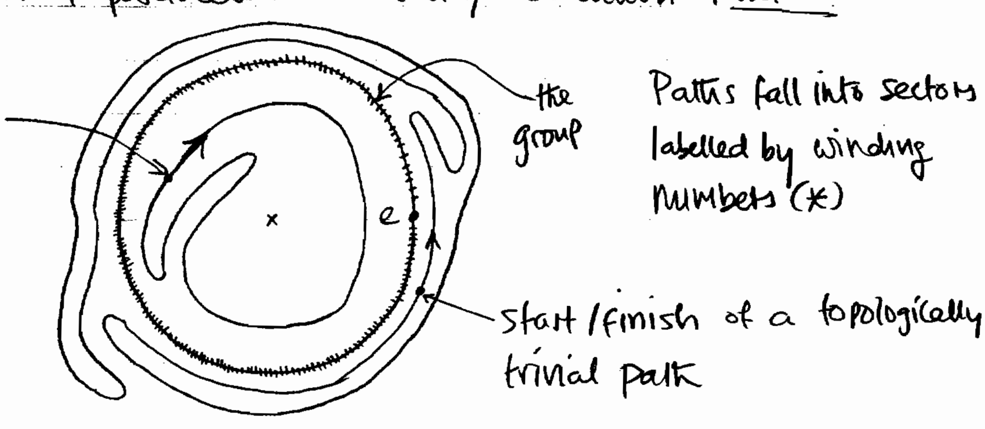


all closed paths can be continuously deformed into a point – we say that the group is topologically trivial

(*) Paths having distinct winding numbers cannot be smoothly deformed into one another

- planar rotations – we must account that the addition of parameters is really addition <u>mod 2π</u>

start/finish of a topologically non trivial path



Paths fall into sectors labelled by winding numbers (*)

Now return to the groups SO(3) and SU(2), which are locally indentical.

First, consider a matrix in the defining rep. of SU(2):

$$U(\omega \underline{n}) = \begin{pmatrix} c - isn_z & -s(in_x + n_y) \\ -s(in_x - n_y) & c + isn_z \end{pmatrix}$$

where $c \equiv \cos\frac{1}{2}\omega$, $s \equiv \sin\frac{1}{2}\omega$

and n_x, n_y, n_z (or n_1, n_2, n_3) are Cartesian components of the axis unit vector $\underline{n}$

This has the form $U(\omega\underline{n}) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$

with a and b being complex-valued entries subject to the single constraint

$$|a|^2 + |b|^2 = 1.$$

In other words, the parameter space for SU(2) has the topology of the three-dimensional surface of a sphere of radius 1 in four dimensions. Any closed path in this space can be shrunk to a point (there are no obstacles to get wrapped around) and hence we say that the group is simply connected.

Now consider the group $SO(3)$, parametrised by ω and $\underline{n}$ with

$$\left.\begin{aligned} n_x &= \sin\theta \cos\phi \\ n_y &= \sin\theta \sin\phi \\ n_z &= \cos\theta \end{aligned}\right\} \text{ the axis of rotation}$$

Then the parameters range over

$$\left.\begin{aligned} 0 \le \theta \le \pi \\ 0 \le \phi \le 2\pi \end{aligned}\right\} \text{ any orientation for } \underline{n}$$

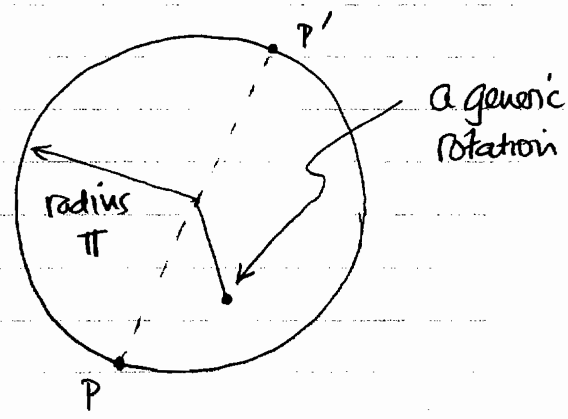
$$\left. 0 \le \omega \le \pi \right\} \text{ up to half way round}$$

But antipodes correspond to identical rotations

$$(\theta, \phi, \pi) \quad \text{and} \quad \underbrace{(\pi-\theta, \phi+\pi, \pi)}_{\text{twist by } \pi \text{ about the opposite axis}}$$

So we can represent the parameter space as the manifold $\{\omega\underline{n}\}$, i.e., the ball of radius π in 3D, provided we regard antipodes as identical points

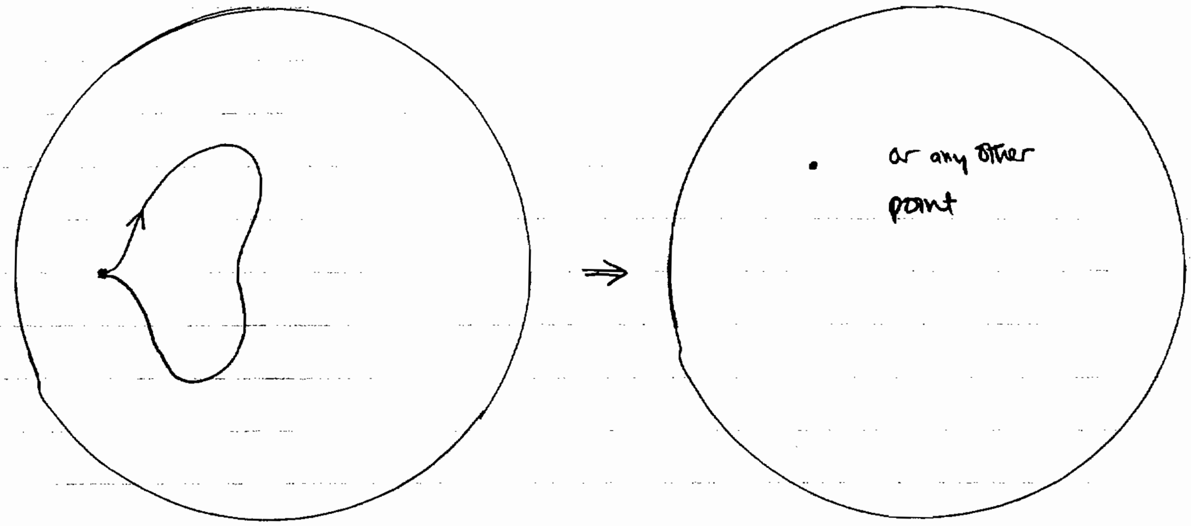
eg P and P' must be identified - they are the same element of the group.
In particular, this means that a path is still continuous if it hits the surface of the ball at P and reappears at P'.



Now we see that there are exactly two (why two?) topologically distinct types of closed path in $SO(3)$

- those that can be continuously deformed to a point; and
- — — — — — — — — — — — — — — — — diameter.

i)



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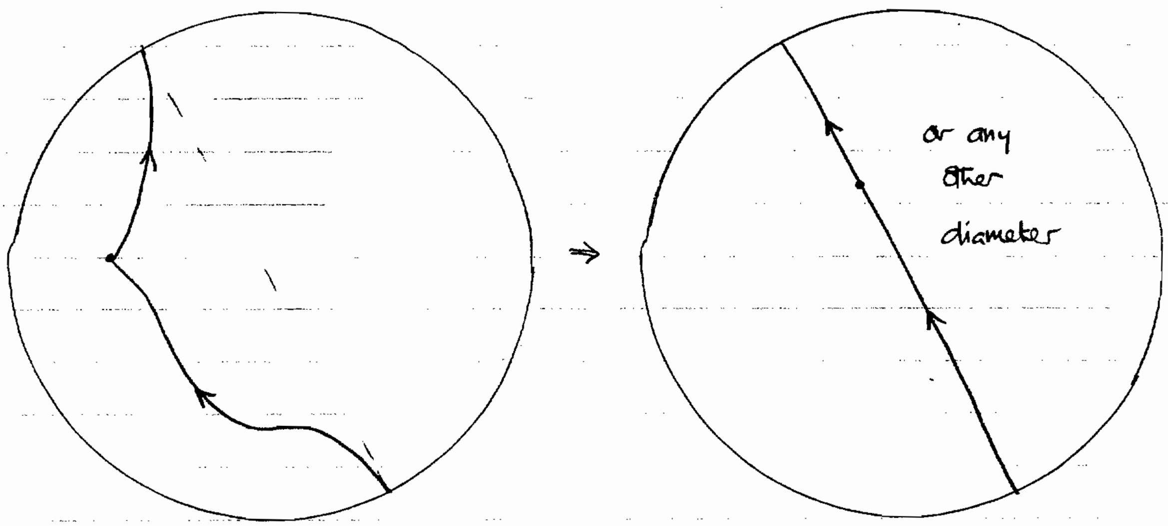

iii)