Georgia Tech PHYS 6124 Fall 2011

Mathematical Methods of Physics I

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Homework #11 due Tuesday November 15 2011, in class

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

[All problems in this set are from Goldbart]

Problem 5) More applications of Cauchy's theorem (**Ablowitz & Fokas**, p. 90-91, p. 231-233)

(a) We wish to evaluate the Fresnel integral $I = \int_0^\infty \exp\left(ix^2\right) dx$. To do this, consider the contour integral $I_R = \int_{C(R)} \exp\left(iz^2\right) dz$, where $C(R)$ is the closed circular sector in the upper half-plane with boundary points 0, R and $R\exp(i\pi/4)$. Show that $I_R = 0$ and that $\lim_{R\to\infty} \int_{C_1(R)} \exp\left(iz^2\right) dz = 0$, where $C_1(R)$ is the contour integral along the circular sector from R to $R\exp(i\pi/4)$. [Hint: use $\sin x \geq (2x/\pi)$ on $0 \leq x \leq \pi/2$.] Then, by breaking up the contour $C(R)$ into three components, deduce that

$$\lim_{R\to\infty}\left(\int_0^R \exp\left(ix^2\right) dx - \mathrm{e}^{i\pi/4}\int_0^R \exp\left(-r^2\right) dr\right) = 0$$

and, from the well-known result of real integration $\int_0^\infty \exp\left(-x^2\right) dx = \sqrt{\pi}/2$, deduce that $I = \mathrm{e}^{i\pi/4}\sqrt{\pi}/2$.

(b) **(optional)** Consider the integral $I = \int_{-\infty}^{\infty} \left(x^2+1\right)^{-1} dx$. Evaluate this integral by considering $\int_{C(R)} \left(z^2+1\right)^{-1} dz$, where $C(R)$ is the closed semicircle in the upper half-plane with end-points at $(-R,0)$ and $(R,0)$ together with the corresponding part of the x axis. [Hint: express the integrand in terms of partial fractions, and show that the contribution from the semicircle vanishes as $R\to\infty$.] Verify your answer by ordinary integration with real variables.

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Optional problems

Problem 1) Complex integration (**Needham**, p. 420; **Carrier et al.**, p. 36-37)

(a) Write down the values of $\oint_C (1/z)\, dz$ for each of the following choices of C:

(i) $|z| = 1$, (ii) $|z-2| = 1$, (iii) $|z-1| = 2$.
(optional) Then confirm the answers the hard way, using parametric evaluation.

(b) Evaluate parametrically the integral of $1/z$ around the square with vertices $\pm 1 \pm i$.

(c) Confirm by parametric evaluation that the integral of z^m around an origin centered circle vanishes, except when the integer $m = -1$.

(d) Evaluate $\int_{1+i}^{3-2i} dz\, \sin z$ in two ways: (i) via the fundamental theorem of (complex) calculus, and (ii) **(optional)** by choosing any path between the end-points and using real integrals.

Problem 2) More complex integration (**Ablowitz & Fokas**, p. 79-81)

(a) By using parametric integration, evaluate $\oint_C f(z)\, dz$, where C is the unit circle enclosing the origin and $f(z)$ is given by: (i) z^2, (ii) $\bar{z}^2$, (iii) $(z+1)/z^2$.

(b) Evaluate $\oint_C f(z)\, dz$, where C is the unit circle enclosing the origin and $f(z)$ is
(i) $1 + 2z + z^2$, (ii) $1/\big(z - (1/2)\big)^2$, (iii) $1/\bar{z}$, (iv) $\exp \bar{z}$.

(c) Let C be the square with vertices $\pm 1 \pm i$. Evaluate $\oint_C f(z)\, dz$, where $f(z)$ is
(i) $\sin z$, (ii) $1/(2z+1)$, (iii) $\bar{z}$, (iv) $\operatorname{Re} z$.

(d) **(optional)** Let C be an arc of a circle of radius R (with $R > 1$) of angle $\pi/3$. Show that
$$\left| \int_C \frac{dz}{z^3+1} \right| \leq \frac{\pi}{3} \left(\frac{R}{R^3 - 1} \right),$$
and hence deduce that $\lim_{R\to\infty} \int_C dz/(z^3+1) = 0$.

Problem 3) Cauchy's theorem via Green's theorem in the plane

Express the integral $\oint_C dz\, f(z)$ of the analytic function $f = u + iv$ around the simple contour C in parametric form, apply the two-dimensional version of Gauss' theorem (a.k.a. Green's theorem in the plane), and invoke the Cauchy-Riemann conditions. Hence establish Cauchy's theorem $\oint_C dz\, f(z) = 0$.

4) Applications of Cauchy's theorem (**Ablowitz & Fokas**, p. 90-91)

(a) Evaluate $\oint_C f(z)\, dz$, where C is the unit circle centered at the origin and $f(z)$ is
(i) $\exp iz$, (ii) $\exp z^2$, (iii) $1/\big(2z-1\big)$, (iv) $1/\big(z^2-4\big)$, (v) $1/\big(2z^2+1\big)$.

(b) By using partial fractions, evaluate $\oint_C f(z)\, dz$, where C is the unit circle centered at the origin and $f(z)$ is given by
(i) $1/z\big(z-2\big)$, (ii) $z/\big(9z^2-1\big)$, (iii) $1/z\big(2z+1\big)\big(z-2\big)$.

(c) Evaluate $\oint_C dz\, z^{-1}(z-\pi)^{-1} \exp(iz)$ for the following origin-centered contours:
  (i) C is the boundary of the annulus between circles of radius 1 and 3.
  (ii) C is the boundary of the annulus between circles of radius 1 and 4.

(iii) C is the circle of radius R where $R > \pi$.
(iv) C is the circle of radius R where $R < \pi$.

(d) **(optional)** Discuss how to evaluate $\oint dz\, z^{-2} \exp\left(z^2\right)$, where C is a simple closed curved enclosing the origin.

6) Nyquist's stability criterion (**Needham**, p. 371)

Let $Q(t)$ be a function of time obeying the linear ordinary differential equation $c_n Q^{(n)} + c_{n-1} Q^{(n-1)} + \cdots + c_1 Q' + c_0 Q = 0$. with constant complex coefficients $\{c_0, \ldots, c_n\}$. Recall that one solves this equation by taking a linear combination of the special solutions of the form $\exp s_j t$.

(a) Show that the s_j are roots of the polynomial equation
$F(s) \equiv c_n s^n + c_{n-1} s^{n-1} + \cdots + c_1 s + c_0 = 0$.

(b) As an aside, consider the case in which the coefficients $\{c_0, \ldots, c_n\}$ are real. Explain why, even though the roots of $F(s)$ may be complex, a real solution may be obtained.

(c) Now revert to the general case, in which the coefficients $\{c_0, \ldots, c_n\}$ may be complex. All solutions $Q(t)$ will decay with time provides that $\mathrm{Re}\, s_j < 0$ for all roots. Thus, the issue of determining whether all solutions decay reduces to the issue of whether all roots of $F(s)$ lie in the half-plane $\mathrm{Re}\, s < 0$ (*i.e.*, whether the polynomial is a *Hurwitz polynomial*). Let $\mathcal{R}$ be the net rotation of the polynomial $F(s)$ as s moves along the imaginary axis from bottom to top. Explain the following result, known as the Nyquist stability criterion: the general solution of the ordinary differential equation will decay away if and only if $\mathcal{R} = n\pi$.

(d) Consider the ordinary differential equation $d^3Q/dt^3 = Q$. Find $\mathcal{R}$ for this equation. Does it satisfy the Nyquist stability criterion? Confirm your conclusion by explicitly solving the ordinary differential equation.

7) Area of an epicycloid (**Needham**, p. 421)

Hold a coin (of radius A) down on a flat surface and roll another coin (of radius B) round it. The curve traced by a point on the rim of the rolling coin is called an *epicycloid*, and closes if $A = nB$, where n is an integer.

(a) With the centre of the fixed coin at the origin, show that the epicycloid can be represented parametrically as $z(t) = B\left[(n+1)\exp(it) - \exp i(n+1)t\right]$.

(b) By evaluating parametrically the integral for the area enclosed, ***i.e.***, $(1/2i) \oint_C \bar{z}\, dz$, show that the area of the epicycloid is given by $\pi B^2 (n+1)(n+2)$.

8) Quaternions (**Needham**, p. 290-291 & 328-329)

Sir William Rowan Hamilton discovered the following four-dimensional generalization of complex numbers, called the quaternions, in which four-component entities can be multiplied and divided.

- Introduce (as analogues of the unit basis "vectors" 1 and i of complex numbers) the four unit basis "vectors" $\{\mathbf{1}, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$.
- Express a general quaternion $\mathcal{V}$ via the four unit basis "vectors" and their real coefficients $\{v, v_1, v_2, v_3\}$ as $\mathcal{V} = v\mathbf{1} + v_1\mathbf{I} + v_2\mathbf{J} + v_3\mathbf{K}$.

- Endow the basis "vectors" with the following (revolutionary—the year was 1843!) non-commutative multiplication structure:

$$\begin{array}{c} \\ \mathbf{1} \\ \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{I} & \mathbf{J} & \mathbf{K} \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \mathbf{I} & -\mathbf{1} & \mathbf{K} & -\mathbf{J} \\ \mathbf{J} & -\mathbf{K} & -\mathbf{1} & \mathbf{I} \\ \mathbf{K} & \mathbf{J} & -\mathbf{I} & -\mathbf{1} \end{pmatrix}, \end{array}$$

  where the entries correspond to the column label pre-multiplied by the row label, *e.g.*, $\mathbf{IJ} = -\mathbf{JI} = \mathbf{K}$.

Sometimes we suppress the identity factor ($\mathbf{1}$), writing v for the scalar part $v\mathbf{1}$, and we write the remaining (vector) part $(v_1\mathbf{I} + v_2\mathbf{J} + v_3\mathbf{K})$ as $\mathbf{V}$. Thus we have $\mathcal{V} = v + \mathbf{V}$.

(a) Show that $\mathcal{V}\mathcal{W} = (vw - \mathbf{V} \cdot \mathbf{W}) + (v\mathbf{W} + w\mathbf{V} + \mathbf{V} \times \mathbf{W})$, where the dot and cross denote the usual scalar and vector products of three-dimensional vector algebra.

(b) The conjugate $\bar{\mathcal{V}}$ of a quaternion $\mathcal{V}$ is given by $\mathcal{V} = v - \mathbf{V}$. The length $|\mathcal{V}|$ of a quaternion $\mathcal{V}$ is defined via $|\mathcal{V}|^2 \equiv \bar{\mathcal{V}}\mathcal{V}$. Show that $|\mathcal{V}|^2 = |\bar{\mathcal{V}}|^2 = v^2 + \mathbf{V} \cdot \mathbf{V}$.

(c) Show that $\overline{\mathcal{V}\mathcal{W}} = \bar{\mathcal{W}}\,\bar{\mathcal{V}}$ and that $|\mathcal{V}\mathcal{W}| = |\mathcal{V}|\,|\mathcal{W}|$.

(d) $\mathcal{V}$ is a *pure* quaternion if $v = 0$. $\mathcal{V}$ is a *unit* quaternion if $|\mathcal{V}| = 1$. Show that $\mathcal{W}$ is a pure unit quaternion if and only if $\mathcal{W}^2 = -1$.

There are interesting and useful connections between three-dimensional rotations, spinors and quaternions; see, *e.g.*, **Needham**, p. 290 *et seq*.