Georgia Tech PHYS 6124 **Fall 2011**

Mathematical Methods of Physics I

Instructor: Predrag Cvitanović

Homework #12 due Tuesday November 22 2011, in class

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

[All problems in this set are from Goldbart]

Problem 3) Cauchy's integral formula (**Needham**, p. 446)

(a) (i) If C is the unit circle, show that

$$\int_0^{2\pi} \frac{dt}{1 - 2a\cos t + a^2} = \oint_C \frac{i\,dz}{(z-a)(az-1)}.$$

(ii) Use Cauchy's integral formula to deduce that if $0 < a < 1$ then the above integrals are given by $2\pi/\left(1-a^2\right)$.

(b) Let $f(z)$ be holomorphic on and inside a circle K defined by $|z-a| = \rho$, and let M be the maximum value of $|f(z)|$ on K.

(i) Use Cauchy's integral formula for derivatives to show that $|f^{(n)}(a)| \leq n!\,M/\rho^n$.

(ii) Suppose that $|f(z)| \leq M$ for all z, where M is some positive constant. By choosing $n = 1$ in the above inequality, derive Liouville's theorem.

(iii) **(optional)** Suppose that $|f(z)| \leq M\,|z^n|$ for all z, where n is some positive integer. Show that $f^{(n+1)}(z) = 0$, and hence deduce that $f(z)$ must be a polynomial whose degree does not exceed n.

(c) **(optional)**

(i) Show that if C is any simple loop around the origin then

$$\frac{1}{2\pi i}\oint_C \frac{(1+z)^n}{z^{r+1}} dz = \begin{pmatrix} n \\ r \end{pmatrix}.$$

(ii) By taking C to be the unit circle, deduce that $\begin{pmatrix} 2n \\ n \end{pmatrix} \leq 4^n$.

Problem 6-2) Evaluation of definite integrals (**Ablowitz & Fokas**, p. 235-237)

(a) Evaluate the following real integrals via residues (for $a^2, b^2, k > 0$):

(i) $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$ (ii) $\displaystyle\int_{-\infty}^\infty \frac{dx\ \cos kx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$ (iii) $\displaystyle\int_0^\infty \frac{dx\,x\ \sin x}{x^2+a^2}$

(iv) $\displaystyle\int_0^\infty \frac{dx\, x^3\, \sin kx}{x^4+a^4}$ (v) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta}$ (vi) $\displaystyle\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2}$

(b) **(optional)** Evaluate the following real integrals via residues (for $a^2, b^2, k, m > 0$):

(i) $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2}$ (ii) $\displaystyle\int_0^\infty \frac{dx}{\left(x^2+a^2\right)^2}$ (iii) $\displaystyle\int_0^\infty \frac{dx}{\left(x^2+a^2\right)\left(x^2+b^2\right)}$

(iv) $\displaystyle\int_{-\infty}^\infty \frac{dx\, x\, \cos kx}{x^2+4x+4}$ (v) $\displaystyle\int_{-\infty}^\infty \frac{dx\, \cos kx\, \cos mx}{x^2+a^2}$ (vi) $\displaystyle\int_0^{\pi/2} d\theta\, \sin^4\theta$

(c) **(optional)** Use an origin-centered sector contour of radius R and angle $2\pi/5$ to show that (for $a > 0$)

$$\int_0^\infty \frac{dx}{x^5+a^5} = \frac{\pi}{5a^4\sin(\pi/5)}.$$

(d) (i) Via a rectangular contour with corners at $b \pm iR$ and $b+1 \pm iR$, show that

$$\lim_{R\to\infty} \int_{b-iR}^{b+iR} \frac{dz}{2\pi i}\, \frac{e^{az}}{\sin \pi z} = \frac{1}{\pi}\frac{1}{1+\exp(-a)} \qquad (0 < b < 1, |\mathrm{Im}\, a| < \pi).$$

(ii) **(optional)** By using a rectangular contour with corners at $\pm R$ and $\pm R + i$, show that

$$\int_0^\infty dx\, \left(\cosh ax/\cosh \pi x\right) = (1/2)\sec(a/2) \qquad (|a| < \pi).$$

(e) **(optional)**

(i) Use a rectangular contour C_N with corners $\left(N+\frac{1}{2}\right)\left(\pm 1 \pm i\right)$ to evaluate

$$\frac{1}{2\pi i}\oint_{C_N} \frac{dz\, \pi\, \cot \pi z\, \coth \pi z}{z^3}.$$

(ii) By considering the $N \to \infty$ limit of your answer to part (i) show that $\sum_{n=1}^{\infty} n^{-3} \coth n\pi = (7/180)\pi^3$.

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Optional problems

Problem 1) Winding numbers and topology (**Needham**, p. 369-372)

(a) Envisage an arbitrarily complicated but nevertheless *simple* contour. By considering the collection of possible values taken by the winding numbers for off-contour points, devise a fast algorithm for establishing whether or not an arbitrary off-contour point lies inside or outside the contour. [Note: You may use this algorithm to impress your friends at dinner parties.]

(b) For each of the following functions $f(z)$, find all the p-points lying inside the specified disc and determine their multiplicities.

(i) $f(z) = \exp 3\pi z$ and $p = i$ for the disc $|z| \leq 4/3$;

(ii) $f(z) = \cos z$ and $p = 1$ for the disc $|z| \leq 5$;
(iii) $f(z) = \sin\left(z^4\right)$ and $p = 0$ for the disc $|z| \leq 2$.

In each case, use a computer to draw the image of the boundary of the circle and, hence, verify the *argument principle*

(c) Use Rouché's theorem to establish the following results:
   (i) If a is greater than 1 then the equation $z^n e^a = e^z$ has n solutions inside the unit circle.
   (ii) If $f(z) = 2z^5$ and $g(z) = 8z - 1$ then all five of the solutions of the equation $f(z) + g(z) = 0$ lie in the disc $|z| < 2$.
   (iii) By reversing the roles of f and g, show that there is only one root in the unit disc. Hence, deduce that there are four roots in the annulus $1 < |z| < 2$.

Problem 2) Cauchy's theorem (**Needham**, p. 421-423)

(a) Let K be a contour that winds once around $z = 1$, once around $z = 0$, twice around $z = -1$, and not around $z = 1 + i$.
   (i) Evaluate the following integral by factoring the denominator and putting the integrand into partial fractions:
   $$\oint_K \frac{z\,dz}{z^2 - iz - 1 - i}.$$
   (ii) Write down the Laurent series (centered at the origin) for $z^{-11}\cos z$. Hence find
   $$\oint_K \frac{\cos z}{z^{11}}\,dz.$$

(b) This exercise illustrates how one type of integral may be evaluated easily using a complex integral. Let L be the straight contour along the real axis from $-R$ to R, and let J be the semicircular contour in the upper half-plane back from R to $-R$. The complete contour $L + J$ is thus a closed loop.
   (i) By using partial fractions, show that the integral
   $$\oint_{L+J} \frac{dz}{z^4 + 1}$$
   vanishes if $R < 1$, and find its value if $R > 1$.
   (ii) By using the fact that $z^4 + 1$ is the complex number from -1 to z^4, write down the minimum of $|z^4 + 1|$ as Z travels around J. Now think of R as large, and use the Darboux inequality to show that the integral of J dies away to zero as R grows to infinity.
   (iii) From the previous parts, deduce the value of
   $$\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}.$$
   (iv) Although it can be evaluated easily by ordinary means, evaluate the integral
   $$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}$$
   by the method used in the previous parts of this exercise.

(v) Likewise, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}.$$

(c) (i) Write down the value of $\int_0^{a+ib} dz\, \mathrm{e}^z$.

(ii) By equating your answer to part (i) to the parametric form of the integral taken along the straight contour from $z = 0$ to $z = a + ib$, deduce the values of the integrals $\int_0^1 dx\, \mathrm{e}^{ax} \sin bx$ and $\int_0^1 dx\, \mathrm{e}^{ax} \cos bx$.

(d) (i) Show that when integrating a product of holomorphic functions we may use the method of integration by parts.

(ii) Let L be a contour between the real numbers $\pm\theta$. Evaluate $\int_L dz\, z\, \mathrm{e}^{iz}$. Verify your result via parametric integration along the line segment between $\pm\theta$.

(e) Let $f(z) = z^{-1}\left(z + z^{-1}\right)^n$, where n is a positive integer.

(i) Use the binomial theorem to find the residue of f at the origin when n is even or odd.

(ii) If n is odd, determine the value of the integral of f around any contour.

(iii) If n is even (and equal to $2m$) and K is a simple contour winding once around the origin, deduce from part (i) that the integral of f around K is given by $2\pi i\,(2m)!/(m!)^2$.

(iv) By taking K to be the unit circle, deduce Wallis' result:

$$\int_0^{2\pi} d\theta\, \cos^{2m}\theta = \frac{2\pi(2m)!}{2^{2m}(m!)^2}.$$

(v) Similarly, by considering functions of the form $z^k f(z)$ with integral k, evaluate $\int_0^{2\pi} d\theta\, \cos^n\theta\, \cos k\theta$ and $\int_0^{2\pi} d\theta\, \cos^n\theta\, \sin k\theta$.

(f) Let E be the elliptical orbit $z(t) = a\cos t + ib\sin t$, where a and b are positive and t varies from 0 to 2π. By considering the integral of $1/z$ around E, show that

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t} = \frac{2\pi}{ab}.$$

Problem 4) Kramers-Krönig relations

(a) The Debye form of the frequency-dependent generalized response function $\epsilon(\omega)$ is given by

$$\epsilon(\omega) = \epsilon_\infty + \frac{\epsilon_0 - \epsilon_\infty}{1 - i\omega T},$$

where ϵ_0, ϵ_∞ and T are real parameters. Show that this form corresponds to the time-dependent generalized reponse function

$$\alpha(\tau) = \epsilon_\infty\, \delta^{(+)}(\tau) + \left(\epsilon_0 - \epsilon_\infty\right) T^{-1}\, \mathrm{e}^{-\tau/T},$$

where $\delta^{(+)}(\tau)$ is understood to mean $\lim_{\tau_0 \to 0^+} \tau_0^{-1} \exp(-\tau/\tau_0)$ with τ_0 real. Confirm that the Debye form obeys the Kramers-Krönig relations.

(b) The Van Vleck-Weisskopf-Fröhlich form of the time-dependent generalized response function $\alpha(\tau)$ is given by

$$\alpha(\tau) = \epsilon_\infty\, \delta^{(+)}(\tau) + \Delta\epsilon\, T^{-1}\, \mathrm{e}^{-\tau/T}\left(\cos\omega_0\tau + \omega_0 T \sin\omega_0\tau\right),$$

where $\Delta\epsilon$ and ω_0 are further real parameters. Determine the corresponding frequency-dependent generalized response function, and confirm that it obeys the Kramers-Krönig relations.