Georgia Tech PHYS 6124

Mathematical Methods of Physics I

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Homework Set #5

due Sept 27 2011, in class

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

[All problems in this set are from Goldbart]

Problem A) Linear homogeneous ODEs with constant coefficients

Consider the general linear n^{th}-order homogeneous ordinary differential equation with constant coefficients: $y^{(n)}(x) + p_{n-1}\, y^{(n-1)}(x) + \cdots + p_0\, y(x) = 0$. The general solution can be obtained by inserting the hypothesis $y(x) = \mathrm{e}^{rx}$, after which the equation becomes an algebraic (rather than differential) equation for r,

$$r^n + p_{n-1}\, r^{n-1} + \cdots p_1\, r^1 + p_0 = 0\,.$$

Now, the fundamental theorem of algebra guarantees that this equation has exactly n roots $\{r_j\}_{j=1}^{n}$ in the complex plane; they occur as complex conjugate pairs if the constant coefficients $\{p_j\}_{j=1}^{n}$ are real. If all the roots are distinct (*i.e.*, all different) then the general solution is $y(x) = \sum_{j=1}^{n} c_j\, \mathrm{e}^{r_j x}$ where $\{c_j\}_{j=1}^{n}$ are constants of integration.

a) Find the general solution of the equation $y^{(2)} - 5y^{(1)} + 4y = 0$.

   If the roots are not all distinct then we must work a little harder because $y = \sum_{j=1}^{n} c_j\, \mathrm{e}^{r_j x}$ is not the most general solution. For example, suppose that the first m roots are equal: $r_1 = r_2 = \ldots = r_m = \rho$. Then the algebraic equation becomes $(r-\rho)^m Q(r) = 0$, where Q is a polynomial of order $n-m$.

b) Show that $(\partial/\partial r)^\ell \mathrm{e}^{rx}|_{r=\rho}$ (for $\ell = 0, 1, \ldots, m-1$) are solutions of the ordinary differential equation.

c) Apply this method by finding the general solution to the ordinary differential equation $y^{(3)} - 3y^{(2)} + 3y^{(1)} - y = 0$.

Problem B) Series solutions of ordinary differential equations

a) Show that if m is not zero or an integer then the equation

$$\frac{d^2u}{dx^2} + \left(\frac{\frac{1}{4} - m^2}{x^2} - \frac{1}{4} \right) u = 0$$

is satisfied by two series about $x = 0$ with leading terms

$$x^{\frac{1}{2}+m} \left(1 + \frac{x^2}{16(1+m)} + \cdots \right), \qquad x^{\frac{1}{2}-m} \left(1 + \frac{x^2}{16(1-m)} + \cdots \right).$$

Determine the recursion relation for the coefficient of the general term in each series, and show that the series converge for all values of x. (From Whittaker and Watson.)

b) By expanding about the regular singular point $x = 1$, find the series solutions of Legendre's equation of order zero:

$$\left(1 - x^2\right) \frac{d^2u}{dx^2} - 2x \frac{du}{dx} + \lambda u = 0.$$

Show that both roots of the indicial equation vanish. Find the first three terms of the regular solution $P(x)$. Suppose that the singular solution $Q(x)$ has the form

$$Q(x) = P(x)\ \ln(x-1) + \left(x - 1\right)\left(b_0 + b_1(x-1) + b_2(x-1)^2 + \cdots\right).$$

Find the coefficients b_0 and b_1 of the singular solution.

c) **Optional**: Show that the solutions of the equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - m^2 u = 0$$

near $x = 0$ are

$$u_1(x) = 1 + \sum_{n=1}^{\infty} \frac{m^{2n} x^{2n}}{2^{2n}(n!)^2}, \qquad u_2(x) = u_1(x)\ \ln x - \sum_{n=1}^{\infty} \frac{m^{2n} x^{2n}}{2^{2n}(n!)^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} \right).$$

Show that these solutions converge for all values of x. (From Whittaker and Watson.)

Optional problems

Problem 2) Separability

Show that Helmholtz's equation $\left(\nabla^2 + k^2\right) u(r, \theta, z) = 0$ remains separable if $\{r, \theta, z\}$ represent circular cylindrical coordinates and k^2 is replaced by $k^2 + f(r) + g(\theta)/r^2 + h(z)$.

Problem 3) Linear homogeneous second order ordinary differential equations

The purpose of this question is to derive some properties of the general linear homogeneous second order ordinary differential equation,

$$\frac{d^2 y}{dx^2} + p(x)\,\frac{dy}{dx} + q(x)\,y = 0\,,$$

and then to exhibit these properties for specific cases. Suppose you have two solutions to the above ordinary differential equation, namely $u_1(x)$ and $u_2(x)$.

a) Show that the condition for $u_1(x)$ and $u_2(x)$ to be linearly independent is that their Wronskian does not vanish identically. Show that the Wronskian does not vary, apart from a multiplicative constant, when $u_1(x)$ and $u_2(x)$ are replaced by alternative linear combinations, *i.e.*,

$$\begin{aligned} u_1(x) &\rightarrow a_1\,u_1(x) + a_2\,u_2(x) \\ u_2(x) &\rightarrow b_1\,u_1(x) + b_2\,u_2(x) \end{aligned}$$

where a_1, a_2, b_1 and b_2 are constants. What property does the new linear combination have if the multiplicative constant makes the Wronskian vanish?
[Note: *vanish identically* means be zero everywhere throughout the range of values of x relevant to the case at hand, not simply at some isolated points.]

b) Show that if the Wronskian of the two solutions vanishes identically then $u_2(x) = c_1\,u_1(x)$, where c_1 is a constant (*i.e.* the solutions are not linearly independent).

The problem of computing the Wronskian reduces to quadratures (*i.e.* to the evaluation of an integral), even for linear homogeneous ordinary differential equations of higher than second order. For the case of second order ordinary differential equations, this becomes particularly useful when one solution is known and a second linearly independent solution is desired.

c) To see this, show that knowledge of the Wronskian $W(x)$ and one solution $u_1(x)$ is sufficient to reduce to quadratures the problem of finding a second, linearly independent, solution.

d) Show that there can be no more than two linearly independent analytic solutions of a second order ordinary differential equation in the neighbourhood of a regular point x_0.

e) Consider Legendre's equation with $\ell = 1$ and $m = 0$, *i.e.*, $\big((1-x^2)y'\big)' + 2y = 0$. Given that $y(x) = x$ is a solution, compute the Wronskian and use the Wronskian and the given solution to construct a linearly independent solution.

f) Compute the Wronskian of the three functions e^x, e^{-x} and $\cosh x$. Are these three functions linearly independent?

Problem 4) The wave equation

Consider the wave equation in one spatial variable x and one temporal variable t,

$$\partial_{tt}\, u - c^2\, \partial_{xx}\, u = 0.$$

a) Transform from the independent variables x and t to the characteristic variables $\xi \equiv (x - ct)$ and $\eta \equiv (x + ct)$, and determine the resulting form of the wave equation.

b) By making use of your answer to part (a) determine the general solution $u(x, t)$ of the wave equation.

c) Interpret the two contributions to $u(x, t)$.

d) Suppose that Cauchy boundary conditions are given for all x at the time $t = 0$, i.e., it is given that $u(x, t)|_{t=0} = \alpha(x)$ and $\partial_t u(x, t)|_{t=0} = \beta(x)$. Determine, in terms of $\alpha(x)$ and $\beta(x)$, the solution to the wave equation throughout space-time.

   Now consider waves in a finite segment of space $0 \le x \le \ell$, and suppose that homogeneous Dirichlet boundary conditions are applied at all times at the ends $x = 0$ and ℓ.

e) Explain how these boundary conditions lead to recursion relations which, together with the initial conditions, determine the solution to the wave equation in the segment $0 \le x \le \ell$ for all time.

Problem 5) Series solution at an ordinary point

Consider the first-order ordinary differential equation: $(1 + x^2)\, y' + 2xy = 0$.

a) Classify the point $x = 0$.

b) Show that $y = c\,(1 + x^2)^{-1}$ is a solution.

c) Suppose you did not know the exact solution. Construct the series solution by expanding around the point $x = 0$. What is the radius of convergence of your series?

d) What feature of the ordinary differential equation is responsible for the finite radius of convergence?

Problem 6) Separation of variables for non-linear partial differential equations

Occasionally, the method of separation of variables is useful for non-linear ordinary differential equations. Although superposition is not now legitimate, it is sometimes possible to obtain a useful solution, as the following example shows. Consider the partial differential equation

$$f(x)\,u_x^2 + g(y)\,u_y^2 = a(x) + b(y),$$

where f, g, a and b are presumed known.

a) By hypothesising a solution with the *additively separated* form $u(x, y,) = \phi(x) + \psi(y)$ derive a solution of this partial differential equation with the form

$$u(x, y,) = \beta + \int_{x_0}^{x} dx' A\left(\frac{a(x') + \alpha}{f(x')}\right) + \int_{y_0}^{y} dy' B\left(\frac{b(y') - \alpha}{g(y')}\right)$$

in terms of one separation constant α, one integration constant β and two functions A and B. State the form of the functions A and B.

b) Apply this method to determine a solution when the equation is specified by $a(z) = b(z) = f(z) = g(z) = z^2$ and the boundary conditions are $u(0, 0) = 0$ and $u(x, y,) = u(y, x)$.

Problem 7) Frullanian integrals

(After Zwillinger: *Handbook of Integration*.) A convergent integral can sometimes be written as the difference of two integrals that each diverge. If these two integrals diverge in the same way, then the difference may be evaluated by certain limiting processes. Consider the convergent integral: $I = \int_0^\infty x^{-2} \sin^3 x\, dx$.

a) Rewrite this integral as $I = \frac{1}{4}\int_0^\infty x^{-2}\,(3\sin x - \sin 3x)\,dx$, and explain why it cannot be rewritten as $I = \frac{3}{4}\int_0^\infty x^{-2} \sin x\, dx - \frac{1}{4}\int_0^\infty x^{-2} \sin 3x\, dx$.

b) Explain why I can, however, be rewritten as:

$$I = \lim_{\delta \to 0} \left(\frac{3}{4} \int_\delta^\infty x^{-2} \sin x \, dx - \frac{1}{4} \int_\delta^\infty x^{-2} \sin 3x \, dx \right).$$

c) Show, by using the change of variables $x \to y = 3x$ in the second integral, that $I = \frac{3}{4} \log 3$.

d) Generalise the above procedure to derive the rule:

$$\int_0^\infty x^{-1} \Big(f(ax) - f(bx) \Big) dx = \Big[f(\infty) - f(0) \Big] \; \log(a/b).$$

e) Use this generalisation to evaluate $\int_0^\infty x^{-1} \Big(\tanh(ax) - \tanh(bx) \Big) dx$.

Problem 8) Coupled ordinary differential equations

a) Consider the system of N coupled first-order ordinary differential equations:

$$\frac{d}{dt} x_i(t) = \sum_{j=1}^{N} A_{ij} \, x_j(t),$$

where we assume that the constant $N \times N$ matrix A has N linearly independent right eigenvectors. Show that the general solution can be written as

$$x_i(t) = \sum_{\ell=1}^{N} C^{(\ell)} \, x_i^{(\ell)} \, \mathrm{e}^{\lambda^{(\ell)} t},$$

where $x_i^{(\ell)}$ and $\lambda^{(\ell)}$ are, respectively the right eigenvectors and eigenvalues of A, and $C^{(\ell)}$ are arbitrary constants that can be chosen to fit the initial conditions.

b) Apply this technique to the problem of finding the general solution of the system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 1 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Problem 9) Lyapunov equation

Consider the following system of ordinary differential equations,

$$\frac{d}{dt} \mathbf{Z}(t) = \mathbf{K}(t) \, \mathbf{Z}(t) + \mathbf{Z}(t) \, \mathbf{K}^{\mathrm{T}}(t) + \mathbf{G}(t) \,,$$

in which $\mathbf{Z}$, $\mathbf{K}$ and $\mathbf{G}$ are $N \times N$ matrix functions of t, with $\mathbf{K}$ and $\mathbf{G}$ supposed known, and $\mathbf{Z}$ sought. The superscript T indicates the transpose of the matrix. Find the solution $\mathbf{Z}(t)$, subject to the boundary condition $\mathbf{Z}(t_0) = \mathbf{Z}_0$, by taking the following steps:

i) Write the solution in the form $\mathbf{Z}(t) = \mathbf{Q}(t)\,\mathbf{Z}_0\,\mathbf{Q}^{\mathrm{T}}(t) + \mathbf{Q}(t)\,\mathbf{W}(t)\,\mathbf{Q}^{\mathrm{T}}(t)$, with $\mathbf{Q}(t)$ satisfying $\frac{d}{dt}\mathbf{Q}(t) = \mathbf{K}(t)\,\mathbf{Q}(t)$, subject to the initial condition $\mathbf{Q}(t_0) = \mathbf{I}$, in which $\mathbf{I}$ is the $N \times N$ identity matrix.

ii) Show that $\mathbf{W}(t)$ then satisfies $\frac{d}{dt}\mathbf{W}(t) = \mathbf{Q}^{-1}(t)\,\mathbf{G}(t)\,\big(\mathbf{Q}^{\mathrm{T}}\big)^{-1}(t)$, subject to the initial condition $\mathbf{W}(t_0) = \mathbf{O}$.

iii) Integrate the preceding equation to obtain

$$\mathbf{Z}(t) = \mathbf{Q}(t)\,\mathbf{Z}_0\,\mathbf{Q}^{\mathrm{T}}(q) + \int_{t_0}^{t} dk\,\mathbf{Q}(t)\,\mathbf{Q}^{-1}(k)\,\mathbf{G}(k)\,\big(\mathbf{Q}^{\mathrm{T}}\big)^{-1}(k)\,\mathbf{Q}^{\mathrm{T}}(t)\,,$$

in terms of (the as yet unknown) matrix $\mathbf{Q}(k)$.

iv) $\mathbf{Q}(t)$ can also be found, at least formally, as $\mathbf{Q}(t) = \hat{T}\exp\left\{\int_{t_0}^{t} dk\,\mathbf{K}(k)\right\}$, where $\hat{T}$ denotes the 'time-ordering' operation.

v) Show that if $\mathbf{K}(k)$ commutes with itself throughout the interval $t_0 \le k \le t$ then the 'time-ordering' operation is redundant, and we have the explicit solution $\mathbf{Q}(t) = \exp\left\{\int_{t_0}^{t} dk\,\mathbf{K}(k)\right\}$. Show that, in this case, the complete solution reduces to

$$\mathbf{Z}(t) = \mathrm{e}^{\int_{t_0}^{t} dk\mathbf{K}(k)}\;\mathbf{Z}_0\;\mathrm{e}^{\int_{t_0}^{t} dk\mathbf{K}^{\mathrm{T}}(k)} + \int_{t_0}^{t} dp\,\mathrm{e}^{\int_{p}^{t} dk\,\mathbf{K}(k)}\,\mathbf{G}(p)\,\mathrm{e}^{\int_{p}^{t} dk\,\mathbf{K}^{\mathrm{T}}(k)}\,.$$