Green's Function

So far we have mostly dealt with the BVP which were homogeneous, i.e.

$$\begin{cases} (pu')' + qu = f \\ A_1 u(a) + B_1 u'(a) = c_1 \\ A_2 u(b) + B_2 u'(b) = c_2 \end{cases} \quad \Bigg\| \quad \begin{aligned} f(x) &= 0 \\ c_1 &= 0 \\ c_2 &= 0 \end{aligned}$$

What happens when either $f(x) \neq 0$ or $c_1 \neq 0$ or $c_2 \neq 0$?

Suppose we find a <u>particular</u> solution $\hat{u}$ such that it satisfies the <u>inhomogeneous</u> problem, then $\tilde{u} = u - \hat{u}$ satisfies the <u>homogeneous</u> problem, i.e.,

$$u = \hat{u} + \tilde{u}$$

$\hat{u}$: particular solution of inhomogeneous BVP

$\tilde{u}$: general solution of homogeneous BVP

How do we find a particular solution of the inhomogeneous BVP?

Assume inhomogeneity is in the diff. eq.: $\mathcal{L}[u] = f$

Expand both $u(x)$ and $f(x)$ in the basis of eigenfunctions of $\mathcal{L}$:

$$u(x) = \sum_n C_n \psi_n(x), \quad f(x) = \sum_n F_n \psi_n(x)$$

$$\Rightarrow \sum_n \lambda_n C_n \psi_n(x) = \sum_n F_n \psi_n(x)$$

Eigenfunctions are linearly independent $\Rightarrow C_n = \dfrac{F_n}{\lambda_n} = \dfrac{\langle \psi_n | f \rangle}{\lambda_n}$

$$u(x) = \sum_n \frac{\langle \psi_n | f \rangle}{\lambda_n} \psi_n(x) = \int \sum_n \frac{\psi_n(x) \psi_n^*(x')}{\lambda_n} f(x') dx'$$

$$\equiv \int G(x,x') f(x') dx', \quad \text{where}$$

$$\boxed{G(x,x') = \sum_n \frac{\psi_n(x) \psi_n^*(x')}{\lambda_n}} \longleftarrow \text{is the Green's function}$$

Notes: 1) Cannot solve the inhomogeneous equation when $\lambda_n = 0$ for some n, unless $\langle \psi_n | f \rangle = 0$, i.e., $f \perp \psi_n$.

2) Green's function is symmetric:

$$G^*(x,x') = G(x',x)$$

3) Calculation of Green's function reduces the solution of an inhomogeneous BVP to quadratures for any $f(x)$.

Example (Loaded String)

$$u'' = f(x), \quad u(0) = u(L) = 0$$

eigenstuff: $\psi_n'' = \lambda_n \psi_n$: $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n}{L} x\right)$, $\lambda_n = -\left(\frac{\pi n}{L}\right)^2$

$$\Rightarrow \quad G(x,x') = \sum_n \frac{2}{L} \frac{\sin(\pi n x/L)\sin(\pi n x'/L)}{-(\pi n/L)^2}$$

$$u(x) = \int_0^L G(x,x') f(x') dx'$$

Formally, we can write:

$$\mathcal{L}[u] = f \quad \Rightarrow \quad u = \mathcal{L}^{-1}[f] =$$

$$= \int G(x,x') f(x') dx' = G[f]$$

differential operator

integral operator

$$\Rightarrow \boxed{G = \mathcal{L}^{-1}}$$

Let us determine the differential equation satisfied by $G(x,x')$:

Take $f(x) = \delta(x - x_0)$:

$$u(x) = \int G(x,x') \delta(x' - x_0) dx' = G(x,x_0)$$

$$\Rightarrow \boxed{\mathcal{L} G(x - x_0) = \delta(x - x_0)} \quad (\mathcal{L} \text{ acts on } x, \text{ not } x_0)$$

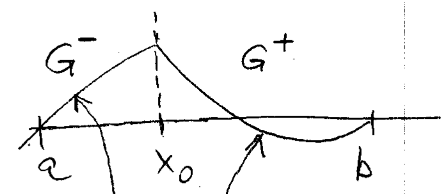
$\Rightarrow$ GF is the solution of the BVP for a unit "point source" $f = \delta(x - x_0)$

Solution in terms of eigenfunctions & eigenvalues is not always convenient (who wants to work with infinite series?)

Alternative way to find the GF:

$$\mathcal{L}G(x,x_0) = \delta(x-x_0) = 0, \quad x \neq x_0.$$

$$G(x,x_0) = \begin{cases} G^-(x), & x < x_0 \\ G^+(x), & x > x_0 \end{cases}$$



Solutions of homogeneous BVP

Two extra boundary conditions:

① $G^-(x_0) = G^+(x_0)$

② $\displaystyle\int_{x_0-\varepsilon}^{x_0+\varepsilon} \mathcal{L}G(x,x_0)\,dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x-x_0)\,dx = 1, \quad \varepsilon \to 0^+$

Example: (Loaded String)

$$u'' = f, \quad u(0) = u(L) = 0$$

$$G'' = \delta(x-x_0): \quad G(x,x_0) = \begin{cases} \overbrace{Ax+B}^{G^-(x)}, & 0 < x < x_0 \\ \underbrace{Cx+D}_{G^+(x)}, & x_0 < x < L \end{cases}$$

B.c.: ① $G(0) = B = 0$

② $G(L) = CL + D = 0 \Rightarrow D = -CL$

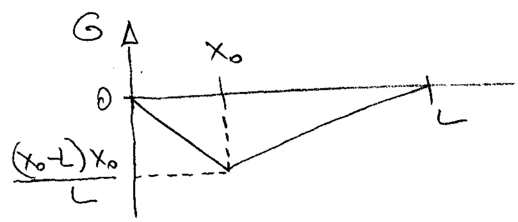
③ $G^+(x_0) = G^-(x_0) \Rightarrow Ax_0 = C(x_0 - L)$

④ $\displaystyle\int_{x_0-\varepsilon}^{x_0+\varepsilon} G''(x,x_0)\,dx = G^+(x_0+\varepsilon)' - G^-(x_0-\varepsilon)' = 1, \quad \varepsilon \to 0^+$

$$\Rightarrow C - A = 1$$

$$\Rightarrow C = 1 + A, \quad Ax_0 = (A+1)(x_0 - L) = Ax_0 + x_0 - AL - L \Rightarrow A = \frac{x_0 - L}{L}$$

$$G(x,x_0) = \begin{cases} \dfrac{(x_0-L)x}{L}, & x < x_0 \\ \dfrac{(x-L)x_0}{L}, & x > x_0 \end{cases}$$



Symmetric, as expected!

$$u(x) = \int_0^L G(x,x_0) f(x_0)\,dx_0 = \int_0^x \frac{(x-L)x_0}{L} f(x_0)\,dx_0 + \int_x^L \frac{(x_0-L)x}{L} f(x_0)\,dx_0$$

Often it is difficult to find solution of $\mathcal{L}G = \delta(x-x')$ which satisfies the boundary conditions explicitly. Split GF into two parts:

$$G(x,x') = U(x,x') + V(x,x'), \quad a < x, x' < b$$

Fundamental solution, $U(x,x')$:

$$\mathcal{L}U = \delta(x-x'), \quad \text{"any" boundary conditions}$$

As soon as $U(x,x')$ is found, solve for $V(x,x')$:

$$\mathcal{L}V = 0, \quad V(a,x') = -U(a,x') \Rightarrow G(a,x') = 0 \quad \checkmark$$

$$V(b,x') = -U(b,x') \Rightarrow G(b,x') = 0. \quad \checkmark$$

ODE: $\mathcal{L}G = \mathcal{L}U + \mathcal{L}V = \delta(x-x') + 0 \quad \checkmark$

<u>Example</u> (Loaded String)

$$u'' = f(x), \quad u(0) = u(L) = 0$$

First find $U(x,x')$: $\quad U'' = \delta(x-x')$

Heaviside (step) function

$$\int(\;)dx: \quad U' = \int \delta(x-x')\,dx' \equiv h(x-x') = \begin{cases} 0, & x < x' \\ 1, & x > x' \end{cases}$$

$$\int(\;)dx: \quad U = \int h(x-x')\,dx = \begin{cases} 0, & x < x' \\ x - x', & x > x' \end{cases}$$

Now find $V(x,x')$: $\quad V'' = 0 \Rightarrow V = Ax + B$

$$\begin{cases} G(0,x') = U(0,x') + V(0,x') = B = 0 \\ G(L,x') = U(L,x') + V(L,x') = L - x' + AL = 0 \end{cases}$$

$$\Rightarrow A = -\frac{L-x'}{L}$$

$$G(x,x') = \begin{cases} 0 - \frac{L-x'}{L}x = \frac{x'-L}{L}x, & x < x' \\ x - x' - \frac{L-x'}{L}x = \frac{x-L}{L}x', & x > x' \end{cases}$$

<u>Note:</u> same as the GF obtained previously.