# From Lagrange to Hamilton

Reminder

- generalized coordinates $q_i, \dot{q}_i$ for each particle.
- Lagrangian $\mathcal{L}(q_1, \dots q_N, \dot{q}_1, \dots, \dot{q}_N)$

for mechanical systems, $\mathcal{L} = \underset{\text{kinetic}}{T} - \underset{\text{potential}}{V}$

- equations of motion: Lagrange's equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \text{for } i = 1, \dots, N$$

remark: $\Leftrightarrow A = \int_{t_1}^{t_2} \mathcal{L}\, dt$ stationary

$\Leftrightarrow$ N second order ordinary differential equations

$(q_i(0), \dot{q}_i(0))$ e.g. for $T = \frac{1}{2}\dot{q}\, \Pi(q)\, \dot{q}$

Hamilton

definition: generalized momenta

for a Lagrangian mechanical system $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

here, $p_j = p_j(q_i, \dot{q}_i, t)$

remark: Lagrange's equations become $\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}$

definition: Hamiltonian (function) = Legendre transform of the Lagrangian

$$H(q, p, t) = \dot{q}_i p_i - \mathcal{L}(q, \dot{q}, t)$$ ⚠ where $\dot{q} = \dot{q}(q, p, t)$

> the new variables are $(q_i, p_i, t)$ instead of $(q_i, \dot{q}_i, t)$

Equations of motion?

$$\frac{\partial H}{\partial p_i} = \dot{q}_i + p_j \frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_i}$$ and using $p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$

one gets $\frac{\partial H}{\partial p_i} = \dot{q}_i$

$$\frac{\partial H}{\partial q_i} = p_j \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial \mathcal{L}}{\partial q_i} - \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} = -\dot{p}_i$$

Hamilton's equations of motion (canonical) : $\dot{q}_i = \frac{\partial H}{\partial p_i}$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

remarks: i/ 2N first order ordinary differential equations

ii/ for $H(q,p)$ i.e. no explicit time dependence

$$\frac{dH}{dt} = \dot{q}_i \frac{\partial H}{\partial q_i} + \dot{p}_i \frac{\partial H}{\partial p_i} = 0$$

H is a conserved quantity

time invariance symmetry $\Rightarrow$ energy conservation

continuous symmetry $\Rightarrow \exists$ conserved quantity (Noether's theorem)

iii/ otherwise: $\frac{dH}{dt} = \frac{\partial H}{\partial t}$

iv/ For simple mechanical systems $\mathcal{L} = \underset{\frac{1}{2}\dot{q}M(q)\dot{q}}{\underset{\parallel}{T(q,\dot{q})}} - V(q)$

then $H = \underbrace{p\dot{q}}_{=2T} - T + V$

v/ H is in general (but not always) the energy of the system.

vi/ $\nabla \cdot v = 0$ (Liouville) incompressibility in phase space.

## Hamilton meets Poisson

definition: canonical Poisson bracket — For functions F, G of the phase space variables $(p_i, q_i)$, bilinear operation:

$$\{F,G\} = \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

properties:
- bilinearity $\{\alpha F + G, H\} = \alpha\{F,H\} + \{G,H\}$
- antisymmetry $\{F,G\} = -\{G,F\}$ or $\{F,F\} = 0$
- Leibniz rule $\{FG,H\} = F\{G,H\} + \{F,H\}G$
- Jacobi identity $\{\{F,G\},H\} + \underset{\{\{H,F\},G\} + \{\{G,H\},F\}}{\underset{\parallel}{\circlearrowleft}} = 0$

Hamilton's equations of motion become $\dot{F} = \{F, H\}$ (*)

$$\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i}$$

F observable $F(q_i, p_i)$

$$\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}$$

Autonomous versus non-autonomous Hamiltonian systems

$H(q_i, p_i, t)$ we introduce a new variable $E$ canonically conjugated to $t$ such that:

$$\dot{E} = -\frac{\partial \mathcal{H}}{\partial t} \qquad \dot{t} = \frac{\partial \mathcal{H}}{\partial E}$$

$\mathcal{H}(q_i, p_i, T, E) = E + H(q_i, p_i, T)$ no longer dependent on the evolution parameter $t$ but on a new variable $T$ such that $\dot{T} = 1$.

→ In what follows, all Hamiltonians will be time-independent (autonomous).

Solution of (*): $F(t) = e^{t\{H\}} F(0)$

where $\{H\}$ is the Liouville operator

$$\{H\}F = -\{H, F\}$$

⇒ formal solution, mainly useless.

definition: Poisson bracket – a bilinear operator satisfying antisymmetry, Leibniz, Jacobi.

exemple: – canonical Poisson bracket
– commutator (quantum mechanics)

definition: Hamiltonian system – dynamical system whose dynamics is given by a scalar function $H$ and a Poisson bracket such that an observable $F$ evolves as

$$\dot{F} = \{F, H\}$$

variables : $z_1, \dots, z_N$ → belong to phase space.
observables : $F(z_1, \dots, z_N)$ scalar functions of the variables

canonical Hamiltonian system ⇒ $N = 2n$    $n$ number of degrees of freedom.

remark : classical mechanics as linear as quantum mechanics.
$F_1, F_2$ two observables

$$\dot{\overline{F_1 + \alpha F_2}} = \{F_1, H\} + \alpha\{F_2, H\} = \{F_1 + \alpha F_2, H\}.$$

Canonical versus non-canonical Hamiltonian systems :
example: charged particle in electromagnetic fields $(\vec{E}, \vec{B})$

- variables : $(p, q)$ + $(t, \mathcal{E})$ if $(\vec{E}, \vec{B})$ depend on time
$p$ canonical momentum

$$H(p, q, t) = \frac{(p - eA)^2}{2} + eV \quad \text{where} \quad B = \nabla \times A, \quad E = -\nabla V - \frac{\partial A}{\partial t}$$

$(A, V)$ not physical : gauge $\tilde{A} = A + \nabla U$
$\tilde{V} = V - \frac{\partial U}{\partial t}$

hence $p$ is not physical

- change of variables : move to physical variables
$v = p - eA$ velocity (or kinetic momentum)

$$\overline{H}(x, v, t) = \frac{v^2}{2} \qquad (\text{when } V = 0 \text{ for simplicity})$$

$$\{\overline{F}, \overline{G}\} = \frac{\partial \overline{F}}{\partial x_i}\frac{\partial \overline{G}}{\partial v_i} - \frac{\partial \overline{F}}{\partial v_i}\frac{\partial \overline{G}}{\partial x_i} + eB \cdot \left(\frac{\partial \overline{F}}{\partial v} \times \frac{\partial \overline{G}}{\partial v}\right)$$

compared with $\{F, G\} = \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$    non-canonical part.

- canonical
  - non-physical
  - simple Poisson bracket

→

- non-canonical
  - physical
  - more complicated Poisson bracket

definition: a change of coordinates which does not change the expression of a Poisson bracket is called a canonical change of coordinates.

example: Hamilton's equations are not affected by a canonical change of coordinates

$$(p_i, q_i) \xrightarrow[\text{canonical}]{} (p_i', q_i')$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{p}_i' = -\frac{\partial H'}{\partial q_i'} \qquad \text{with } H(p_i, q_i) = H'(p_i', q_i')$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{q}_i' = \frac{\partial H'}{\partial p_i'}$$

$$\{F, G\} = \frac{\partial F}{\partial z_i} \mathbb{J}_{ij}(z) \frac{\partial G}{\partial z_j} \qquad \text{we show that } \mathbb{J}_{ij}(z) = \{z_i, z_j\}$$

$\mathbb{J}$ Poisson matrix, antisymmetric.

remarks: i) if N odd then $\mathbb{J}$ is not invertible

ii) as a consequence, there exists a new class of invariants called Casimir invariants $C(z)$ such that $\{F, C\} = 0$ for all observable F.

$\nabla C$ belongs to the kernel of $\mathbb{J}$.

iii) the number of variables is not necessarily finite

↳ field theory

example: $\rho(x)$ fluid density
$v(x)$ fluid velocity

continuity equation: $\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v)$

momentum equation: $\frac{\partial v}{\partial t} = -(v \cdot \nabla) v - \frac{\nabla P}{\rho}$

$$H = \int d^3x \left[\rho \frac{v^2}{2} + \rho U(\rho)\right] \qquad P = \rho^2 \frac{\partial U}{\partial \rho}$$

$$\{F, G\} = -\int d^3x \left[F_\rho \nabla \cdot G_v - \nabla \cdot F_v G_\rho + \frac{\nabla \times v}{\rho} \cdot (F_v \times G_v)\right]$$

functional derivatives.