Matrices and ODEs

ODE with constant coefficients:

$$y^{(n)} + a_n\, y^{(n-1)} + \dots + a_3 \ddot{y} + a_2 \dot{y} + a_1 y = 0.$$

homogeneous

can be converted into a system of n linear ODEs, if we define n new functions:

$$y^{(k-1)} = y_k\,, \quad k = 1, \dots, n$$

$$\Rightarrow \begin{cases} \dot{y}_k = y_{k+1}\,, & k = 1, \dots, n-1 \\ \dot{y}_n = -a_n y_n - \dots - a_2 y_2 - a_1 y_1 = -\sum_{l=1}^{n} a_l y_l \end{cases}$$

or in matrix form:

$$\dot{\vec{y}} \equiv \begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_n \end{pmatrix} = A\vec{y}\,, \qquad A = \left(\begin{array}{ccc|c} 0 & 1 & 0 & \\ & 0 & 1 & 0 \\ 0 & & \ddots & \\ \hline -a_1 & \cdots & -a_{n-1} & -a_n \end{array}\right)$$

Find solution:

Example: stability of solutions to nonlin. laser equations

1) Diagonalize A:

$A\vec{e}_i = \lambda_i \vec{e}_i$, $\det(A - \lambda \mathbb{I}) = 0$ – Secular equation

$$\Downarrow$$

$$\vec{e}_i:\ S = [\vec{e}_1 \cdots \vec{e}_n] \xrightarrow{\lambda_i} S^{-1}AS = \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

2) Change coordinates: $\vec{x} = S^{-1}\vec{y}$

$$\dot{\vec{x}} = S^{-1}\dot{\vec{y}} = S^{-1}A\vec{y} = (S^{-1}AS)S^{-1}\vec{y} = \Lambda\vec{x}$$

$$\Rightarrow \dot{x}_i = \lambda_i x_i \Rightarrow x_i = e^{\lambda_i t} x_i(0) \Rightarrow \vec{x} = e^{\Lambda t}\vec{x}(0)$$

3) Change coordinates back:

$$\vec{y}(t) = S\vec{x} = S e^{\Lambda t} S^{-1} \vec{y}(0) = S\left(1 + \Lambda t + \Lambda^2 \frac{t^2}{2} + \dots\right) S^{-1} \vec{y}(0) =$$

$$= \left(1 + (S\Lambda S^{-1})t + (S\Lambda S^{-1})(S\Lambda S^{-1})\frac{t^2}{2} + \dots\right)\vec{y}(0) = e^{At}\vec{y}(0)$$

Evolution operator

General solution for <u>n-th order ODE</u> or <u>system of n ODEs</u>:

$$\vec{y} = e^{At}\vec{y}(0)$$

$(y_1(0), \dots, y_n(0)) \leftarrow$ n-constants

$$y \equiv y_1 = (S\vec{x})_1 = \sum_k \underbrace{S_{1k} x_k(0)}_{\alpha_k} e^{\lambda_k t} = \sum_k \alpha_k e^{\lambda_k t}$$

Unfortunately, life is not always so simple.

Suppose, A is degenerate: $\lambda_i = \lambda_j$, $i \neq j$

$$\alpha_i e^{\lambda_i t} + \alpha_j e^{\lambda_j t} = (\alpha_i + \alpha_j) e^{\lambda_i t}$$

just <u>one</u> independent constant!

What is going on?

<u>Example</u> (First glimpse of perturbation theory)

$$\ddot{y} - 2\dot{y} + y = 0, \quad y(0) = y_0, \quad \dot{y}(0) = v_0$$

Convert to a system of 2 ODEs: $\dot{y} = x \Rightarrow \dot{x} - 2x + y = 0$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} +2x - y \\ x \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det \begin{pmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = (2-\lambda)(-\lambda) + 1 = \lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = 1$$

Let us take $\lambda_1 = 1$, $\lambda_2 = 1 + \varepsilon \neq \lambda_1$ and then $\varepsilon \to 0$.

■ $y = (A - B)e^t + Be^{(1+\varepsilon)t}$

$y(0) = A - B + B = y_0 \Rightarrow A = y_0$

$\dot{y}(0) = (A - B) + B(1 + \varepsilon) = A + B\varepsilon = v_0 \Rightarrow B = \frac{v_0}{\varepsilon} - \frac{y_0}{\varepsilon}$

$$\Rightarrow y = \left(y_0 - \frac{v_0}{\varepsilon} + \frac{y_0}{\varepsilon}\right)e^t + \left(\frac{v_0}{\varepsilon} - \frac{y_0}{\varepsilon}\right)e^{\varepsilon t} e^t =$$

$$= \left(y_0 - \frac{v_0}{\varepsilon} + \frac{y_0}{\varepsilon} + \left(\frac{v_0}{\varepsilon} - \frac{y_0}{\varepsilon}\right)\left(1 + \varepsilon t + \varepsilon^2 \frac{t^2}{2} + \dots\right)\right)e^t =$$

$$= \left(y_0 - \cancel{\frac{v_0}{\varepsilon}} + \cancel{\frac{y_0}{\varepsilon}} + \cancel{\frac{v_0}{\varepsilon}} - \cancel{\frac{y_0}{\varepsilon}} + v_0 t - y_0 t + O(\varepsilon)\right)e^t \xrightarrow[\varepsilon \to 0]{} y_0 e^t + (v_0 - y_0)te^t$$

secular term

How do we find solutions, if more than two eigenvalues (or roots of the characteristic equation) are degenerate?

Let us assume that:

$$\dot{\vec{y}} = A\vec{y}, \text{ where } \quad ① \ A \text{ is } n\times n$$
$$② \ \lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$$

Now proceed like in the case of diagonalizable A:

① Find generalized eigenvectors:

$$A\vec{e}_1 = \lambda\vec{e}_1, \quad A\vec{e}_k = \vec{e}_{k-1} + \lambda\vec{e}_k, \quad k = 2, \dots, n$$

Convert A to Jordan normal form:

$$S = [\vec{e}_1 \dots \vec{e}_n]: \quad S^{-1}AS = \Lambda = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

② In new variables

$$\vec{x} = S^{-1}\vec{y}: \quad \dot{\vec{x}} = \Lambda\vec{x}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = \lambda x_1 + x_2 \\ \dot{x}_2 = \lambda x_2 + x_3 \\ \dots \\ \dot{x}_{n-1} = \lambda x_{n-1} + x_n \\ \dot{x}_n = \lambda x_n \end{cases}$$

Solve starting from the last equation:

$$x_n = a_n e^{\lambda t}$$

$$\dot{x}_{n-1} = \lambda x_{n-1} + x_n = \lambda x_{n-1} + a_n e^{\lambda t}$$

Seek solution in the form $x_{n-1} = f(t)e^{\lambda t}$:

$$\dot{f}e^{\lambda t} + \lambda f e^{\lambda t} = \lambda f e^{\lambda t} + a_n e^{\lambda t} \Rightarrow \dot{f} = a_n \Rightarrow f = a_n t + a_{n-1}$$

$$\Rightarrow x_{n-1}(t) = (a_{n-1} + a_n t)e^{\lambda t}$$

$$\dot{x}_{n-2} = \lambda x_{n-2} + x_{n-1} = \lambda x_{n-2} + (a_{n-1} + a_n t)e^{\lambda t}$$

Same procedure: $x_{n-2} = f(t)e^{\lambda t}$, $\dot{f} = a_{n-1} + a_n t$

$$\Rightarrow f = a_{n-2} + a_{n-1}t + a_n\frac{t^2}{2} \Rightarrow x_{n-2} = \left(a_{n-2} + a_{n-1}t + a_n\frac{t^2}{2}\right)e^{\lambda t}$$

and so on:

$$\vec{x} = \begin{pmatrix} a_1 + a_2 t + \dots + a_n \frac{t^{n-1}}{(n-1)!} \\ \dots \\ a_{n-1} + a_n t \\ a_n \end{pmatrix} e^{\lambda t}$$

③ Go back to original variables:

$$y(t) = (\vec{y})_1 = (S\vec{x})_1 = \sum_k S_{1k}x_k(t) = \left(\alpha_1 + \alpha_2 t + \dots + \alpha_n t^{n-1}\right)e^{\lambda t}$$

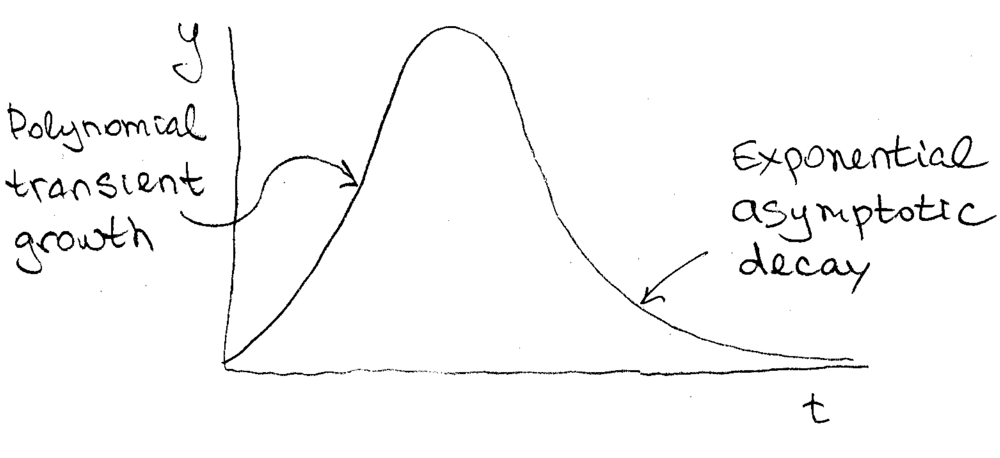
n-constants

$\Downarrow$

can satisfy n initial conditions.

Transients

Even if all $\lambda_k < 0$ (such that the state $y=0$ is stable) the solution can grow initially (sometimes by many orders of magnitude) before eventually decaying to zero



Example:
Laminar $\rightarrow$ Turbulent transition in shear flows