mathematical methods - week 2

Eigenvalue problems

Georgia Tech PHYS-6124

Homework HW #2 due Tuesday, September 2, 2014

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Exercise 2.1 *Three masses on a loop* 8 points

Bonus points
Exercise 2.2 *A simple stable/unstable manifolds pair* 4 points
Exercise X *Normal modes of colinear CO_2 molecule*
Explain why there are 7 zero modes, not 6 3 points

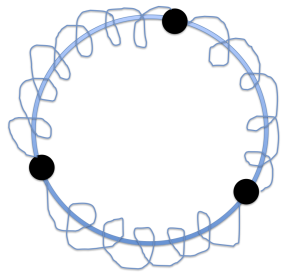

Figure 2.1: Three identical masses are constrained to move on a hoop, connected by three identical springs such that the system wraps completely around the hoop. Find the normal modes.

2014-08-26 Predrag Lecture 3

Recap from Lecture 2: state the moment of inertia tensor formula (they substitute for Grigoriev p. 6.2 Mechanics, inertia tensor)

Work through Grigoriev notes p. 6.6 crankshaft; Predrag notes - derivation of $\mathbf{e}^{(1)}$ eigenvector; sketch eigenvectors.
Sect. 1.2 *Matrix-valued functions*
Sect. 1.3 *A linear diversion*

2014-08-28 Predrag Lecture 4

Kimberly: Intro to normal modes, a 2-modes example
Work through Grigoriev notes 8 Normal modes and Arfken and Weber Example 3.6.1.

Exercises

2.1. **Three masses on a loop.** Three identical masses, connected by three identical springs, are constrained to move on a circle hoop as shown in figure 2.1. Find the normal modes. Hint: write down coupled harmonic oscillator equations, guess the form of oscillatory solutions. Then use basic matrix methods, i.e., find zeros of a characteristic determinant, find the eigenvectors, etc..

(Kimberly Y. Short)

2.2. **A simple stable/unstable manifolds pair.** Integrate flow (1.45), verify (1.46). Check that the projection matrices $\mathbf{P}_i$ (1.49) are orthonormal and complete. Use them to construct right and left eigenvectors; check that they are mutually orthogonal. Explain why is (1.50) the equation for the stable manifold. (N. Lebovitz)

Chapter 2 solutions: Eigenvalue problems

Solution 2.1 - Three masses on a loop. *As the masses and springs are identical, the equilibrium positions, x_1, x_2, and x_3, of the masses are equally spaced on the hoop, i.e., separated by $120°$ or $2\pi/3$ rads. The equations of motion are*

$$\begin{aligned} m\ddot{x}_1 + k(x_1 - x_2) + k(x_1 - x_3) &= 0 \\ m\ddot{x}_2 + k(x_2 - x_3) + k(x_2 - x_1) &= 0 \\ m\ddot{x}_3 + k(x_3 - x_1) + k(x_3 - x_2) &= 0 \end{aligned} \tag{2.1}$$

Anticipating oscillatory solutions, we introduce trial solutions of the form

$$x_1 = A_1 e^{i\alpha t}, \quad x_2 = A_2 e^{i\alpha t}, \quad x_3 = A_3 e^{i\alpha t}$$

Plugging these expressions into our system of equations yields

$$\begin{pmatrix} -\alpha^2 + 2\omega^2 & -\omega^2 & -\omega^2 \\ -\omega^2 & -\alpha^2 + 2\omega^2 & -\omega^2 \\ -\omega^2 & -\omega^2 & -\alpha^2 + 2\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.2}$$

Taking the determinant yields a cubic equation for α^2:

$$-(\alpha^2)^3 + 6(\alpha^2)^2\omega^2 - 9(\alpha^2)\omega^4 = 0 \tag{2.3}$$

The three solutions are

$$\alpha^2 = 0, \qquad \alpha^2 = 3\omega^2(\text{multiplicity} = 2)$$

Consider first the solution corresponding to $\alpha^2 = 0$ with eigenvector $(1,1,1)^T/\sqrt{3}$. In this case, the normal mode is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (At + B) \tag{2.4}$$

where have absorbed the $1/\sqrt{3}$ factor into the coefficients A and B. This normal mode has zero frequency and corresponds to the masses sliding around the hoop, equally spaced, at constant speed. The normal coordinates are simply $x_1 + x_2 + x_3$.

The remaining two roots corresponding to $\alpha^2 = 3\omega^2$ describe oscillations. Any vector of the form (a, b, c) satisfying $a + b + c = 0$ is a valid normal mode with frequency $\sqrt{3}\omega$. Here we have chosen vectors $(0, 1, -1)^T/\sqrt{2}$ and $(1, 0, -1)^T/\sqrt{2}$ as the basis for the two-dimensional subspace of normal modes. Consequently, the normal modes may be written as linear combinations of the vectors

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_1) \tag{2.5}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = C_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega t + \phi_2) \tag{2.6}$$

with normal coordinates $x_1 - 2x_2 + x_3$ and $-2x_1 + x_2 + x_3$, respectively.

(Kimberly Y. Short)

Solution 2.2 - A simple stable/unstable manifolds pair. *(No solution available.)*