mathematical methods - week 3

Eigenvectors and eigenvalues

Georgia Tech PHYS-6124

Homework HW #3 due Tuesday, September 9, 2014

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Exercise 3.1 *Stability of a 2-dimensional flow*	10 points
Exercise 3.2 *Complex logarithm*	2 points

Bonus points

Exercise 3.3 *A limit cycle with analytic Floquet exponent*	6 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2014-09-02 Kimberly Lecture 5

Watch a movie of a chemical oscillator

Discuss the Brusselator, the minimal mathematical model that leads to nonlinear oscillating behavior:

1. The Brusselator is two-dimensional (x, y), with parameters a and b
2. find its equilibria
3. linearize it
4. compute its stability matrix A
5. find the eigenvalues of the stability matrix A
6. classify the stabilities of its equilibria
7. vary (a, b) to change the stability of the equilibrium; a limit cycle

2014-09-04 Predrag Lecture 6

Complex variables: History; algebraic and geometric insights; De Moivre's formula; roots of unity; functions of complex variables as mappings; differentiation of complex functions; Cauchy-Riemann relations.

Arfken and Weber [3.1] Chapter 6 on complex numbers.

References

[3.1] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists: A Comprehensive Guide*, 6 ed. (Academic Press, New York, 2005).

Exercises

3.1. **Stability of a 2-dimensional flow.** Consider the system

$$\begin{aligned} \dot{x} &= ax - \alpha xy \\ \dot{y} &= -cy + \gamma xy \,. \end{aligned} \tag{3.1}$$

where all the parameters are real and strictly positive $a, c, \alpha, \gamma > 0$.

(a) How many of these parameters are independent, i.e., set as many of them as possible equal to 1 by rescaling x, y and t.

(b) The origin $(0, 0)$ is an equilibrium point for the system. Find the eigenvalues and eigenvectors of the stability matrix A in the linearized neighborhood of $(0, 0)$ and describe qualitatively different types of the stability of the origin as function of parameters.

(c) Find the general solution $x(t)$ and $y(t)$ passing thorough the origin, $[x(0), y(0)] = [0, 0]$, to the coupled differential equations (3.1). Describe, in words and a plot, the behavior of nearby trajectories as they pass by the origin.

(d) Find the other equilibrium for the system. Linearize in the neighborhood of this second critical point, find the eigenvalues of the stability matrix A, and describe the stability of this second equilibrium.

(e) Show that for all parameter values (?) the trajectories of the linearized system in the vicinity of the second equilibrium are ellipses. Express the equation of this family of ellipses in a standard form.

(Kimberly Y. Short)

3.2. **Complex logarithm.** Show that the phase of $f(z) = u + iv$ is equal to the imaginary part of the logarithm of $f(z)$.

3.3. **A limit cycle with analytic Floquet exponent.** There are only two examples of nonlinear flows for which the Floquet multipliers can be evaluated analytically. Both are cheats. One example is the 2-dimensional flow

$$\begin{aligned} \dot{q} &= p + q(1 - q^2 - p^2) \\ \dot{p} &= -q + p(1 - q^2 - p^2)\,. \end{aligned}$$

Determine all periodic solutions of this flow, and determine analytically their Floquet exponents. Hint: go to polar coordinates $(q, p) = (r\cos\theta, r\sin\theta)$. G. Bard Ermentrout