mathematical methods - week 5

Integration in complex plane

Georgia Tech PHYS-6124

Homework HW #5 due Tuesday, September 23, 2014

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Exercise 5.1 *Complex integration* (a) 4; (b) 2; (c) 2; and (d) 3 points
Exercise 5.2 *Fresnel integral* 7 points

Bonus points
Exercise 5.3 *Cauchy's theorem via Green's theorem in the plane* 6 points

Total of 18 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2014-09-16 Predrag Lecture 10 Calculus of residues

Arfken & Weber Chapter 6 - Laurent expansion; chapter 7 - residues

2014-09-18 Predrag Lecture 11 Integration in complex plane

Arfken & Weber Chapter 7, section 7.1
Grigoriev notes: Evaluation of integrals

Exercises

5.1. **Complex integration.**

(a) Write down the values of $\oint_C (1/z)\, dz$ for each of the following choices of C:
(i) $|z| = 1$, (ii) $|z-2| = 1$, (iii) $|z-1| = 2$.
Then confirm the answers the hard way, using parametric evaluation.

(b) Evaluate parametrically the integral of $1/z$ around the square with vertices $\pm 1 \pm i$.

(c) Confirm by parametric evaluation that the integral of z^m around an origin centered circle vanishes, except when the integer $m = -1$.

(d) Evaluate $\int_{1+i}^{3-2i} dz\, \sin z$ in two ways: (i) via the fundamental theorem of (complex) calculus, and (ii) (optional) by choosing any path between the end-points and using real integrals.

5.2. **Fresnel integral.**
We wish to evaluate the $I = \int_0^\infty \exp\left(ix^2\right) dx$. To do this, consider the contour integral $I_R = \int_{C(R)} \exp\left(iz^2\right) dz$, where $C(R)$ is the closed circular sector in the upper half-plane with boundary points 0, R and $R\exp(i\pi/4)$. Show that $I_R = 0$ and that $\lim_{R\to\infty} \int_{C_1(R)} \exp\left(iz^2\right) dz = 0$, where $C_1(R)$ is the contour integral along the circular sector from R to $R\exp(i\pi/4)$. [Hint: use $\sin x \geq (2x/\pi)$ on $0 \leq x \leq \pi/2$.] Then, by breaking up the contour $C(R)$ into three components, deduce that

$$\lim_{R\to\infty} \left(\int_0^R \exp\left(ix^2\right) dx - \mathrm{e}^{i\pi/4} \int_0^R \exp\left(-r^2\right) dr \right) = 0$$

and, from the well-known result of real integration $\int_0^\infty \exp\left(-x^2\right) dx = \sqrt{\pi}/2$, deduce that $I = \mathrm{e}^{i\pi/4}\sqrt{\pi}/2$.

5.3. **Cauchy's theorem via Green's theorem in the plane.** Express the integral $\oint_C dz\, f(z)$ of the analytic function $f = u+iv$ around the simple contour C in parametric form, apply the two-dimensional version of Gauss' theorem (a.k.a. Green's theorem in the plane), and invoke the Cauchy-Riemann conditions. Hence establish Cauchy's theorem $\oint_C dz\, f(z) = 0$.