mathematical methods - week 6

Method of steepest descent

Georgia Tech PHYS-6124

Homework HW #6 due Tuesday, October 7, 2014

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Exercise 6.1 *In high dimensions any two vectors are (nearly) orthogonal* 16 points

Bonus points
Exercise 6.2 *Airy function for large arguments* 10 points

Total of 16 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2014-09-23 Predrag Lecture 11 Asymptotic evaluation of integrals
Jensen's theorem, saddle point method. Arfken & Weber section 7.3. A propos Jensen: the most popular Danish family names are 1. Jensen 303,089 2. Nielsen 296,850 3. Hansen 248,968. This out of population of 5.5 million.

2014-09-25 Predrag Lecture 12 The Gamma, Airy function estimates
Arfken& Weber Chapter 8 has interesting tidbits about the Gamma function. Beta function is also often encountered. Grigoriev discusses Airy function in his notes.

6.1 Literature

Optional reading for this course: M. Stone and P. Goldbart [6.1], Mathematics for Physics (Cambridge University Press, Cambridge 2004), offers a very engaging, physics focused approach. A pre-publication draft can be found here.

References

[6.1] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge, 2009).

[6.2] J. F. Gibson, J. Halcrow, and P. Cvitanović, Visualizing the geometry of state space in plane Couette flow, J. Fluid Mech. **611**, 107 (2008), arXiv:0705.3957.

Exercises

6.1. **In high dimensions any two vectors are (nearly) orthogonal.** Among humble plumbers laboring with extremely high-dimensional ODE discretizations of fluid and other PDEs, there is an inclination to visualize the ∞-dimensional state space flow by projecting it onto a basis constructed from a few random coordinates, let's say the 2nd Fourier mode along the spatial x direction against the 4th Chebyshev mode along the y direction. It's easy, as these are typically the computational degrees of freedom. As we will now show, it's easy but it might be stupid, with vectors representing the dynamical states of interest being almost orthogonal to any such random basis.

Suppose your state space $\mathcal{M}$ is a real 10 247-dimensional vector space, and you pick from it two vectors $x_1, x_2 \in \mathcal{M}$ at random. What is the angle between them likely to be?

By asking for 'angle between two vectors' we have implicitly assumed that there exist is a dot product

$$x_1{}^\top \cdot x_2 = |\,x_1\,|\,|\,x_2\,|\cos(\theta_{12})\,,$$

so let's make these vectors unit vectors, $|\,x_j\,| = 1$. When you think about it, you would be hard put to say what 'uniform probability' would mean for a vector $x \in \mathcal{M} = \mathbb{R}^{10\,247}$, but for a unit vector it is obvious: probability that x direction lies within a solid angle $d\Omega$ is $d\Omega/(\text{unit hyper-sphere surface})$.

So what is the surface of the unit sphere (or, the total solid angle) in d dimensions? One way to compute it is to evaluate the Gaussian integral

$$I_d = \int_{-\infty}^{\infty} dx_1 \cdots dx_d \, e^{-\frac{1}{2}\left(x_1^2+\cdots+x_d^2\right)}$$

in cartesian and polar coordinates. Show that

(a) In cartesian coordinates $I_d = (2\pi)^{d/2}$.

(b) Recast the integrals in polar coordinate form. You know how to compute this integral in 2 and 3 dimensions. Show by induction that the surface S_{d-1} of unit d-ball, or the total solid angle in even and odd dimensions is given by

$$S_{2k} = \frac{2\pi^k}{(k-1)!}\,, \qquad S_{2k+1} = \frac{2(2\pi)^k}{(2k-1)!!}\,.$$

(c) Show, by examining the form of the integrand in the polar coordinates, that for arbitrary, perhaps even complex dimension $d \in \mathbb{C}$

$$S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)\,.$$

In Quantum Field Theory this is called the 'dimensional regularization'.

(d) Check your formula for $d = 2$ (1-sphere, or the circle) and $d = 3$ (2-sphere, or the sphere).

(e) (bonus) What limit does S_d does tend to for large d? (Hint: it's not what you think. Try Sterling's formula).

So now that we know the volume of a sphere, what is a the most likely angle between two vectors x_1, x_2 picked at random? We can rotate coordinates so that x_1 is aligned with the 'z-axis' of the hypersphere. An angle θ then defines a meridian around the 'z-axis'.

(f) Show that probability $P(\theta)d\theta$ of finding two vectors at angle θ is given by the area of the meridional strip of width $d\theta$, and derive the formula for it:

$$P(\theta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(d/2)}{\Gamma((d-1)/2)}\,.$$

(One can write analytic expression for this in terms of beta functions, but it is unnecessary for the problem at hand).

(g) Show that for large d the probability $P(\theta)$ tends to a normal distribution with mean $\theta = \pi$ and variance $1/d$.

So, in d-dimensional vector space the two random vectors are nearly orthogonal, within accuracy of $\theta = \pi/2 \pm 1/d$.

If you are a humble plumber, and the notion of a vector space is some abstract hocus-pocus to you, try thinking this way. Your 2nd Fourier mode basis vector is something that wiggles twice along your computation domain. Your turbulent state is very wiggly. The product of the two functions integrated over the computational domain will average to zero, with a small leftover. We have just estimated that with dumb choices of coordinate bases this leftover will be of order of $1/10\,247$, which embarrassingly small for displaying a phenomenon of order ≈ 1.

Several intelligent choices of coordinates for state space projections are described in Gibson *et al.* [6.2] and the web tutorial ChaosBook.org/tutorials.

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6.2. **Airy function for large arguments.** Important contributions as stationary phase points may arise from extremal points where the first non-zero term in a Taylor expansion of the phase is of third or higher order. Such situations occur, for example, at bifurcation points or in diffraction effects, (such as waves near sharp corners, waves creeping around obstacles, etc.). In such calculations, one meets Airy functions integrals of the form

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy \, e^{i(xy - \frac{y^3}{3})} \,. \tag{6.1}$$

Calculate the Airy function $Ai(x)$ using the stationary phase approximation. What happens when considering the limit $x \to 0$. Estimate for which value of x the stationary phase approximation breaks down.