mathematical methods - week 8

Fourier transform

2014-10-07 Predrag Lecture 15

Arfken and Weber chapter 14. Fourier Series.

Roger Penrose [8.1] chapter on Fourier transforms is too pretty to pass up.

2014-10-09 Mohammad Farazmand Lecture 16

Farazmand notes on Fourier transforms.

Grigoriev notes

4. Integral transforms, 4.3-4.4 square wave, Gibbs phenomenon;

5. Fourier transform: 5.1-5.6 inverse, Parseval's identity, ..., examples

Georgia Tech PHYS-6124

Homework HW #8 due Thursday, October 9, 2014

== show all your work for maximum credit,

== put labels, title, legends on any graphs

== acknowledge study group member, if collective effort

Exercise 8.1 *D-dimensional Gaussian integrals* 5 points

Exercise 8.2 *Convolution of Gaussians* 5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

References

[8.1] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (A. A. Knopf, New York, 2005).

[8.2] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals* (Dover, New York, 1986).

Exercises

8.1. **D-dimensional Gaussian integrals.** Show that the Gaussian integral in D-dimensions is given by

$$\frac{1}{(2\pi)^{d/2}} \int d^d\phi e^{-\frac{1}{2}\phi^\top \cdot M^{-1} \cdot \phi + \phi \cdot J} \quad = \quad |\det M|^{\frac{1}{2}}\, e^{\frac{1}{2} J^\top \cdot M \cdot J}\,, \tag{8.1}$$

where M is a real positive definite $[d \times d]$ matrix, i.e., a matrix with strictly positive eigenvalues. x, J are D-dimensional vectors, and $x^\top$ is the transpose of x.

8.2. **Convolution of Gaussians.** Show that the Fourier transform of convolution

$$[f * g](x) = \int d^d y\, f(x-y) g(y)$$

of two Gaussians

$$f(x) = e^{-\frac{1}{2}x^\top \cdot \frac{1}{\Delta_1} \cdot x}\,, \qquad g(x) = e^{-\frac{1}{2}x^\top \cdot \frac{1}{\Delta_2} \cdot x}$$

(a) factorizes as

$$[f * g](x) = \frac{1}{(2\pi)^d} \int dk\, F(k) G(k) e^{ik \cdot x}\,, \tag{8.2}$$

where

$$\begin{aligned} F(k) &= \frac{1}{(2\pi)^d} \int d^d x\, f(x) e^{-ik\cdot x} = |\det \Delta_1|^{1/2} e^{\frac{1}{2}k^\top \cdot \Delta_1 \cdot k} \\ G(k) &= \frac{1}{(2\pi)^d} \int d^d x\, g(x) e^{-ik\cdot x} = |\det \Delta_2|^{1/2} e^{\frac{1}{2}k^\top \cdot \Delta_2 \cdot k}\,. \end{aligned}$$

(b) Show that

$$\begin{aligned} [f * g](x) &= \frac{1}{(2\pi)^d} |\det \Delta_1 \det \Delta_1|^{1/2} \int d^d p\, e^{\frac{1}{2}p^\top \cdot (\Delta_1 + \Delta_2) \cdot p + ip \cdot x} \\ &= \left| \frac{\det \Delta_1 \det \Delta_2}{\det (\Delta_1 + \Delta_2)} \right|^{1/2} e^{-\frac{1}{2}x^\top \cdot (\Delta_1 + \Delta_2)^{-1} \cdot x}\,. \end{aligned} \tag{8.3}$$