Chapter Nine

Unitary groups

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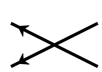
$U(n)$ is the group of all transformations that leave invariant the norm $\overline{q}q = \delta_b^a q^b q_a$ of a complex vector q. For $U(n)$ there are no other invariant tensors beyond those constructed of products of Kronecker deltas. They can be used to decompose the tensor reps of $U(n)$. For purely covariant or contravariant tensors, the symmetric group can be used to construct the Young projection operators. In sections. 9.1–9.2 we show how to do this for 2- and 3-index tensors by constructing the appropriate characteristic equations.

For tensors with more indices it is easier to construct the Young projection operators directly from the Young tableaux. In section 9.3 we review the Young tableaux, and in section 9.4 we show how to construct Young projection operators for tensors with any number of indices. As examples, 3- and 4-index tensors are decomposed in section 9.5. We use the projection operators to evaluate $3n$-j coefficients and characters of $U(n)$ in sections. 9.6–9.9, and we derive new sum rules for $U(n)$ 3-j and 6-j symbols in section 9.7. In section 9.8 we consider the consequences of the Levi-Civita tensor being an extra invariant for $SU(n)$.

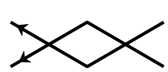
For mixed tensors the reduction also involves index contractions and the symmetric group methods alone do not suffice. In sections. 9.10–9.12 the mixed $SU(n)$ tensors are decomposed by the projection operator techniques introduced in chapter 3. $SU(2)$, $SU(3)$, $SU(4)$, and $SU(n)$ are discussed from the "invariance group" perspective in chapter 15.

9.1 TWO-INDEX TENSORS

Consider 2-index tensors $q^{(1)} \otimes q^{(2)} \in \otimes V^2$. According to (6.1), all permutations are represented by invariant matrices. Here there are only two permutations, the identity and the flip (6.2),

$\sigma =$  .

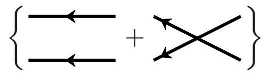
The flip satisfies

$\sigma^2 =$  $= \mathbf{1}$,

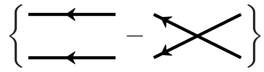
$$(\sigma + \mathbf{1})(\sigma - \mathbf{1}) = 0\,. \qquad (9.1)$$

The eigenvalues are $\lambda_1 = 1, \lambda_2 = -1$, and the corresponding projection operators (3.48) are

$$\mathbf{P}_1 = \frac{\sigma - (-1)\mathbf{1}}{1-(-1)} = \frac{1}{2}(\mathbf{1}+\sigma) = \frac{1}{2}\left\{ \right\}, \qquad (9.2)$$

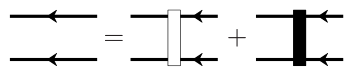


$$\mathbf{P}_2 = \frac{\sigma - \mathbf{1}}{-1-1} = \frac{1}{2}(\mathbf{1}-\sigma) = \frac{1}{2}\left\{ \right\}. \qquad (9.3)$$



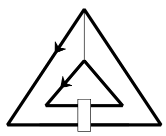
We recognize the symmetrization, antisymmetrization operators (6.4), (6.15); $\mathbf{P}_1 = \mathbf{S}$, $\mathbf{P}_2 = \mathbf{A}$, with subspace dimensions $d_1 = n(n+1)/2$, $d_2 = n(n-1)/2$. In other words, under general linear transformations the symmetric and the antisymmetric parts of a tensor x_{ab} transform separately:

$$x = \mathbf{S}x + \mathbf{A}x\,,$$
$$x_{ab} = \frac{1}{2}(x_{ab}+x_{ba}) + \frac{1}{2}(x_{ab}-x_{ba})$$

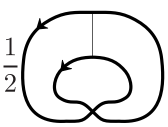


(9.4)

The Dynkin indices for the two reps follow by (7.29) from $6j's$:



$$= \frac{1}{2}(0) + \frac{1}{2} \qquad = \frac{N}{2}$$



$$\ell_1 = \frac{2\ell}{n}\cdot d_1 + \frac{2\ell}{N}\cdot\frac{N}{2}$$
$$= \ell(n+2)\,. \qquad (9.5)$$

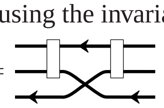
Substituting the defining rep Dynkin index $\ell^{-1} = C_A = 2n$, computed in section 2.2, we obtain the two Dynkin indices

$$\ell_1 = \frac{n+2}{2n}\,, \qquad \ell_2 = \frac{n-2}{2n}\,. \qquad (9.6)$$

9.2 THREE-INDEX TENSORS

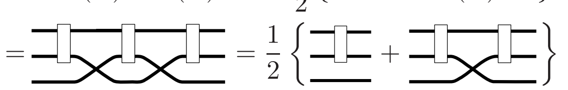
Three-index tensors can be reduced to irreducible subspaces by adding the third index to each of the 2-index subspaces, the symmetric and the antisymmetric. The results of this section are summarized in figure 9.1 and table 9.1. We mix the third index into the symmetric 2-index subspace using the invariant matrix

$$\mathbf{Q} = \mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12} = \qquad . \qquad (9.7)$$



Here projection operators $\mathbf{S}_{12}$ ensure the restriction to the 2-index symmetric subspace, and the transposition $\sigma_{(23)}$ mixes in the third index. To find the characteristic equation for $\mathbf{Q}$, we compute $\mathbf{Q}^2$:

$$\mathbf{Q}^2 = \mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12} = \frac{1}{2}\left\{\mathbf{S}_{12} + \mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12}\right\} = \frac{1}{2}\mathbf{S}_{12} + \frac{1}{2}\mathbf{Q}$$

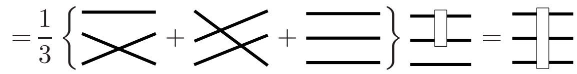


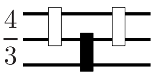
Hence, $\mathbf{Q}$ satisfies

$$(\mathbf{Q}-1)(\mathbf{Q}+1/2)\mathbf{S}_{12}=0\,, \tag{9.8}$$

and the corresponding projection operators (3.48) are

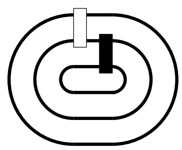
$$\mathbf{P}_1=\frac{\mathbf{Q}+\frac{1}{2}\mathbf{1}}{1+\frac{1}{2}}\mathbf{S}_{12}=\frac{1}{3}\left\{\sigma_{(23)}+\sigma_{(123)}+\mathbf{1}\right\}\mathbf{S}_{12}=\mathbf{S}$$


(9.9)

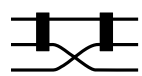
$$\mathbf{P}_2=\frac{\mathbf{Q}-\mathbf{1}}{-\frac{1}{2}-1}\mathbf{S}_{12}=\frac{4}{3}\mathbf{S}_{12}A_{23}\mathbf{S}_{12}=\frac{4}{3}$$

. (9.10)

Hence, the symmetric 2-index subspace combines with the third index into a symmetric 3-index subspace (6.13) and a mixed symmetry subspace with dimensions

$$d_1=\mathrm{tr}\,\mathbf{P}_1 = n(n+1)(n+2)/3! \tag{9.11}$$

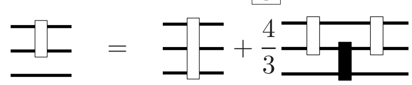
$$d_2=\mathrm{tr}\,\mathbf{P}_2=\frac{4}{3}$$

$$= n(n^2-1)/3\,. \tag{9.12}$$

The antisymmetric 2-index subspace can be treated in the same way using the invariant matrix

$$\mathbf{Q}=\mathbf{A}_{12}\sigma_{(23)}\mathbf{A}_{12}=$$

. (9.13)

The resulting projection operators for the antisymmetric and mixed symmetry 3-index tensors are given in figure 9.1. Symmetries of the subspace are indicated by the corresponding Young tableaux, table 9.2. For example, we have just constructed

| 1 | 2 | ⊗ | 3 | = | 1 | 2 | 3 | ⊕ | 1 2 / 3 |



$$\frac{n^2(n+1)}{2}=\frac{n(n+1)(n+2)}{3!}+\frac{n(n^2-1)}{3}\,. \tag{9.14}$$

The projection operators for tensors with up to 4 indices are shown in figure 9.1, and in figure 9.2 the corresponding stepwise reduction of the irreps is given in terms of Young standard tableaux (defined in section 9.3.1).

9.3 YOUNG TABLEAUX

We have seen in the examples of sections. 9.1–9.2 that the projection operators for 2-index and 3-index tensors can be constructed using characteristic equations. For tensors with more than three indices this method is cumbersome, and it is much simpler to construct the projection operators directly from the Young tableaux. In this section we review the Young tableaux and some aspects of symmetric group representations that will be important for our construction of the projection operators in section 9.4.

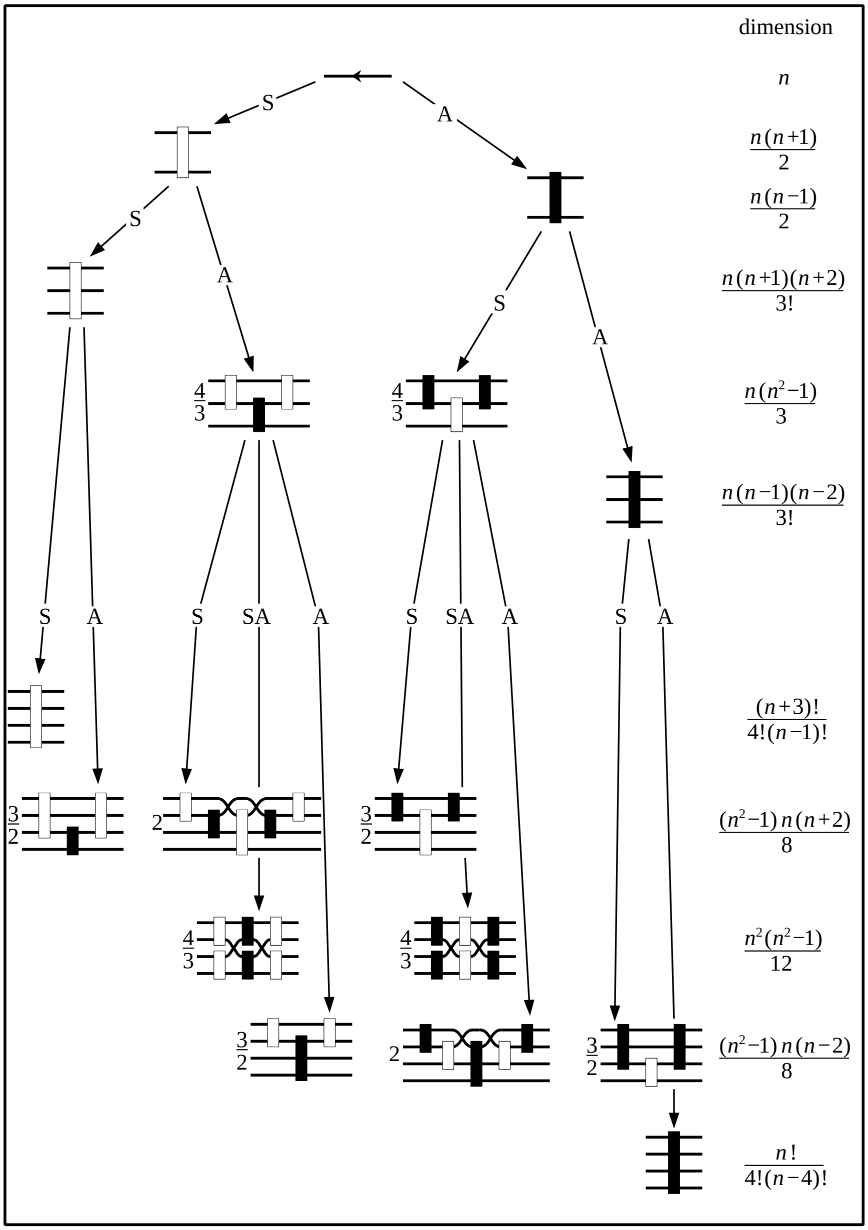


Figure 9.1 Projection operators for 2-, 3-, and 4-index tensors in $U(n)$, $SU(n)$, $n \geq p =$ number of indices.

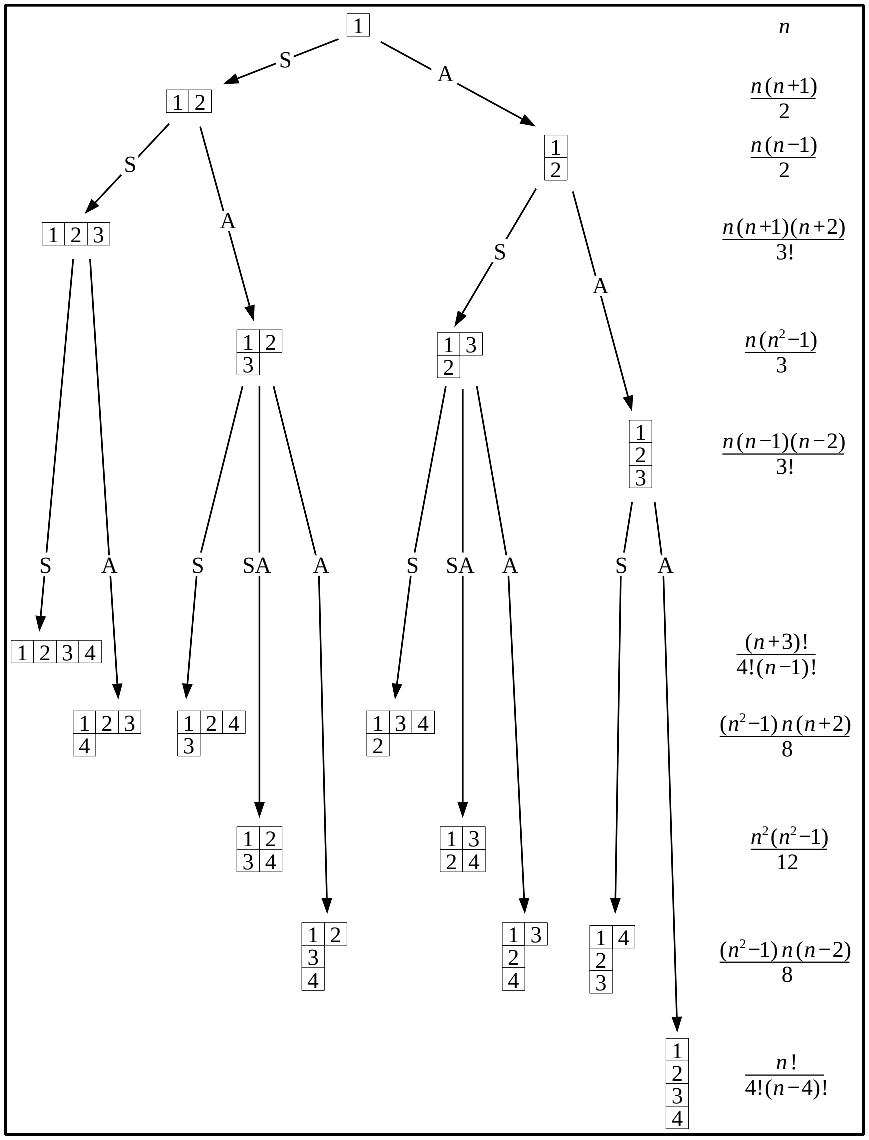
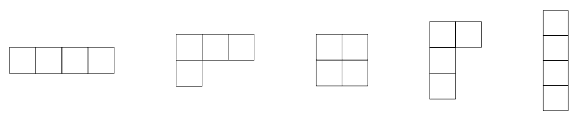


Figure 9.2 Young tableaux for the irreps of the symmetric group for 2-, 3-, and 4-index tensors. Rows correspond to symmetrizations, columns to antisymmetrizations. The reduction procedure is not unique, as it depends on the order in which the indices are combined; this order is indicated by labels 1, 2, 3 , ..., p in the boxes of Young tableaux.

9.3.1 Definitions

Partition k identical boxes into D subsets, and let λ_m, $m = 1, 2, \ldots, D$, be the number of boxes in the subsets ordered so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_D \geq 1$. Then the partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_D]$ fulfills $\sum_{m=1}^{D} \lambda_m = k$. The diagram obtained by drawing the D rows of boxes on top of each other, left aligned, starting with λ_1 at the top, is called a *Young diagram* Y.

Examples:
The ordered partitions for $k = 4$ are $[4], [3, 1], [2, 2], [2, 1, 1]$ and $[1, 1, 1, 1]$. The corresponding Young diagrams are



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Inserting a number from the set $\{1, \ldots, n\}$ into every box of a Young diagram Y_λ in such a way that numbers increase when reading a column from top to bottom, and numbers do not decrease when reading a row from left to right, yields a *Young tableau* Y_a. The subscript a labels different tableaux derived from a given Young diagram, *i.e.*, different admissible ways of inserting the numbers into the boxes.

A *standard tableau* is a k-box Young tableau constructed by inserting the numbers $1, \ldots, k$ according to the above rules, but using each number exactly once. For example, the 4-box Young diagram with partition $\lambda = [2, 1, 1]$ yields three distinct standard tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \,,\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \,,\quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \,. \tag{9.15}$$

An alternative labeling of a Young diagram are Dynkin labels, the list of numbers b_m of columns with m boxes: $(b_1 b_2 \ldots)$. Having k boxes we must have $\sum_{m=1}^{k} m b_m = k$. For example, the partition $[4, 2, 1]$ and the labels $(21100 \cdots)$ give rise to the same Young diagram, and so do the partition $[2, 2]$ and the labels $(020 \cdots)$.

We define the *transpose* diagram Y^t as the Young diagram obtained from Y by interchanging rows and columns. For example, the transpose of $[3, 1]$ is $[2, 1, 1]$,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}^{\,t} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \,,$$

or, in terms of Dynkin labels, the transpose of $(210 \ldots)$ is $(1010 \ldots)$.

The Young tableaux are useful for labeling irreps of various groups. We shall use the following facts (see for instance ref. [153]):

1. The k-box *Young diagrams* label all irreps of the symmetric group S_k.

2. The *standard tableaux* of k-box Young diagrams with no more than n rows label the irreps of $GL(n)$, in particular they label the irreps of $U(n)$.

3. The *standard tableaux* of k-box Young diagrams with no more than $n-1$ rows label the irreps of $SL(n)$, in particular they label the irreps of $SU(n)$.

In this section, we consider the Young tableaux for reps of S_k and $U(n)$, while the case of $SU(n)$ is postponed to section 9.8.

9.3.2 Symmetric group S_k

The irreps of the symmetric group S_k are labeled by the k-box Young diagrams. For a given Young diagram, the basis vectors of the corresponding irrep can be labeled by the standard tableaux of Y; consequently the dimension Δ_{Y} of the irrep is the number of standard tableaux that can be constructed from the Young diagram Y. The example (9.15) shows that the irrep $\lambda = [2,1,1]$ of S_4 is 3-dimensional.

As an alternative to counting standard tableaux, the dimension Δ_{Y} of the irrep of S_k corresponding to the Young diagram Y can be computed easily as

$$\Delta_{\mathrm{Y}} = \frac{k!}{|\mathrm{Y}|}\,, \qquad (9.16)$$

where the number $|\mathrm{Y}|$ is computed using a "hook" rule: Enter into each box of the Young diagram the number of boxes below and to the right of the box, including the box itself. Then $|\mathrm{Y}|$ is the product of the numbers in all the boxes. For instance,

$$\mathrm{Y} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \\ \cline{1-3} & \\ \cline{1-2} \end{array} \quad \longrightarrow \quad |\mathrm{Y}| = \begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & 1 \\ \cline{1-3} 2 & 1 \\ \cline{1-2} \end{array} = 6!\,3\,. \qquad (9.17)$$

The hook rule (9.16) was first proven by Frame, de B. Robinson, and Thrall [123]. Various proofs can be found in the literature [295, 170, 133, 142, 21]; see also Sagan [302] and references therein.

We now discuss the regular representation of the symmetric group. The elements $\sigma \in S_k$ of the symmetric group S_k form a basis of a $k!$-dimensional vector space V of elements

$$s = \sum_{\sigma \in S_k} s_\sigma\, \sigma \in V\,, \qquad (9.18)$$

where s_σ are the components of a vector s in the given basis. If $s \in V$ has components (s_σ) and $\tau \in S_k$, then τs is an element in V with components $(\tau s)_\sigma = s_{\tau^{-1}\sigma}$. This action of the group elements on the vector space V defines an $k!$-dimensional matrix representation of the group S_k, the *regular representation*.

The regular representation is reducible, and each irrep λ appears Δ_λ times in the reduction; Δ_λ is the dimension of the subspace V_λ corresponding to the irrep λ. This gives the well-known relation between the order of the symmetric group $|S_k| = k!$ (the dimension of the regular representation) and the dimensions of the irreps,

$$|S_k| = \sum_{\text{all irreps } \lambda} \Delta_\lambda^2\,.$$

Using (9.16) and the fact that the Young diagrams label the irreps of S_k, we have

$$1 = k! \sum_{(k)} \frac{1}{|Y|^2}\,, \qquad (9.19)$$

where the sum is over all Young diagrams with k boxes. We shall use this relation to determine the normalization of Young projection operators in appendix B.3.

The reduction of the regular representation of S_k gives a completeness relation,

$$\mathbf{1} = \sum_{(k)} \mathbf{P}_{\mathrm{Y}} \,,$$

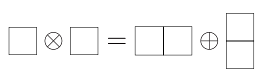
in terms of projection operators

$$\mathbf{P}_{\mathrm{Y}} = \sum_{\mathrm{Y}_a \in \mathrm{Y}} \mathbf{P}_{\mathrm{Y}_a} \,.$$

The sum is over all standard tableaux derived from the Young diagram Y. Each $\mathbf{P}_{\mathrm{Y}_a}$ projects onto a corresponding invariant subspace V_{Y_a}: for each Y there are Δ_{Y} such projection operators (corresponding to the Δ_{Y} possible standard tableaux of the diagram), and each of these project onto one of the Δ_{Y} invariant subspaces V_{Y} of the reduction of the regular representation. It follows that the projection operators are orthogonal and that they constitute a complete set.

9.3.3 Unitary group $U(n)$

The irreps of $U(n)$ are labeled by the k-box Young standard tableaux with no more than n rows. A k-index tensor is represented by a Young diagram with k boxes — one typically thinks of this as a k-particle state. For $U(n)$, a 1-index tensor has n-components, so there are n 1-particle states available, and this corresponds to the n-dimensional fundamental rep labeled by a 1-box Young diagram. There are n^2 2-particle states for $U(n)$, and as we have seen in section 9.1 these split into two irreps: the symmetric and the antisymmetric. Using Young diagrams, we write the reduction of the 2-particle system as



(9.20)

Except for the fully symmetric and the fully antisymmetric irreps, the irreps of the k-index tensors of $U(n)$ have mixed symmetry. Boxes in a row correspond to indices that are symmetric under interchanges (symmetric multiparticle states), and boxes in a column correspond to indices antisymmetric under interchanges (antisymmetric multiparticle states). Since there are only n labels for the particles, no more than n particles can be antisymmetrized, and hence only standard tableaux with up to n rows correspond to irreps of $U(n)$.

The number of standard tableaux Δ_{Y} derived from a Young diagram Y is given in (9.16). In terms of irreducible tensors, the Young diagram determines the symmetries of the indices, and the Δ_{Y} distinct standard tableaux correspond to the independent ways of combining the indices under these symmetries. This is illustrated in figure 9.2.

For a given $U(n)$ irrep labeled by some standard tableau of the Young diagram Y, the basis vectors are labeled by the Young tableaux Y_a obtained by inserting the numbers $1, 2, \ldots, n$ into Y in the manner described in section 9.3.1. Thus the dimension of an irrep of $U(n)$ equals the number of such Young tableaux, and we note that all irreps with the same Young diagram have the same dimension. For $U(2)$, the $k = 2$ Young tableaux of the symmetric and antisymmetric irreps are

$$\begin{array}{|c|c|}\hline 1&1\\\hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 2&2\\\hline\end{array}\,,\quad \text{and}\quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,,$$

so the symmetric state of $U(2)$ is 3-dimensional and the antisymmetric state is 1-dimensional, in agreement with the formulas (6.4) and (6.15) for the dimensions of the symmetry operators. For $U(3)$, the counting of Young tableaux shows that the symmetric 2-particle irrep is 6-dimensional and the antisymmetric 2-particle irrep is 3-dimensional, again in agreement with (6.4) and (6.15). In section 9.4.3 we state and prove a dimension formula for a general irrep of $U(n)$.

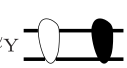
9.4 YOUNG PROJECTION OPERATORS

Given an irrep of $U(n)$ labeled by a k-box standard tableaux Y, we construct the corresponding Young projection operator $\mathbf{P}_\text{Y}$ in birdtrack notation by identifying each box in the diagram with a directed line. The operator $\mathbf{P}_\text{Y}$ is a block of symmetrizers to the left of a block of antisymmetrizers, all imposed on the k lines. The blocks of symmetry operators are dictated by the Young *diagram*, whereas the attachment of lines to these operators is specified by the particular standard tableau.

The Kronecker delta is invariant under unitary transformations: for $U \in U(n)$, we have $(U^\dagger)_a{}^{a'} \delta_{a'}^{b'} U_{b'}{}^b = \delta_a^b$. Consequently, any combination of Kronecker deltas, such as a symmetrizer, is invariant under unitary transformations. The symmetry operators constitute a complete set, so any $U(n)$ invariant tensor built from Kronecker deltas can be expressed in terms of symmetrizers and antisymmetrizers. In particular, the invariance of the Kronecker delta under $U(n)$ transformations implies that the same symmetry group operators that project the irreps of S_k also yield the irreps of $U(n)$.

The simplest examples of Young projection operators are those associated with the Young tableaux consisting of either one row or one column. The corresponding Young projection operators are simply the symmetrizers or the antisymmetrizers respectively. As projection operators for S_k, the symmetrizer projects onto the 1-dimensional subspace corresponding to the fully symmetric representation, and the antisymmetrizer projects onto the fully antisymmetric representation (the alternating representation).

A Young projection operator for a mixed symmetry Young tableau will here be constructed by first antisymmetrizing subsets of indices, and then symmetrizing other subsets of indices; the Young tableau determines which subsets, as will be explained shortly. Schematically,

$$\mathbf{P}_{\text{Y}_a} = \alpha_\text{Y}$$  , (9.21)

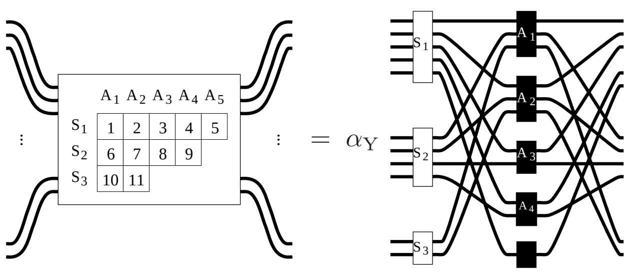
where the white (black) blob symbolizes a set of (anti)symmetrizers. The normalization constant α_Y (defined below) ensures that the operators are idempotent, $\mathbf{P}_{\text{Y}_a}\mathbf{P}_{\text{Y}_b} = \delta_{ab}\mathbf{P}_{\text{Y}_a}$.

This particular form of projection operators is not unique: in section 9.2 we built 3-index tensor Young projection operators that were symmetric under transposition.

The Young projection operators constructed in this section are particularly convenient for explicit $U(n)$ computations, and another virtue is that we can write down the projectors explicitly from the standard tableaux, without having to solve a characteristic equation. For multiparticle irreps, the Young projection operators of this section will generally be different from the ones constructed from characteristic equations (see sections. 9.1–9.2); however, the operators are equivalent, since the difference amounts to a choice of basis.

9.4.1 Construction of projection operators

Let Y_a be a k-box standard tableau. Arrange a set of symmetrizers corresponding to the rows in Y_a, and to the right of this arrange a set of antisymmetrizers corresponding to the columns in Y_a. For a Young diagram Y with s rows and t columns we label the rows S_1, S_2, …, S_s and to the columns A_1, A_2, …, A_t. Each symmetry operator in $\mathbf{P}_\mathrm{Y}$ is associated to a row/column in Y, hence we label a symmetry operator after the corresponding row/column, for example,



(9.22)

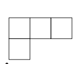
Let the lines numbered 1 to k enter the symmetrizers as described by the numbers in the boxes in the standard tableau and connect the set of symmetrizers to the set of antisymmetrizers in a nonvanishing way, avoiding multiple intermediate lines prohibited by (6.17). Finally, arrange the lines coming out of the antisymmetrizers such that if the lines all passed straight through the symmetry operators, they would exit in the same order as they entered. This ensures that upon expansion of all the symmetry operators, the identity appears exactly once.

We denote by $|\mathrm{S}_i|$ or $|\mathrm{A}_i|$ the *length* of a row or column, respectively, that is the number of boxes it contains. Thus $|\mathrm{A}_i|$ also denotes the number of lines entering the antisymmetrizer A_i. In the above example we have $|\mathrm{S}_1| = 5$, $|\mathrm{A}_2| = 3$, *etc.*

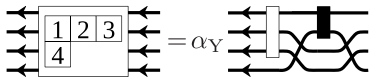
The normalization α_Y is given by

$$\alpha_\mathrm{Y} = \frac{\left(\prod_{i=1}^{s} |\mathrm{S}_i|!\right)\left(\prod_{j=1}^{t} |\mathrm{A}_j|!\right)}{|\mathrm{Y}|}\,, \tag{9.23}$$

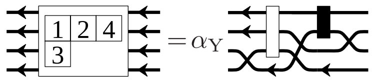
where $|\mathrm{Y}|$ is related through (9.16) to Δ_Y, the dimension of irrep Y of S_k, and is a hook rule S_k combinatoric number. The normalization depends only on the shape of the Young diagram, not the particular tableau.

Example: The Young diagram  tells us to use one symmetrizer of length three, one of length one, one antisymmetrizer of length two, and two of length one.

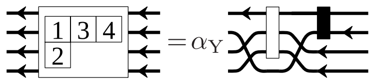
There are three distinct k-standard arrangements, each corresponding to a projection operator



(9.24)



(9.25)



(9.26)

where the normalization constant is $\alpha_Y = 3/2$ by (9.23). More examples of Young projection operators are given in section 9.5.

9.4.2 Properties

We prove in appendix B that the above construction yields well defined projection operators. In particular, the internal connection between the symmetrizers and antisymmetrizers is unique up to an overall sign (proof in appendix B.1). We fix the overall sign by requiring that when all symmetry operators are expanded, the identity appears with a positive coefficient. Note that by construction (the lines exit in the same order as they enter) the identity appears exactly once in the full expansion of any of the Young projection operators.

We list here the most important properties of the Young projection operators:

1. The Young projection operators are *orthogonal*: If Y and Z are two distinct standard tableaux, then $\mathbf{P}_Y\mathbf{P}_Z = 0 = \mathbf{P}_Z\mathbf{P}_Y$.

2. With the normalization (9.23), the Young projection operators are indeed *projection operators*, *i.e.*, they are idempotent: $\mathbf{P}_Y^2 = \mathbf{P}_Y$.

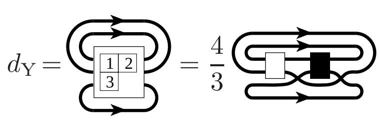
3. For a given k the Young projection operators constitute a complete set such that $\mathbf{1} = \sum \mathbf{P}_Y$, where the sum is over all standard tableaux Y with k boxes.

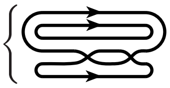
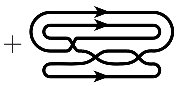
The proofs of these properties are given in appendix B.

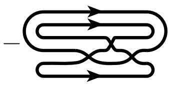
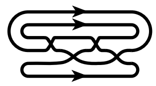
9.4.3 Dimensions of $U(n)$ irreps

The dimension d_Y of a $U(n)$ irrep Y can be computed diagrammatically as the trace of the corresponding Young projection operator, $d_Y = \operatorname{tr} \mathbf{P}_Y$. Expanding the symmetry operators yields a weighted sum of closed-loop diagrams. Each loop is worth n, and since the identity appears precisely once in the expansion, the dimension d_Y of a irrep with a k-box Young tableau Y is a degree k polynomial in n.

Example: We compute we dimension of the $U(n)$ irrep $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$:



$$= \frac{4}{3}\left(\frac{1}{2!}\right)^2 \Bigg\{$$  $+$ 

$-$  $-$  $\Bigg\}$

$$= \frac{1}{3}(n^3 + n^2 - n^2 - n) = \frac{n(n^2-1)}{3} \,. \tag{9.27}$$

In practice, this is unnecessarily laborious. The dimension of a $U(n)$ irrep Y is given by

$$d_{\rm Y} = \frac{f_{\rm Y}(n)}{|Y|} \,. \tag{9.28}$$

Here $f_{\rm Y}(n)$ is a polynomial in n obtained from the Young diagram Y by multiplying the numbers written in the boxes of Y, according to the following rules:

1. The upper left box contains an n.
2. The numbers in a row increase by one when reading from left to right.
3. The numbers in a column decrease by one when reading from top to bottom.

Hence, if k is the number of boxes in Y, $f_{\rm Y}(n)$ is a polynomial in n of degree k. The dimension formula (9.28) is well known (see for instance ref. [138]).

Example: In the above example with the irrep $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, we have

$$d_{\rm Y} = \frac{f_{\rm Y}(n)}{|Y|} = \frac{n(n^2-1)}{3}$$

in agreement with the diagrammatic trace calculation (9.27).

Example: With Y = [4,2,1], we have

$$f_{\rm Y}(n) = \begin{array}{|c|c|c|c|}\hline n & n+1 & n+2 & n+3 \\ \hline n-1 & n \\ \cline{1-2} n-2 \\ \cline{1-1}\end{array} = n^2(n^2-1)(n^2-4)(n+3),$$

$$|{\rm Y}| = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array} = 144, \tag{9.29}$$

hence,

$$d_{\rm Y} = \frac{n^2(n^2-1)(n^2-4)(n+3)}{144} \,. \tag{9.30}$$

Using $d_{\rm Y} = {\rm tr}\, \mathbf{P}_{\rm Y}$, the dimension formula (9.28) can be proven diagrammatically by induction on the number of boxes in the irrep Y. The proof is given in appendix B.4.

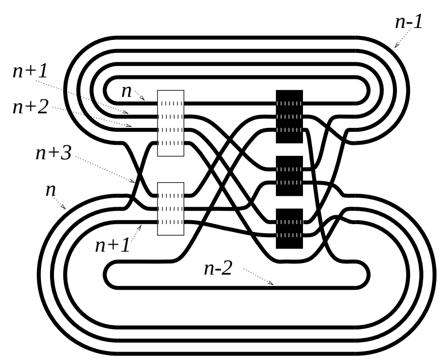
The polynomial $f_{\mathrm{Y}}(n)$ has an intuitive interpretation in terms of strand colorings of the diagram for $\mathrm{tr}\,\mathbf{P}_{\mathrm{Y}}$. Draw the trace of the Young projection operator. Each line is a strand, a closed line, which we draw as passing straight through all of the symmetry operators. For a k-box Young diagram, there are k strands. Given the following set of rules, we count the number of ways to color the k strands using n colors. The top strand (corresponding to the leftmost box in the first row of Y) may be colored in n ways. Color the rest of the strands according to the following rules:

1. If a path, which could be colored in m ways, enters an antisymmetrizer, the lines below it can be colored in $m-1$, $m-2$, … ways.

2. If a path, which could be colored in m ways, enters a symmetrizer, the lines below it can be colored in $m+1$, $m+2$, … ways.

Using this coloring algorithm, the number of ways to color the strands of the diagram is $f_{\mathrm{Y}}(n)$.

Example: For Y = $\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5&7\\\cline{1-3} 8\\\cline{1-1}\end{array}$, the strand diagram is



(9.31)

Each strand is labeled by the number of admissible colorings. Multiplying these numbers and including the factor $1/|\mathrm{Y}|$, we find

$$d_{\mathrm{Y}} = (n-2)\,(n-1)\,n^2\,(n+1)^2\,(n+2)\,(n+3) \Big/ \begin{array}{|c|c|c|c|}\hline 6&4&3&1\\\hline 4&2&1\\\cline{1-3} 1\\\cline{1-1}\end{array}$$

$$= \frac{n\,(n+1)\,(n+3)!}{2^6\,3^2\,(n-3)!}\,,$$

in agreement with (9.28).

9.5 REDUCTION OF TENSOR PRODUCTS

We now work out several explicit examples of decomposition of direct products of Young diagrams/tableaux in order to motivate the general rules for decomposition

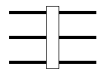
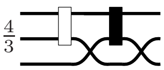
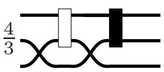
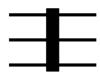
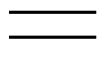
Y_a	$\mathbf{P}_{\mathrm{Y}_a}$	d_{Y_a}
$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\ \hline\end{array}$		$\frac{n(n+1)(n+2)}{6}$
$\begin{array}{\|c\|c\|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}$	$\frac{4}{3}$ 	$\frac{n(n^2-1)}{3}$ (both)
$\begin{array}{\|c\|c\|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}$	$\frac{4}{3}$ 	
$\begin{array}{\|c\|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}$		$\frac{(n-2)(n-1)n}{6}$
$\boxed{1}\otimes\boxed{2}\otimes\boxed{3}$		n^3

Table 9.1 Reduction of 3-index tensor. The last row shows the direct sum of the Young tableaux, the sum of the dimensions of the irreps adding up to n^3, and the sum of the projection operators adding up to the identity as verification of completeness (3.51).

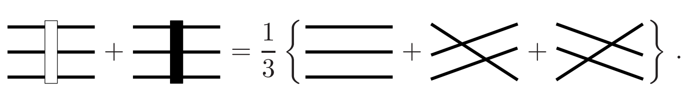
of direct products stated below, in section 9.5.1. We have already treated the decomposition of the 2-index tensor into the symmetric and the antisymmetric tensors, but we shall reconsider the 3-index tensor, since the projection operators are different from those derived from the characteristic equations in section 9.2.

The 3-index tensor reduces to

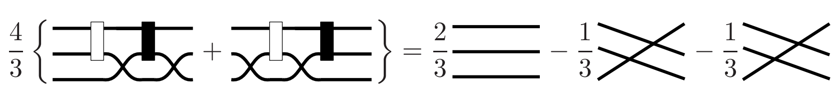
$$\begin{aligned}\boxed{1}\otimes\boxed{2}\otimes\boxed{3} &= \left(\begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}\oplus\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\right)\otimes\boxed{3}\\ &= \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}\oplus\begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\oplus\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\,.\end{aligned} \tag{9.32}$$

The corresponding dimensions and Young projection operators are given in table 9.1. For simplicity, we neglect the arrows on the lines where this leads to no confusion.

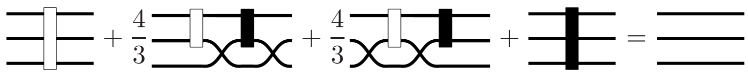
The Young projection operators are orthogonal by inspection. We check completeness by a computation. In the sum of the fully symmetric and the fully antisymmetric tensors, all the odd permutations cancel, and we are left with



Expanding the two tensors of mixed symmetry, we obtain



Adding the two equations we get

 ,(9.33)

verifying the completeness relation.

For 4-index tensors the decomposition is performed as in the 3-index case, resulting in table 9.2.

Acting with any permutation on the fully symmetric or antisymmetric projection operators gives ± 1 times the projection operator (see (6.8) and (6.18)). For projection operators of mixed symmetry the action of a permutation is not as simple, because the permutations will mix the spaces corresponding to the distinct tableaux. Here we shall need only the action of a permutation within a $3n$-j symbol, and, as we shall show below, in this case the result will again be simple, a factor ± 1 or 0.

9.5.1 Reduction of direct products

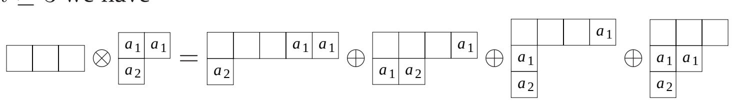
We state the rules for general decompositions of direct products such as (9.20) in terms of Young diagrams:

Draw the two diagrams next to each other and place in each box of the second diagram an a_i, $i = 1, \ldots, k$, such that the boxes in the first row all have a_1 in them, second row boxes have a_2 in them, *etc.* The boxes of the second diagram are now added to the first diagram to create new diagrams according to the following rules:

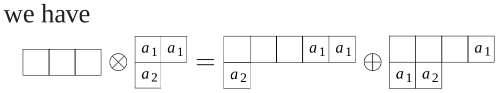
1. Each diagram must be a Young diagram.
2. The number of boxes in the new diagram must be equal to the sum of the number of boxes in the two initial diagrams.
3. For U(n) no diagram has more than n rows.
4. Making a journey through the diagram starting with the top row and entering each row from the right, at any point the number of a_i's encountered in any of the attached boxes must not exceed the number of previously encountered a_{i-1}'s.
5. The numbers must not increase when reading across a row from left to right.
6. The numbers must decrease when reading a column from top to bottom.

Rules 4–6 ensure that states that were previously symmetrized are not antisymmetrized in the product, and vice versa. Also, the rules prevent counting the same state twice.

For example, consider the direct product of the partitions $[3]$ and $[2, 1]$. For $U(n)$ with $n \geq 3$ we have

 ,

while for $n = 2$ we have

 .

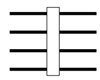
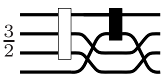
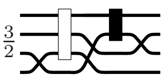
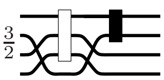
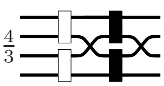
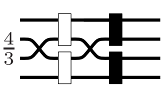
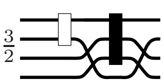
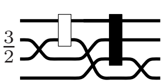
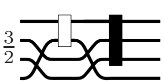
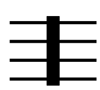
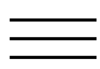
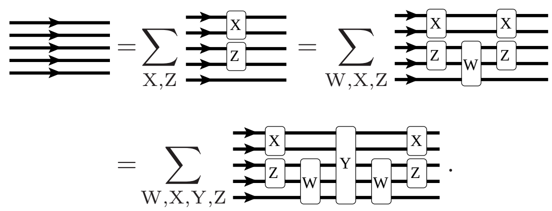
Y_a	$\mathbf{P}_{\mathrm{Y}_a}$	d_{Y_a}
1 2 3 4		$\frac{n(n+1)(n+2)(n+3)}{24}$
1 2 3 4	$\frac{3}{2}$ 	
1 2 4 3	$\frac{3}{2}$ 	$\frac{(n-1)n(n+1)(n+2)}{8}$
1 3 4 2	$\frac{3}{2}$ 	
1 2 3 4	$\frac{4}{3}$ 	$\frac{n^2(n^2-1)}{12}$
1 3 2 4	$\frac{4}{3}$ 	
1 2 3 4	$\frac{3}{2}$ 	
1 3 2 4	$\frac{3}{2}$ 	$\frac{(n-2)(n-1)n(n+1)}{8}$
1 4 2 3	$\frac{3}{2}$ 	
1 2 3 4		$\frac{n(n-1)(n-2)(n-3)}{24}$
1 ⊗ 2 ⊗ 3 ⊗ 4		n^4

Table 9.2 Reduction of 4-index tensors. Note the symmetry under $n \leftrightarrow -n$.

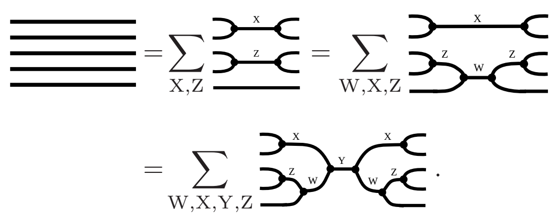
As a check that a decomposition is correct, one can compute the dimensions for the product of irreps on the LHS and the sums of the irreps on the RHS to see that they match. Methods for calculating the dimension of a $U(n)$ irreps are discussed in section 9.4.3.

9.6 $U(n)$ RECOUPLING RELATIONS

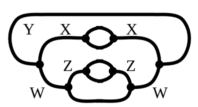
For $U(n)$ (as opposed to $SU(n)$; see section 9.8) we have no antiparticles, so in recoupling relations the total particle number is conserved. Consider as an example the step-by-step reduction of a 5-particle state in terms of the Young projection operators:



More generally, we can visualize any sequence of $U(n)$ pairwise Clebsch-Gordan reductions as a flow with lines joining into thicker and thicker projection operators, always ending in a maximal $\mathbf{P}_{\mathrm{Y}}$ that spans across all lines. In the clebsches notation of section 5.1, this can be redrawn more compactly as

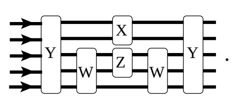


The trace of each term in the final sum of the 5-particle state is a 12-j symbol of the form



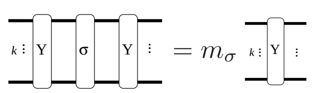
(9.34)

In the trace (9.34) we can use the idempotency of the projection operators to double the maximal Young projection operator $\mathbf{P}_{\mathrm{Y}}$, and sandwich by it all smaller projection operators:



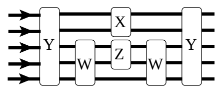
(9.35)

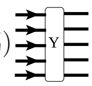
From uniqueness of connection between the symmetry operators (see appendix B.1), we have for any permutation $\sigma \in S_k$:



(9.36)

where $m_\sigma = 0, \pm 1$. Expressions such as (9.35) can be evaluated by expanding the projection operators $\mathbf{P}_\mathrm{W}$, $\mathbf{P}_\mathrm{X}$, $\mathbf{P}_\mathrm{Z}$ and determining the value of m_σ of (9.36) for each permutation σ of the expansion. The result is



$= M(\mathrm{Y};\mathrm{W},\mathrm{X},\mathrm{Z})$


, (9.37)

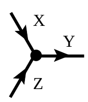
where the factor $M(\mathrm{Y};\mathrm{W},\mathrm{X},\mathrm{Z})$ *does not depend* on n and is determined by a purely symmetric group calculation. Examples follow.

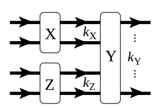
9.7 $U(n)$ $3n$-j SYMBOLS

In this section, we construct $U(n)$ 3-j and 6-j symbols using the Young projection operators, and we give explicit examples of their evaluation. Sum rules for 3-j's and 6-j's are derived in section 9.7.3.

9.7.1 3-j symbols

Let X, Y, and Z be irreps of $U(n)$. In terms of the Young projection operators $\mathbf{P}_\mathrm{X}$, $\mathbf{P}_\mathrm{Y}$, and $\mathbf{P}_\mathrm{Z}$, a $U(n)$ 3-vertex (5.4) is obtained by tying together the three Young projection operators,

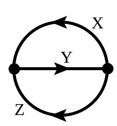


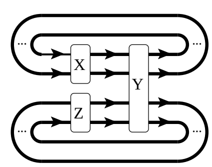
$=$


. (9.38)

Since there are no antiparticles, the construction requires $k_\mathrm{X} + k_\mathrm{Z} = k_\mathrm{Y}$.

A 3-j coefficient constructed from the vertex (9.38) is then



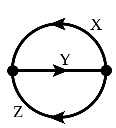
$=$


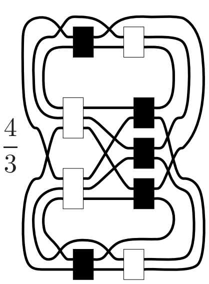
. (9.39)

As an example, take

$$\mathrm{X} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \quad \mathrm{Y} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}\,, \quad \text{and} \quad \mathrm{Z} = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 6 \\ \cline{1-1}\end{array}\,.$$

Then

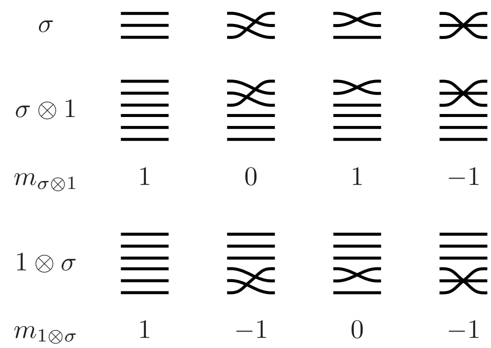


$= \frac{4}{3} \cdot 2 \cdot \frac{4}{3}$

$= M \cdot d_\mathrm{Y}\,,$ (9.40)

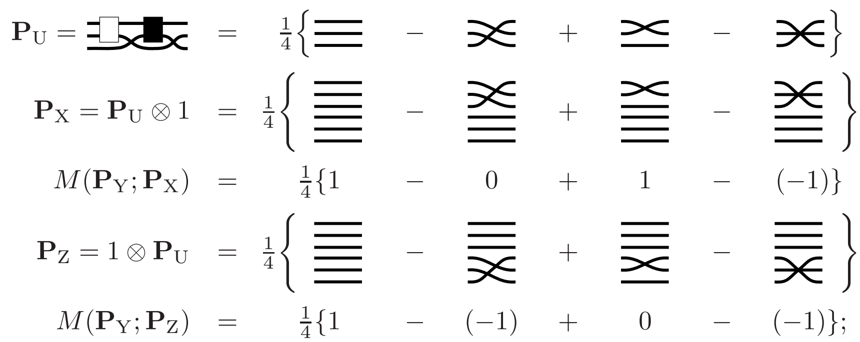
where $M = 1$ here. Below we derive that d_Y (the dimension of the irrep Y) is indeed the value of this 3-j symbol.

In principle the value of a 3-j symbol (9.39) can be computed by expanding out all symmetry operators, but that is not recommended as the number of terms in such expansions grows combinatorially with the total number of boxes in the Young diagram Y. One can do a little better by carefully selecting certain symmetry operators to expand. Then one simplifies the resulting diagrams using rules such as (6.7), (6.8), (6.17), and (6.18) before expanding more symmetry operators. However, a much simpler method exploits (9.36) and leads to the answer — in the case of (9.40) it is $d_Y = (n^2-1)n^2(n+1)(n+2)/144$ – much faster.

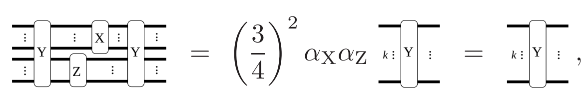
The idea for evaluating a 3-j symbol (9.39) using (9.36) is to expand the projections $\mathbf{P}_X$ and $\mathbf{P}_Z$ and determine the value of m_σ in (9.36) for each permutation σ of the expansion. As an example, consider the 3-j symbol (9.40). With $\mathbf{P}_Y$ as in (9.40) we find



so



and hence

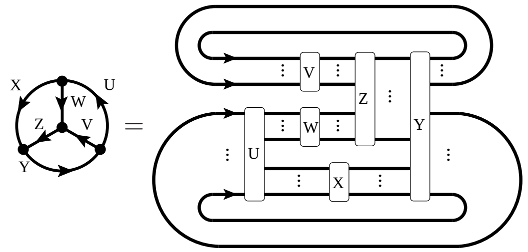


and the value of the 3-j is d_Y as claimed in (9.40). That the eigenvalue happens to be 1 is an accident — in tabulations of 3-j symbols [112] it takes a range of values.

The relation (9.36) implies that the value of any $U(n)$ 3-j symbol (9.39) is $M(Y; X, Z)d_Y$, where d_Y is the dimension of the maximal irrep Y. Again we remark that $M(Y; X, Z)$ is *independent* of n.

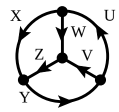
9.7.2 6-j symbols

A general $U(n)$ 6-j symbol has form



(9.41)

Using the relation (9.36) we immediately see that



$$= M\, d_{\mathrm{Y}}\,, \tag{9.42}$$

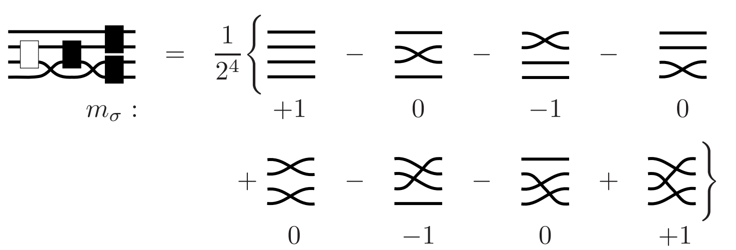
where M is a pure symmetric group $S_{k_{\mathrm{Y}}}$ number, independent of $U(n)$; it is surprising that the only vestige of $U(n)$ is the fact that the value of a 6-j symbol is proportional to the dimension d_{Y} of its largest projection operator.

Example: Consider the 6-j constructed from the Young tableaux

$$\mathrm{U} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\,, \quad \mathrm{V} = \begin{array}{|c|}\hline 1 \\ \hline\end{array}\,, \quad \mathrm{W} = \begin{array}{|c|}\hline 2 \\ \hline\end{array}\,,$$

$$\mathrm{X} = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array}\,, \quad \mathrm{Y} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\,, \quad \mathrm{Z} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\,.$$

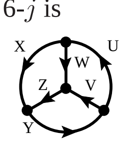
Using the idempotency we can double the projection $\mathbf{P}_{\mathrm{Y}}$ and sandwich the other operators, as in (9.35). Several terms cancel in the expansion of the sandwiched operator, and we are left with



We have listed the symmetry factors m_σ of (9.36) for each of the permutations σ sandwiched between the projection operators $\mathbf{P}_{\mathrm{Y}}$. We find that in this example the symmetric group factor M of (9.42) is

$$M = \frac{4}{2^4}\,\alpha_{\mathrm{U}}\,\alpha_{\mathrm{V}}\,\alpha_{\mathrm{W}}\,\alpha_{\mathrm{X}}\,\alpha_{\mathrm{Z}} = \frac{1}{3}\,,$$

so the value of the 6-j is



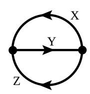
$$= \frac{1}{3} d_{\mathrm{Y}} = \frac{n\,(n^2-1)\,(n-2)}{4!}\,.$$

The method generalizes to evaluations of any $3n$-j symbol of $U(n)$.

Challenge: We have seen that there is a coloring algorithm for the dimensionality of the Young projection operators. *Open question:* Find a coloring algorithm for the 3-j's and 6-j's of $SU(n)$.

9.7.3 Sum rules

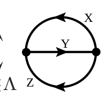
Let Y be a standard tableau with $k_{\rm Y}$ boxes, and let Λ be the set of all standard tableaux with one or more boxes (this excludes the trivial $k = 0$ representation). Then the 3-j symbols obey the sum rule

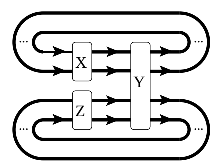
$$\sum_{\rm X,Z\in\Lambda}$$ 

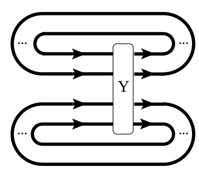
$$= \quad (k_{\rm Y} - 1)d_{\rm Y}. \qquad (9.43)$$

The sum is finite, because the 3-j is nonvanishing only if the number of boxes in X and Z add up to $k_{\rm Y}$, and this happens only for a finite number of tableaux.

To prove the 3-j sum rule (9.43), recall that the Young projection operators constitute a complete set, $\sum_{{\rm X}\in\Lambda_k} \mathbf{P}_{\rm X} = \mathbf{1}$, where $\mathbf{1}$ is the $[k \times k]$ unit matrix and Λ_k the set of all standard tableaux of Young diagrams with k boxes. Hence:

$$\sum_{\rm X,Z\in\Lambda}$$ 

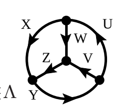
$$= \sum_{k_{\rm X}=1}^{k_{\rm Y}-1} \sum_{\substack{{\rm X}\in\Lambda_{k_{\rm X}} \\ {\rm Z}\in\Lambda_{k_{\rm Y}-k_{\rm X}}}}$$ 

$$= \sum_{k_{\rm X}=1}^{k_{\rm Y}-1}$$ 

$$= \sum_{k_{\rm X}=1}^{k_{\rm Y}-1} d_{\rm Y} \;=\; (k_{\rm Y} - 1)d_{\rm Y}\,.$$

The sum rule offers a useful cross-check on tabulations of 3-j values.

There is a similar sum rule for the 6-j symbols:

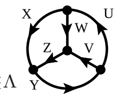
$$\sum_{\rm X,Z,U,V,W\in\Lambda}$$ 

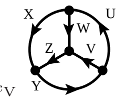
$$= \frac{1}{2}(k_{\rm Y} - 1)(k_{\rm Y} - 2)\, d_{\rm Y}\,. \qquad (9.44)$$

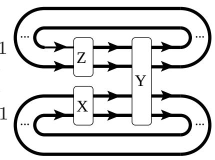
Referring to the 6-j (9.41), let $k_{\rm U}$ be the number of boxes in the Young diagram U, $k_{\rm X}$ be the number of boxes in X, *etc.*

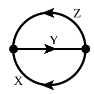
Let $k_{\rm Y}$ be given. From (9.41) we see that $k_{\rm X}$ takes values between 1 and $k_Y - 2$, and $k_{\rm Z}$ takes values between 2 and $k_Y - 1$, subject to the constraint $k_{\rm X} + k_{\rm Z} = k_{\rm Y}$. We now sum over all tableaux U, V, and W keeping $k_{\rm Y}$, $k_{\rm X}$, and $k_{\rm Z}$ fixed. Note that

k_V can take values $1, \ldots, k_Z - 1$. Using completeness, we find

$$\sum_{U,V,W\in\Lambda}$$


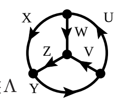
$$= \sum_{k_V=1}^{k_Z-1} \sum_{V\in\Lambda_{k_V}} \sum_{W\in\Lambda_{k_Z-k_V}} \sum_{U\in\Lambda_{k_Y-k_V}}$$


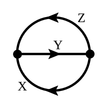
$$= \sum_{k_V=1}^{k_Z-1}$$


$$= (k_Z - 1)$$


.

Now sum over all tableaux X and Z to find

$$\sum_{X,Z,U,V,W\in\Lambda}$$


$$= \sum_{k_Z=2}^{k_Y-1} (k_Z - 1) \sum_{Z\in\Lambda_{k_Z}} \sum_{X\in\Lambda_{k_Y-k_Z}}$$


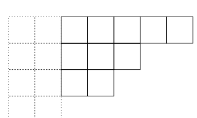
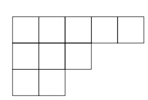
$$= \frac{1}{2}(k_Y - 1)(k_Y - 2)\, d_Y \,,$$

verifying the sum rule (9.44) for 6-j symbols.

9.8 $SU(n)$ AND THE ADJOINT REP

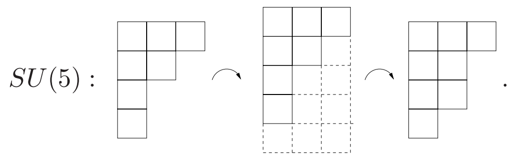
The $SU(n)$ group elements satisfy $\det G = 1$, so $SU(n)$ has an additional invariant, the Levi-Civita tensor $\varepsilon_{a_1 a_2 \ldots a_n} = G_{a_1}{}^{a'_1} G_{a_2}{}^{a'_2} \cdots G_{a_n}{}^{a'_n} \varepsilon_{a'_1 a'_2 \ldots a'_n}$. The diagrammatic notation for the Levi-Civita tensors was introduced in (6.27).

While the irreps of $U(n)$ are labeled by the standard tableaux with no more than n rows (see section 9.3), the standard tableaux with a maximum of $n-1$ rows label the irreps of $SU(n)$. The reason is that in $SU(n)$, a column of length n can be removed from any diagram by contraction with the Levi-Civita tensor (6.27). For example, for $SU(4)$

 $\rightarrow$  . (9.45)

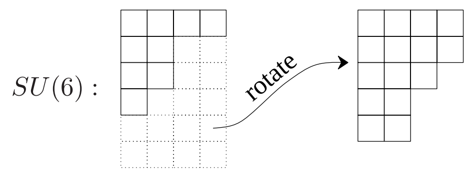
Standard tableaux that differ only by columns of length n correspond to equivalent irreps. Hence, for the standard tableaux labeling irreps of $SU(n)$, the highest column is of height $n-1$, which is also the rank of $SU(n)$. A rep of $SU(n)$, or A_{n-1} in the Cartan classification (table 7.6) is characterized by $n-1$ *Dynkin labels* $b_1 b_2 \ldots b_{n-1}$. The corresponding Young diagram (defined in section 9.3.1) is then given by $(b_1 b_2 \ldots b_{n-1} 00 \ldots)$, or $(b_1 b_2 \ldots b_{n-1})$ for short.

For $SU(n)$ a column with k boxes (antisymmetrization of k covariant indices) can be converted by contraction with the Levi-Civita tensor into a column of $(n-k)$ boxes (corresponding to $(n-k)$ contravariant indices). This operation associates with each diagram a conjugate diagram. Thus the *conjugate* of a $SU(n)$ Young diagram Y is constructed from the missing pieces needed to complete the rectangle of n rows,



(9.46)

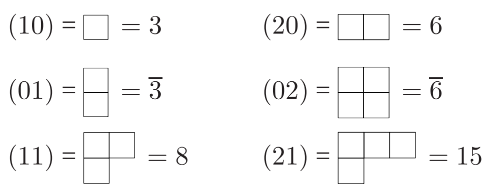
To find the conjugate diagram, add squares below the diagram of Y such that the resulting figure is a rectangle with height n and width of the top row in Y. Remove the squares corresponding to Y and rotate the rest by 180 degrees. The result is the conjugate diagram of Y. For example, for $SU(6)$ the irrep (20110) has (01102) as its conjugate rep:



(9.47)

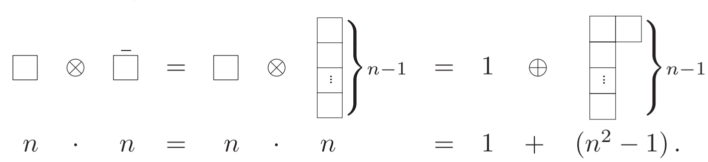
In general, the $SU(n)$ reps $(b_1 b_2 \ldots b_{n-1})$ and $(b_{n-1} \ldots b_2 b_1)$ are conjugate. For example, $(10 \ldots 0)$ stands for the defining rep, and its conjugate is $(00 \ldots 01)$, *i.e.*, a column of $n-1$ boxes.

The Levi-Civita tensor converts an antisymmetrized collection of n–1 "in"-indices into 1 "out"-index, or, in other words, it converts an $(n$–$1)$-particle state into a single antiparticle state. We use $\overline{\square}$ to denote the single antiparticle state; it is the conjugate of the fundamental representation $\square$ single particle state. For example, for $SU(3)$ we have



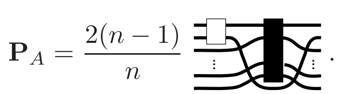
(9.48)

The product of the fundamental rep $\square$ and the conjugate rep $\overline{\square}$ of $SU(n)$ decomposes into a singlet and the *adjoint representation*:



Note that the conjugate of the diagram for the adjoint is again the adjoint.

Using the construction of section 9.4, the birdtrack Young projection operator for the adjoint representation A can be written



Using $\mathbf{P}_A$ and the definition (9.38) of the 3-vertex, $SU(n)$ group theory weights involving quarks, antiquarks, and gluons can be calculated by expansion of the symmetry operators or by application of the recoupling relation. For this reason, we prefer to keep the conjugate reps conjugate, rather than replacing them by columns of $(n-1)$ defining reps, as this will give us $SU(n)$ expressions valid for any n.

9.9 AN APPLICATION OF THE NEGATIVE DIMENSIONALITY THEOREM

An $SU(n)$ invariant scalar is a fully contracted object (vacuum bubble) consisting of Kronecker deltas and Levi-Civita symbols. Since there are no external legs, the Levi-Civitas appear only in pairs, making it possible to combine them into antisymmetrizers. In the birdtrack notation, an $SU(n)$ invariant scalar is therefore a vacuum bubble graph built only from symmetrizers and antisymmetrizers.

The negative dimensionality theorem for $SU(n)$ states that for any $SU(n)$ invariant scalar exchanging symmetrizers and antisymmetrizers is equivalent to replacing n by $-n$:

$$\mathrm{SU}(n) = \overline{\mathrm{SU}}(-n) \,, \tag{9.49}$$

where the bar on $\overline{\mathrm{SU}}$ indicates transposition, *i.e.*, exchange of symmetrizations and antisymmetrizations. The theorem also applies to $U(n)$ invariant scalars, since the only difference between $U(n)$ and $SU(n)$ is the invariance of the Levi-Civita tensor in $SU(n)$. The proof of this theorem is given in chapter 13.

We can apply the negative dimensionality theorem to computations of the dimensions of the $U(n)$ irreps, $d_{\mathrm{Y}} = \mathrm{tr}\,\mathbf{P}_{\mathrm{Y}}$. Taking the transpose of a Young diagram interchanges rows and columns, and it is therefore equivalent to interchanging the symmetrizers and antisymmetrizers in $\mathrm{tr}\,\mathbf{P}_{\mathrm{Y}}$. The dimension of the irrep corresponding to the transpose Young diagram Y^t can then be related to the dimension of the irrep labeled by Y as $d_{\mathrm{Y}^t}(n) = d_{\mathrm{Y}}(-n)$ by the negative dimensionality theorem.

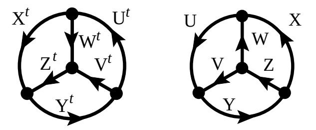
Example: $[3,1]$ is the transpose of $[2,1,1]$,

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \right)^t = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \,.$$

Note the $n \to -n$ duality in the dimension formulas for these and other tableaux (table 9.2).

Now for standard tableaux X, Y, and Z, compare the diagram of the 3-j constructed from X, Y, and Z to that constructed from X^t, Z^t, and Y^t. The diagrams are related by a reflection in a vertical line, reversal of all the arrows on the lines, and interchange of symmetrizers and antisymmetrizers. The first two operations do not change the value of the diagram, and by the negative dimensionality theorem the values of two 3-j's are related by $n \leftrightarrow -n$ (and possibly an overall sign; this sign is fixed by requiring that the highest power of n comes with a positive coefficient). In tabulations, it suffices to calculate approximately half of all 3-j's. Furthermore, the 3-j sum rule (9.43) provides a cross-check.

The two 6-j symbols

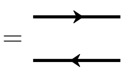


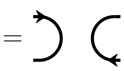
(9.50)

are related by a reflection in a vertical line, reversal of all the arrows on the lines, and interchange of symmetrizers and antisymmetrizers — this can be seen by writing out the 6-j symbols in terms of the Young projection operators as in (9.41). By the negative dimensionality theorem, the values of the two 6-j symbols are therefore related by $n \leftrightarrow -n$.

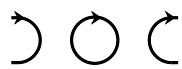
9.10 $SU(n)$ MIXED TWO-INDEX TENSORS

We now return to the construction of projection operators from characteristic equations. Consider mixed tensors $q^{(1)} \otimes \overline{q}^{(2)} \in V \otimes \overline{V}$. The Kronecker delta invariants are the same as in section 9.1, but now they are drawn differently (we are looking at a "cross channel"):

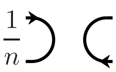
identity: $\mathbf{1} = 1_{a,d}^{b\ c} = \delta_a^c \delta_d^b =$  ,

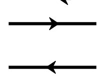
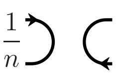
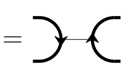
trace: $\mathbf{T} = T_{a,d}^{b\ c} = \delta_a^b \delta_d^c =$  . (9.51)

The $\mathbf{T}$ matrix satisfies a trivial characteristic equation

$\mathbf{T}^2 =$  $= n\mathbf{T}\,,$ (9.52)

i.e., $\mathbf{T}(\mathbf{T} - n\mathbf{1}) = 0$, with roots $\lambda_1 = 0,\ \lambda_2 = n$. The corresponding projection operators (3.48) are

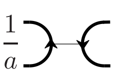
$\mathbf{P}_1 = \frac{1}{n}\mathbf{T} = \frac{1}{n}$  , (9.53)

$\mathbf{P}_2 = \mathbf{1} - \frac{1}{n}\mathbf{T} =$  $- \frac{1}{n}$  $=$  , (9.54)

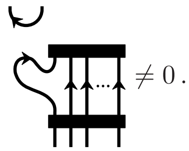
with dimensions $d_1 = \mathrm{tr}\,\mathbf{P}_1 = 1,\ d_2 = \mathrm{tr}\,\mathbf{P}_2 = n^2 - 1$. $\mathbf{P}_2$ is the projection operator for the adjoint rep of $SU(n)$. In this way, the invariant matrix $\mathbf{T}$ has resolved the space of tensors $x_b^a \in V \otimes \overline{V}$ into a singlet and a traceless part,

$$\mathbf{P}_1 x = \frac{1}{n} x_c^c \delta_a^b\,, \qquad \mathbf{P}_2 x = x_a^b - \left(\frac{1}{n} x_c^c\right) \delta_a^b\,. \tag{9.55}$$

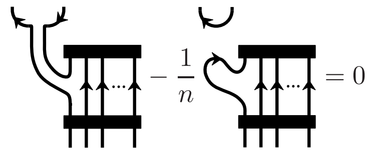
Both projection operators leave δ_b^a invariant, so the generators of the unitary transformations are given by their sum

$U(n):\quad \frac{1}{a}$  $=$, (9.56)

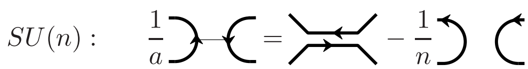
and the dimension of the $U(n)$ adjoint rep is $N = \mathrm{tr}\,\mathbf{P}_A = \delta_a^a \delta_b^b = n^2$. If we extend the list of primitive invariants from the Kronecker delta to the Kronecker delta and the Levi-Civita tensor (6.27), the singlet subspace does not satisfy the invariance condition (6.56)



For the traceless subspace (9.54), the invariance condition is



This is the same relation as (6.25), as can be shown by expanding the antisymmetrization operator using (6.19), so the invariance condition is satisfied. The adjoint rep is given by



$$\frac{1}{a}(T_i)^a_b\,(T_i)^d_c = \delta^a_c\delta^d_b - \frac{1}{n}\delta^a_b\delta^d_c\,. \tag{9.57}$$

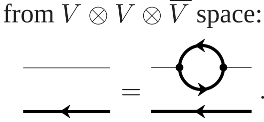
The special unitary group $SU(n)$ is, by definition, the invariance group of the Levi-Civita tensor (hence "special") and the Kronecker delta (hence "unitary"), and its dimension is $N = n^2 - 1$. The defining rep Dynkin index follows from (7.27) and (7.28)

$$\ell^{-1} = 2n \tag{9.58}$$

(This was evaluated in the example of section 2.2.) The Dynkin index for the singlet rep (9.55) vanishes identically, as it does for any singlet rep.

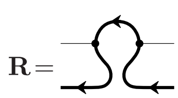
9.11 $SU(n)$ MIXED DEFINING $\otimes$ ADJOINT TENSORS

In this and the following section we generalize the reduction by invariant matrices to spaces other than the defining rep. Such techniques will be very useful later on, in our construction of the exceptional Lie groups. We consider the defining $\otimes$ adjoint tensor space as a projection from $V \otimes V \otimes \overline{V}$ space:

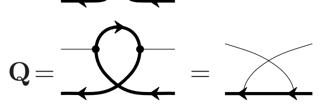


(9.59)

The following two invariant matrices acting on $V^2 \otimes \overline{V}$ space contract or interchange defining rep indices:

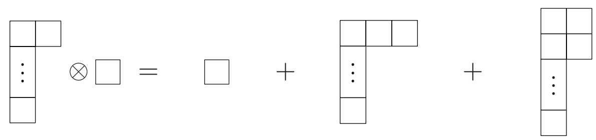


(9.60)



(9.61)

	$\underline{A} \otimes \underline{q}$	$=$	$\underline{V_1}$	$\oplus$	$\underline{V_2}$	$\oplus$	$\underline{V_3}$
Dynkin labels	$(10\ldots1) \otimes (10\ldots)$	$=$	$(10\ldots)$	$\oplus$	$(200\ldots01)$	$\oplus$	$(010\ldots01)$
Dimensions:	$(n^2-1)n$	$=$	n	$+$	$\frac{n(n-1)(n+2)}{2}$	$+$	$\frac{n(n+1)(n-2)}{2}$
Indices:	$n + \frac{n^2-1}{2n}$	$=$	$\frac{1}{2n}$	$+$	$\frac{(n+2)(3n-1)}{4n}$	$+$	$\frac{(n-2)(3n+1)}{4n}$
SU(3) example:							
Dimensions:	$8 \cdot 3$	$=$	3	$+$	15	$+$	6
Indices:	$13/3$	$=$	$1/6$	$+$	$10/3$	$+$	$5/6$
SU(4) example:							
Dimensions:	$15 \cdot 4$	$=$	4	$+$	36	$+$	20
Indices:	$47/8$	$=$	$1/8$	$+$	$33/8$	$+$	$13/8$



Projection operators:

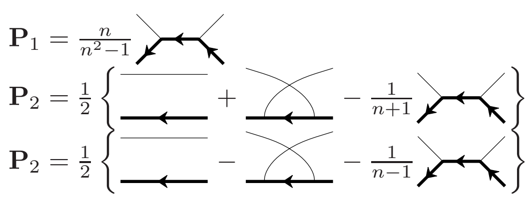
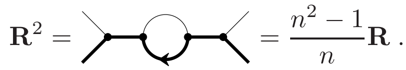


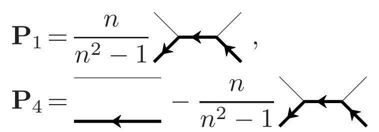
Table 9.3 $SU(n)\ V \otimes A$ Clebsch-Gordan series.

R projects onto the defining space and satisfies the characteristic equation



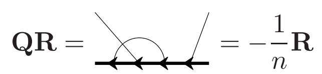
(9.62)

The corresponding projection operators (3.48) are



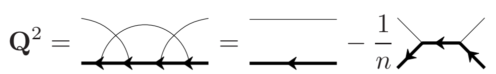
(9.63)

Q takes a single eigenvalue on the $\mathbf{P}_1$ subspace



(9.64)

$\mathbf{Q}^2$ is computed by inserting the adjoint rep projection operator (9.57):

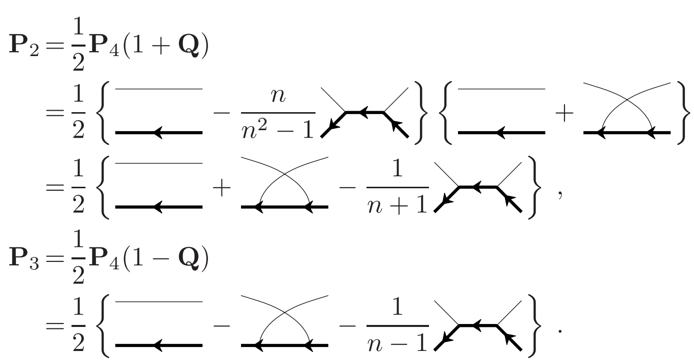


(9.65)

The projection on the $\mathbf{P}_4$ subspace yields the characteristic equation

$$\mathbf{P}_4(\mathbf{Q}^2 - 1) = 0 \,, \tag{9.66}$$

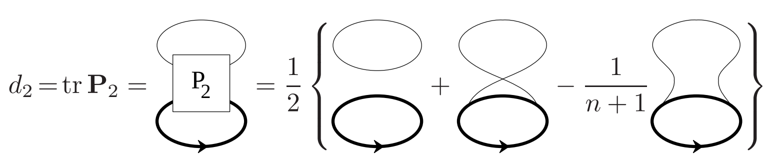
with the associated projection operators



(9.67)

(9.68)

The dimensions of the two subspaces are computed by taking traces of their projection operators:

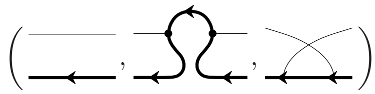


$$= \frac{1}{2}\left(nN + N - N/(n+1)\right) = \frac{1}{2}(n-1)n(n+2) \tag{9.69}$$

and similarly for d_3. This is tabulated in table 9.3.

9.11.1 Algebra of invariants

Mostly for illustration purposes, let us now perform the same calculation by utilizing the algebra of invariants method outlined in section 3.4. A possible basis set, picked from the $V \otimes A \to V \otimes A$ linearly independent tree invariants, consists of

$$(\mathbf{e}, \mathbf{R}, \mathbf{Q}) =$$  . (9.70)

The multiplication table (3.42) has been worked out in (9.62), (9.64), and (9.65). For example, the $(t_\alpha)_\beta{}^\gamma$ matrix rep for $\mathbf{Qt}$ is

$$\sum_{\gamma \in \mathcal{T}} (\mathbf{Q})_\beta{}^\gamma \mathbf{t}_\gamma = \mathbf{Q} \begin{pmatrix} \mathbf{e} \\ \mathbf{R} \\ \mathbf{Q} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1/n & 0 \\ 1 & -1/n & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{R} \\ \mathbf{Q} \end{pmatrix} \tag{9.71}$$

and similarly for $\mathbf{R}$. In this way, we obtain the $[3 \times 3]$ matrix rep of the algebra of invariants

$$\{\mathbf{e}, \mathbf{R}, \mathbf{Q}\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & n - \frac{1}{n} & 0 \\ 0 & -1/n & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1/n & 0 \\ 1 & -1/n & 0 \end{pmatrix} \right\} . \tag{9.72}$$

From (9.62) we already know that the eigenvalues of $\mathbf{R}$ are $\{0, 0, n - 1/n\}$. The last eigenvalue yields the projection operator $\mathbf{P}_1 = (n - 1/n)^{-1}$, but the projection operator $\mathbf{P}_4$ yields a 2-dimensional degenerate rep. $\mathbf{Q}$ has three distinct eigenvalues $\{-1/n, 1, -1\}$ and is thus more interesting; the corresponding projection operators fully decompose the $V \otimes A$ space. The $-1/n$ eigenspace projection operator is again $\mathbf{P}_1$, but $\mathbf{P}_4$ is split into two subspaces, verifying (9.68) and (9.67):

$$\mathbf{P}_2 = \frac{(\mathbf{Q} + \mathbf{1})(\mathbf{Q} + \frac{1}{n}\mathbf{1})}{(1+1)(1+1/n)} = \frac{1}{2}\left(\mathbf{1} + \mathbf{Q} - \frac{1}{n+1}\mathbf{R}\right)$$

$$\mathbf{P}_3 = \frac{(\mathbf{Q} - \mathbf{1})(\mathbf{Q} + \frac{1}{n}\mathbf{1})}{(-1-1)(-1+1/n)} = \frac{1}{2}\left(\mathbf{1} - \mathbf{Q} - \frac{1}{n-1}\mathbf{R}\right) . \tag{9.73}$$

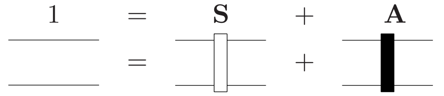
We see that the matrix rep of the algebra of invariants is an alternative tool for implementing the full reduction, perhaps easier to implement as a computation than an out and out birdtracks evaluation.

To summarize, the invariant matrix $\mathbf{R}$ projects out the 1-particle subspace $\mathbf{P}_1$. The particle exchange matrix $\mathbf{Q}$ splits the remainder into the irreducible $V \otimes A$ subspaces $\mathbf{P}_2$ and $\mathbf{P}_3$.

9.12 $SU(n)$ TWO-INDEX ADJOINT TENSORS

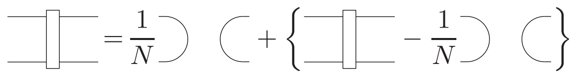
Consider the Kronecker product of two adjoint reps. We want to reduce the space of tensors $x_{ij} \in A \otimes A$, with $i = 1, 2, \ldots N$. The first decomposition is the obvious decomposition (9.4) into the symmetric and antisymmetric subspaces,

$$1 = \mathbf{S} + \mathbf{A}$$

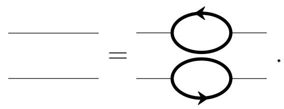
 . (9.74)

The symmetric part can be split into the trace and the traceless part, as in (9.54):

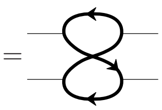
$$\mathbf{S} = \frac{1}{N}\mathbf{T} + \mathbf{P}_S$$

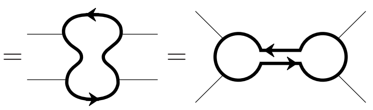
 (9.75)

Further decomposition can be effected by studying invariant matrices in the $V^2 \otimes \overline{V}^2$ space. We can visualize the relation between $A \otimes A$ and $V^2 \otimes \overline{V}^2$ by the identity

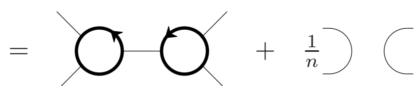
 (9.76)

This suggests the introduction of two invariant matrices:

$\mathbf{Q} =$  (9.77)

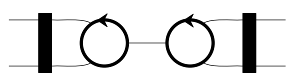
$\mathbf{R} =$  (9.78)

$\mathbf{R}$ can be decomposed by (9.54) into a singlet and the adjoint rep

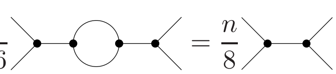
$\mathbf{R} =$  (9.79)

$$= \mathbf{R}' + \frac{1}{n}\mathbf{T}\,.$$

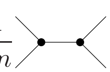
The singlet has already been taken into account in the trace-traceless tensor decomposition (9.75). The $\mathbf{R}'$ projection on the antisymmetric subspace is

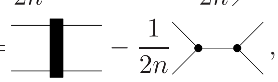
$\mathbf{AR'A} =$  (9.80)

By the Lie algebra (4.47),

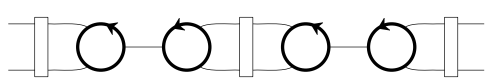
$(\mathbf{AR'A})^2 = \frac{1}{16}$  $= \frac{n}{2}\mathbf{AR'A}\,,$ (9.81)

and the associated projection operators,

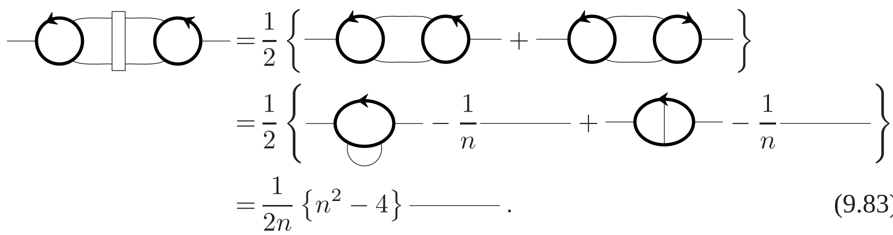
$(\mathbf{P}_5)_{ij,kl} = \frac{1}{2n} C_{ijm} C_{mlk} = \frac{1}{2n}$ 

$\mathbf{P}_a =$  (9.82)

split the antisymmetric subspace into the adjoint rep and a remainder. On the symmetric subspace (9.75), $\mathbf{R}'$ acts as $\mathbf{P}_S\mathbf{R}'\mathbf{P}_S$. As $\mathbf{R}'\mathbf{T} = 0$, this is the same as $\mathbf{SR'S}$. Consider

$(\mathbf{SR'S})^2 =$ 

We compute

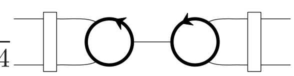


(9.83)

Hence, $\mathbf{SR'S}$ satisfies the characteristic equation

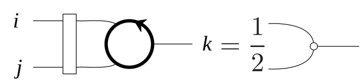
$$\left(\mathbf{SR'S} - \frac{n^2-4}{2n}\right)\mathbf{SR'S} = 0\,. \tag{9.84}$$

The associated projection operators split up the traceless symmetric subspace (9.75) into the adjoint rep and a remainder:

$$\mathbf{P}_2 = \frac{2n}{n^2-4}\mathbf{SR'S} = \frac{2n}{n^2-4}$$  , (9.85)

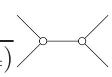
$$\mathbf{P}_{2'} = \mathbf{P}_S - \mathbf{P}_2\,. \tag{9.86}$$

The Clebsch-Gordan coefficients for $\mathbf{P}_2$ are known as the Gell-Mann d_{ijk} tensors [137]:

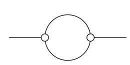


$$= \frac{1}{2}d_{ijk}\,. \tag{9.87}$$

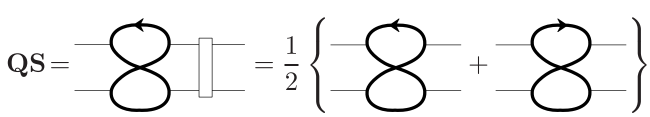
For $SU(3)$, $\mathbf{P}_2$ is the projection operator $(\underline{8}\otimes\underline{8})$ symmetric $\rightarrow \underline{8}$. In terms of d_{ijk}'s, we have

$$(\mathbf{P}_2)_{ij,k\ell} = \frac{n}{2(n^2-4)}d_{ijm}d_{mk\ell} = \frac{n}{2(n^2-4)}$$  , (9.88)

with the normalization

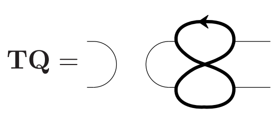
$$d_{ijk}d_{kj\ell} =$$  $$= \frac{2(n^2-4)}{n}\delta_{i\ell}\,. \tag{9.89}$$

Next we turn to the decomposition of the symmetric subspace induced by matrix $\mathbf{Q}$ (9.77). $\mathbf{Q}$ commutes with $\mathbf{S}$:



$$= \mathbf{SQ} = \mathbf{SQS}\,. \tag{9.90}$$

On the 1-dimensional subspace in (9.75), it takes eigenvalue $-1/n$

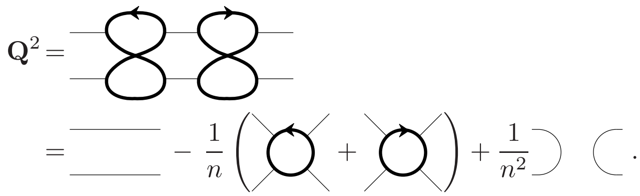


$$= -\frac{1}{n}\mathbf{T}\,; \tag{9.91}$$

so $\mathbf{Q}$ also commutes with the projection operator $\mathbf{P}_S$ from (9.75),

$$\mathbf{QP}_S = \mathbf{Q}\left(\mathbf{S} - \frac{1}{n^2-1}T\right) = \mathbf{P}_S\mathbf{Q}\,. \tag{9.92}$$

$\mathbf{Q}^2$ is easily evaluated by inserting the adjoint rep projection operators (9.54)

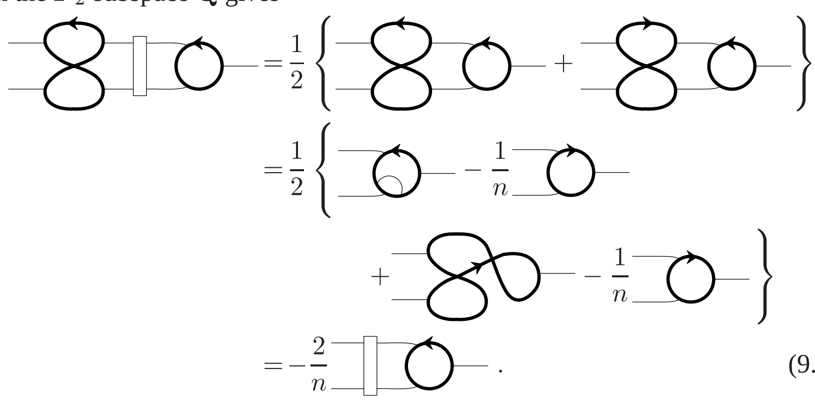


(9.93)

Projecting on the traceless symmetric subspace gives

$$\mathbf{P}_S\left(\mathbf{Q}^2 - 1 + \frac{n^2-4}{n^2}\mathbf{P}_2\right) = 0\,. \tag{9.94}$$

On the $\mathbf{P}_2$ subspace $\mathbf{Q}$ gives



(9.95)

Hence, $\mathbf{Q}$ has a single eigenvalue,

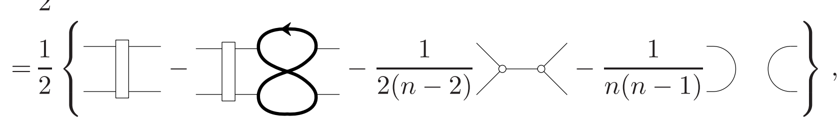
$$\mathbf{Q}\mathbf{P}_2 = -\frac{2}{n}\mathbf{P}_2\,, \tag{9.96}$$

and does not decompose the $\mathbf{P}_2$ subspace; this is as it should be, as $\mathbf{P}_2$ is the adjoint rep and is thus irreducible. On $\mathbf{P}_{2'}$ subspace (9.93) yields a characteristic equation

$$\mathbf{P}_{2'}(\mathbf{Q}^2 - 1) = 0\,,$$

with the associated projection operators

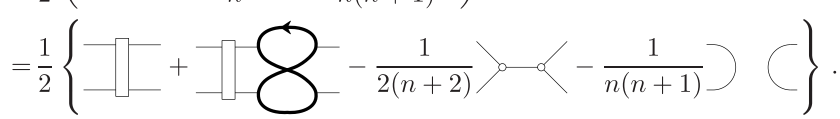
$$\mathbf{P}_3 = \frac{1}{2}\mathbf{P}_{2'}(1-\mathbf{Q}) \tag{9.97}$$



$$\mathbf{P}_4 = \frac{1}{2}\mathbf{P}_{2'}(1+\mathbf{Q}) = \frac{1}{2}(\mathbf{P}_S - \mathbf{P}_1)(1+\mathbf{Q})$$

$$= \frac{1}{2}\left(\mathbf{P}_S - \mathbf{P}_1 + \mathbf{S}\mathbf{Q} - \frac{1}{n^2-1}\mathbf{T}\mathbf{Q} + \frac{2}{n}\mathbf{P}_1\right)$$

$$= \frac{1}{2}\left(\mathbf{S} + \mathbf{S}\mathbf{Q} - \frac{n-2}{n}\mathbf{P}_1 - \frac{1}{n(n+1)}\mathbf{T}\right) \tag{9.98}$$



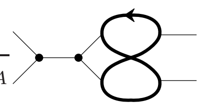
This completes the reduction of the symmetric subspace in (9.74). As in (9.90), $\mathbf{Q}$ commutes with $\mathbf{A}$

$$\mathbf{QA} = \mathbf{AQ} = \mathbf{AQA}\ . \qquad (9.99)$$

On the antisymmetric subspace, the $\mathbf{Q}^2$ equation (9.93) becomes

$$0 = \mathbf{A}\left(\mathbf{Q}^2 - 1 + \frac{2}{n}\mathbf{R}\right)\ , \qquad \mathbf{A} = \mathbf{A}(\mathbf{Q}^2 - 1 - \mathbf{P}_A)\ . \qquad (9.100)$$

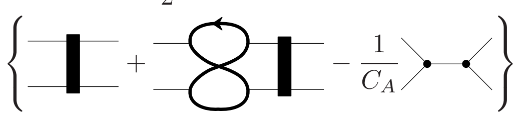
The adjoint rep (9.82) should be irreducible. Indeed, it follows from the Lie algebra, that $\mathbf{Q}$ has zero eigenvalue for any simple group:

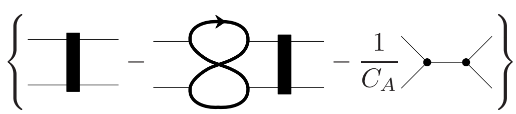
$$\mathbf{P}_5\mathbf{Q} = \frac{1}{C_A}$$

$$= 0\ . \qquad (9.101)$$

On the remaining antisymmetric subspace $\mathbf{P}_a$ (9.100) yields the characteristic equation

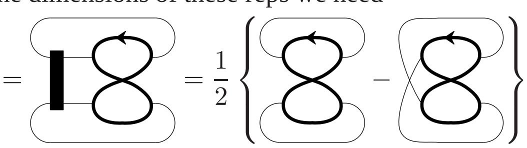
$$\mathbf{P}_a(\mathbf{Q}^2 - 1) = 0\ , \qquad (9.102)$$

with corresponding projection operators

$$\mathbf{P}_6 = \frac{1}{2}\mathbf{P}_a(1 + \mathbf{Q}) = \frac{1}{2}\mathbf{A}(1 + \mathbf{Q} - \mathbf{P}_A)$$
$$= \frac{1}{2}$$

$$, \qquad (9.103)$$

$$\mathbf{P}_7 = \frac{1}{2}\mathbf{P}_a(1 - \mathbf{Q})$$
$$= \frac{1}{2}$$

$$. \qquad (9.104)$$

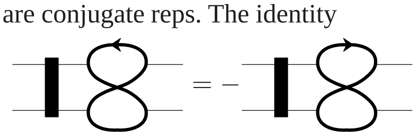
To compute the dimensions of these reps we need

$$\mathrm{tr}\,\mathbf{AQ} =$$

$$= 0\ , \qquad (9.105)$$

so both reps have the same dimension

$$d_6 = d_7 = \frac{1}{2}(\mathrm{tr}\,\mathbf{A} - \mathrm{tr}\,\mathbf{P}_A) = \frac{1}{2}\left\{\frac{(n^2-1)(n^2-2)}{2} - n^2 - 1\right\}$$
$$= \frac{(n^2-1)(n^2-4)}{4}\ . \qquad (9.106)$$

Indeed, the two reps are conjugate reps. The identity

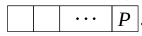

$$, \qquad (9.107)$$

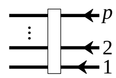
obtained by interchanging the two left adjoint rep legs, implies that the projection operators (9.103) and (9.104) are related by the reversal of the loop arrow. This is the birdtrack notation for complex conjugation (see section 4.1).

This decomposition of two $SU(n)$ adjoint reps is summarized in table 9.4.

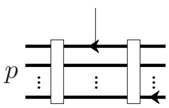
9.13 CASIMIRS FOR THE FULLY SYMMETRIC REPS OF $SU(n)$

In this section we carry out a few explicit birdtrack casimir evaluations.

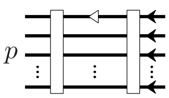
Consider the fully symmetric Kronecker product of p particle reps. Its Dynkin label (defined on page 106) is $(p, 0, 0 \ldots 0)$, and the corresponding Young tableau is a row of p boxes: . The projection operator is given by (6.4)

$\mathbf{P}_S = \mathbf{S} =$ ,

and the generator (4.40) in the symmetric rep is

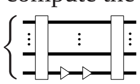
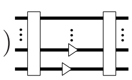
$T^i = p$  . (9.108)

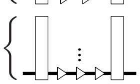
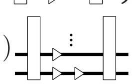
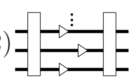
To compute the casimirs, we introduce matrices:

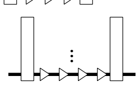
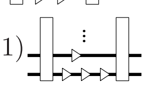
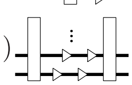
$X = x_i T^i = p$ 

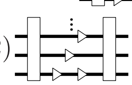
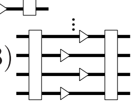
$X_a^b = x_i (T^i)_a^b = a$ b . (9.109)

We next compute the powers of X:

$X^2 = p \Big\{$  $+ (p-1)$  $\Big\}$

$X^3 = p \Big\{$  $+ 3(p-1)$  $+ (p-1)(p-2)$  $\Big\}$

$X^4 = p \Big\{$  $+ 4(p-1)$  $+ 3(p-1)$ 

$+6(p-1)(p-2)$  $+ (p-1)(p-2)(p-3)$  $\Big\}$

$\vdots$ (9.110)

The $\mathrm{tr}\, X^k$ are then

$$\mathrm{tr}\, X^0 = d_s \binom{n+p-1}{p} \qquad \text{(see (6.13))} \tag{9.111}$$

$$\mathrm{tr}\, X = 0 \qquad \text{(semisimplicity)} \tag{9.112}$$

$$\mathrm{tr}\, X^2 = d_s \frac{p(p+n)}{n(n+1)} \mathrm{tr}\, x^2 \tag{9.113}$$

$$\begin{aligned} \mathrm{tr}\, X^3 &= \frac{d_s}{n} p \left(1 + 3\frac{p-1}{n+1} + 2\frac{(p-1)(p-2)}{(n+1)(n+2)} \right) \mathrm{tr}\, x^3 \\ &= \frac{(n+p)!(n+2p)}{(n+2)!(p-1)!} \mathrm{tr}\, x^3 = d_s \frac{p(n+p)(n+2p)}{n(n+1)(n+2)} \mathrm{tr}\, x^3 \end{aligned} \tag{9.114}$$

$$\begin{aligned} \mathrm{tr}\, X^4 = d\frac{p}{n} \Big\{ &\left(1 + 7\frac{p-1}{n+1} + 12\frac{p-1}{n+1}\frac{p-2}{n+2} + 6\frac{p-1}{n+1}\frac{p-2}{n+2}\frac{p-3}{n+3} \right) \mathrm{tr}\, x^4 \\ &+ \frac{p-1}{n+1} \left(3 + 6\frac{p-2}{n+2} + 3\frac{p-2}{n+2}\frac{p-3}{n+3} \right) \left(\mathrm{tr}\, x^2\right)^2 \Big\} \, . \end{aligned} \tag{9.115}$$

The quadratic Dynkin index is given by the ratio of $\mathrm{tr}\, X^2$ and $\mathrm{tr}_A\, X^2$ for the adjoint rep (7.30):

$$\ell_2 = \frac{\mathrm{tr}\, X^2}{\mathrm{tr}_A\, X^2} = \frac{d_s p(p+n)}{2n^2(n+1)} \,. \tag{9.116}$$

To take a random example from the Patera-Sankoff tables [273], the $SU(6)$ rep dimension and Dynkin index

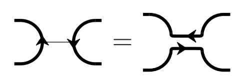
rep	dim	ℓ_2
(0,0,0,0,0,14)	11628	6460

(9.117)

check with the above expressions.

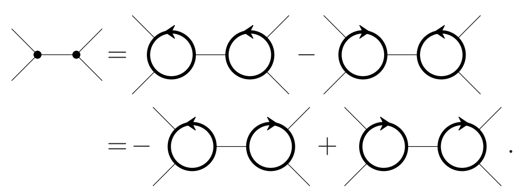
9.14 $SU(n)$, $U(n)$ EQUIVALENCE IN ADJOINT REP

The following simple observation speeds up evaluation of pure adjoint rep group-theoretic weights $(3n\text{-}j)$'s for $SU(n)$: The adjoint rep weights for $U(n)$ and $SU(n)$ are identical. This means that we can use the $U(n)$ adjoint projection operator

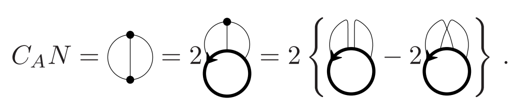
$U(n)$:  (9.118)

instead of the traceless $SU(n)$ projection operator (9.54), and halve the number of terms in the expansion of each adjoint line.

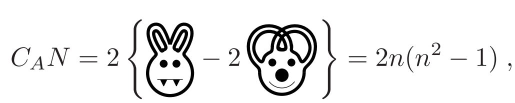
Proof: Any internal adjoint line connects two C_{ijk}'s:



The trace part of (9.54) cancels on each line; hence, it does not contribute to the pure adjoint rep diagrams. As an example, we reevaluate the adjoint quadratic casimir for $SU(n)$:



Now substitute the $U(n)$ adjoint projection operator (9.118):



in agreement with the first exercise of section 2.2.

9.15 SOURCES

Sections 9.3–9.9 of this chapter are based on Elvang *et al.* [113]. The introduction to the Young tableaux folows ref. [113], which, in turn, is based on Lichtenberg [214] and Hamermesh [153]. The rules for reduction of direct products follow Lichtenberg [214], stated here as in ref. [112]. The construction of the Young projection operators directly from the Young tableaux is described in van der Waerden [334], who ascribes the idea to von Neumann.

R. Penrose's papers are the first (known to the authors) to cast the Young projection operators into a diagrammatic form. Here we use Penrose diagrammatic notation for symmetrization operators [280], Levi-Civita tensors [282], and "strand networks" [281]. For several specific, few-particle examples, diagrammatic Young projection operators were constructed by Canning [41], Mandula [227], and Stedman [318]. A diagrammatic construction of the $U(n)$ Young projection operators for *any* Young tableau was outlined in the unpublished ref. [186], without proofs; the proofs of appendix B that the Young projection operators so constructed are unique were given in ref. [112].

			Symmetric								Antisymmetric				
	$\underline{V}_A \otimes \underline{V}_A$	$=$	V_1	$\oplus$	V_2	$\oplus$	V_3	$\oplus$	V_4	$\oplus$	V_5	$\oplus$	V_6	$\oplus$	V_7
Dimensions	$(n^2-1)^2$	$=$	1	$+$	(n^2-1)	$+$	$\frac{n^2(n-3)(n+1)}{4}$	$+$	$\frac{n^2(n+3)(n-1)}{4}$	$+$	(n^2-1)	$+$	$\frac{(n^2-1)(n^2-4)}{4}$	$+$	$\frac{(n^2-1)(n^2-4)}{4}$
Dynkin indices	$2(n^2-1)$	$=$	0	$+$	1	$+$	$\frac{n(n-3)}{2}$	$+$	$\frac{n(n+3)}{2}$	$+$	1	$+$	$\frac{n^2-4}{2}$	$+$	$\frac{n^2-4}{2}$
SU(3) example:															
Dimensions	8^2	$=$	1	$+$	8	$+$	0	$+$	27	$+$	8	$+$	10	$+$	$\overline{10}$
Indices	$2 \cdot 8$	$=$	0	$+$	1	$+$	0	$+$	9	$+$	1	$+$	$\frac{5}{2}$	$+$	$\frac{5}{2}$
SU(4) example:															
	$(101) \otimes (101)$	$=$	(000)	$\oplus$	(101)	$\oplus$	(020)	$\oplus$	(202)	$\oplus$	(101)	$\oplus$	(012)	$\oplus$	(210)
Dimensions	15^2	$=$	1	$+$	15	$+$	20	$+$	84	$+$	15	$+$	45	$+$	$\overline{45}$
Indices	$2 \cdot 15$	$=$	0	$+$	1	$+$	2	$+$	14	$+$	1	$+$	6	$+$	6

Projection operators

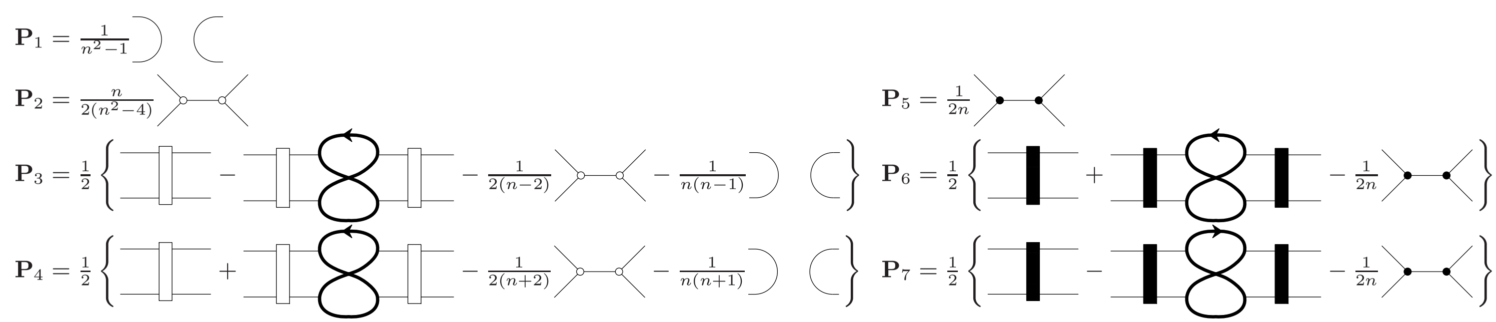


Table 9.4 $SU(n)$, $n \geq 3$ Clebsch-Gordan series for $A \otimes A$.