PHYS-7147 QFT

P. Cvitanović
G. Tech, April 2003

Final exam – takehome

(1) Dimensional Analysis with $\hbar = c = 1$

(answer questions (a) – (f))

We have set $\hbar = c = 1$. This allows us to convert a time T to a length L via $T = L/c$, and a length L to a mass M via $M = \hbar c^{-1}/L$. Thus any quantity A can be thought of as having units of mass to some some power (positive, negative, or zero) that we will call $[A]$. For example,

$$
\begin{aligned}
[m] &= +1\,, && (289)\\
[x^\mu] &= -1\,, && (290)\\
[\partial^\mu] &= +1\,, && (291)\\
[d^d x] &= -d\,. && (292)
\end{aligned}
$$

In the last line, we have generalized our considerations to theories in d spacetime dimensions.

Let us now consider a scalar field in d spacetime dimensions with lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 - \sum_{n=3}^{N} \tfrac{1}{n!} g_n \varphi^n\,. \tag{293}$$

The action is

$$S = \int d^d x\, \mathcal{L}\,, \tag{294}$$

and the path integral is

$$Z(J) = \int \mathcal{D}\varphi\, \exp\left[i\int d^d x\, (\mathcal{L} + J\varphi)\right]\,. \tag{295}$$

From eq. (295), we see that the action S must be dimensionless, because it appears as the argument of the exponential function. Therefore

$$[S] = 0\,. \tag{296}$$

What is:

(a) $[\mathcal{L}] = ?$

(b) $[\varphi] = ?$

(c) $[g_n] = ?$

If you got the dimensions right, for $\boxed{\phi^3 \text{ theory}}$

$$[g_3] = \frac{\varepsilon}{2} \quad , \quad \varepsilon = 6-d$$

So in general the coupling constant is "dimensionfull"

$$g_3 = g\,\tilde{\mu}^{\varepsilon/2}$$

dimensionless number "mass" scale

At high energies, $p^2 >> m$

4-momentum rest mass

the only scale is p^2, so

$$g_3(p^2) \simeq g\,|p^2|^{\varepsilon/2}$$

<u>what happens is</u>

(d) $\varepsilon < 0$?

(e) $\varepsilon = 0$

(f) $\varepsilon > 0$?

$\stackrel{?}{=}$ {nonrenormalizable, renormalizable, "trivial"} ?

if you got this right, you want to study in more detail:

$$\boxed{\phi^3 \text{ in } 6 \text{ dimensions}}$$

1- Loop Corrections to the Propagator

The (connected) propagator is related to the 1-P-I propagator by P.C. "Field Theory" eq. (2.32):

$$\tilde{\Delta}(k^2) = \frac{1}{k^2+m^2-\Pi(k^2)} \qquad (306)$$

if in doubt, remember; $m^2 \to m^2 - i\varepsilon$

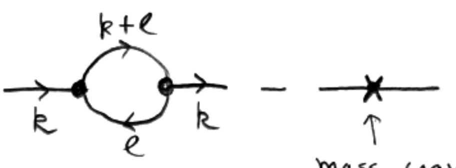
Physical mass-shell condition $k^2 = -m^2$ pole of $\tilde{\Delta}(k^2)$

consistent with eq. (306) if and only if

$$\Pi(-m^2) = 0\,, \qquad (307)$$
$$\Pi'(-m^2) = 0\,, \qquad (308)$$

One-loop contributions

$i\Pi(k^2) = \frac{1}{2}$



mass, wave fct. renormalization counter terms

$$i\Pi(k^2) = \tfrac{1}{2}(ig)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^d\ell}{(2\pi)^d}\,\tilde{\Delta}(\ell+k)\tilde{\Delta}(\ell) - i(Ak^2+Bm^2) + O(g^4)\,. \qquad (303)$$

Here we have written the integral appropriate for d spacetime dimensions; for now we will leave d arbitrary, but later we will want to focus on $d = 6$, where the coupling g is dimensionless.

(2)

Prove the *Feynman's formula* to combine denominators,

$$\frac{1}{a_1 \dots a_n} = \int dF_n \, (x_1 a_1 + \dots + x_n a_n)^{-n} \,, \tag{309}$$

where the integration measure over the *Feynman parameters* x_i is

$$\int dF_n = (n-1)! \int_0^1 dx_1 \dots dx_n \, \delta(x_1 + \dots + x_n - 1) \,. \tag{310}$$

verify that the measure is normalized so that

$$\int dF_n \, 1 = 1 \,. \tag{311}$$

Eq. (309) can be proven by direct evaluation for $n = 2$, and by induction for $n > 2$.

(I prefer going to the Schwinger-exponential-parametric representation, where this formula is a triviality)

(3) show that

$$\begin{aligned} \tilde{\Delta}(k+\ell)\tilde{\Delta}(\ell) &= \frac{1}{(\ell^2 + m^2)((\ell + k)^2 + m^2)} \\ &= \int_0^1 dx \left[q^2 + D\right]^{-2} . \end{aligned} \tag{312}$$

In the last line we have defined $q \equiv \ell + xk$ and

$$D \equiv x(1-x)k^2 + m^2 \,. \tag{313}$$

Wick rotation:

evaluate $\int_{-\infty}^{\infty} dq_0$ integral by drawing a contour in the complex q_0 plane avoiding the $m^2 - i\varepsilon$ pole and closing it along $\int_{-i\infty}^{i\infty} dq_0$. Define $iq_0 = \bar{q}_d$, $q_i = \bar{q}_i$ otherwise and a Eulidean vector

$$[\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_d]$$

(4) such that $q^2 = \bar{q}^2$, and $d^d q = i d^d \bar{q}$. Check that (303) becomes

$$\Pi(k^2) = \tfrac{1}{2} g^2 I(k^2) - Ak^2 - Bm^2 + O(g^4)\,, \tag{316}$$

where

$$I(k^2) \equiv \int_0^1 dx \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^2}\,. \tag{317}$$

evaluation of $I(k^2)$, eq. (317).

The angular part of the integral over $\bar{q}$ yields the area Ω_d of the unit sphere in d dimensions, which is $\Omega_d = 2\pi^{d/2}/\Gamma(\frac{1}{2}d)$. (This is most easily verified by computing the ussian integral $\int d^d\bar{q}\, e^{-\bar{q}^2}$ in both cartesian and spherical coordinates.) The radial part of the $\bar{q}$ integral can also be evaluated in terms of gamma functions.

(5) Derive

The overall result (generalized slightly for later use) is

$$\int \frac{d^d\bar{q}}{(2\pi)^d} \frac{(\bar{q}^2)^a}{(\bar{q}^2 + D)^b} = \frac{\Gamma(b-a-\frac{1}{2}d)\Gamma(a+\frac{1}{2}d)}{(4\pi)^{d/2}\Gamma(b)\Gamma(\frac{1}{2}d)} D^{-(b-a-d/2)} . \qquad (325)$$

In the case of interest, eq. (317), we have $a = 0$ and $b = 2$.

Useful Integrals:

You will find useful to know the following integrals:

$$\frac{1}{A^n B^m} = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^1 dx \, \frac{x^{n-1}(1-x)^{m-1}}{(xA + (1-x)B)^{n+m}} \qquad (5)$$

$$I_{D,n} = \int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + 2p\cdot q + m_0^2)^n} = \frac{1}{2} S_D \frac{\Gamma(\frac{D}{2})\Gamma(n - \frac{D}{2})}{\Gamma(n)} \left(m_0^2 - q^2\right)^{\frac{D}{2}-n} \qquad (6)$$

where S_D is the volume of the D-dimensional unit hypersphere

$$S_D = [2^{D-1}\, \pi^{D/2}\, \Gamma(\frac{D}{2})]^{-1} \qquad (7)$$

and $\Gamma(s)$ is the Γ-function

$$\Gamma(s) = \int_0^\infty dt\, t^{s-1} e^{-t} \qquad (8)$$

For $s \to 0$, the Γ-function behaves like

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} \approx \frac{1}{s} + \text{finite terms} \qquad (9)$$

$\Gamma(x)$ is the Euler gamma function; for a nonnegative integer n and small x,

$$\Gamma(n+1) = n! \,, \qquad (322)$$

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!}{n!2^n}\sqrt{\pi} \,, \qquad (323)$$

$$\Gamma(-n+x) = \frac{(-1)^n}{n!}\left[\frac{1}{x} - \gamma + \sum\nolimits_{k=1}^n k^{-1} + O(x)\right] , \qquad (324)$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant.

(6) verify

We now return to eq. (317), use eq. (324), and set $d = 6 - \varepsilon$; we get

$$\boxed{I(k^2) = \frac{\Gamma(-1+\frac{\varepsilon}{2})}{(4\pi)^3} \int_0^1 dx\, D \left(\frac{4\pi}{D}\right)^{\varepsilon/2} .} \qquad (328)$$

(7) show that in $\varepsilon \to 0$ limit ($\alpha \equiv \frac{g^2}{(4\pi)^3}$).

$$\begin{aligned} \Pi(k^2) &= -\tfrac{1}{2}\alpha\left[\left(\tfrac{2}{\varepsilon}+1\right)\left(\tfrac{1}{6}k^2+m^2\right) + \int_0^1 dx\, D \ln\left(\frac{4\pi\tilde{\mu}^2}{e^\gamma D}\right)\right] \\ &\quad - Ak^2 - Bm^2 + O(\alpha^2) . \end{aligned} \qquad (332)$$

we must still impose the conditions $\Pi(-m^2) = 0$ and $\Pi'(-m^2) = 0$. The easiest way to do this is to first note that, schematically,

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln D + \text{linear in } k^2 \text{ and } m^2 + O(\alpha^2) . \qquad (338)$$

We can then impose $\Pi(-m^2) = 0$ via

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln(D/D_0) + \text{linear in } (k^2+m^2) + O(\alpha^2) . \qquad (339)$$

where

$$D_0 \equiv D\Big|_{k^2=-m^2} = [1-x(1-x)]m^2 . \qquad (340)$$

(8) show that:

that $\Pi'(-m^2)$ vanishes for

$$\Pi(k^2) = \tfrac{1}{2}\alpha \int_0^1 dx\, D \ln(D/D_0) - \tfrac{1}{12}\alpha(k^2+m^2) + O(\alpha^2) . \qquad (341)$$

This is our final formula for the $O(\alpha)$ term in $\Pi(k^2)$.

What have we learned: the physical "polarization" correction is finite, and we have it's k^2 dependence.

Loop Corrections to the Vertex

We can define an exact three-point vertex function $ig\mathbf{V}_3(k_1, k_2, k_3)$ as the sum of one-particle irreducible diagrams with three external lines carrying momenta k_1, k_2, and k_3, all incoming, with $k_1 + k_2 + k_3 = 0$ by momentum conservation. (In adopting this convention, we allow k_i^0 to have either sign; if k_i is the momentum of an external particle, then the sign of k_i^0 is positive if the particle is incoming, and negative if it is outgoing.) The original vertex $iZ_g g$ is the first term in this sum,

followed by

(9) Show that

$$\tilde{\Delta}(\ell-k_1)\tilde{\Delta}(\ell+k_2)\tilde{\Delta}(\ell) = \int dF_3 \left[q^2 + D\right]^{-3} . \tag{357}$$

In the last line, we have defined $q \equiv \ell - x_1 k_1 + x_2 k_2$, and

$$\begin{aligned} D &\equiv x_1(1-x_1)k_1^2 + x_2(1-x_2)k_2^2 + 2x_1x_2 k_1\cdot k_2 + m^2 \\ &= x_3x_1k_1^2 + x_1x_2k_2^2 + x_2x_3k_3^2 + m^2 . \end{aligned} \tag{358}$$

After making a Wick rotation of the q^0 contour, we have

$$\mathbf{V}_3(k_1, k_2, k_3) = Z_g + g^2 \int dF_3 \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^3} + O(g^4) , \tag{359}$$

(10) Derive this relation:

$$\int \frac{d^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^3} = \frac{\Gamma(3-\frac{1}{2}d)}{2(4\pi)^{d/2}} D^{-(3-d/2)} . \tag{360}$$

Then we set $d = 6 - \varepsilon$. To keep g dimensionless, we make the replacement $g \to g\,\tilde{\mu}^{\varepsilon/2}$. Then we have

$$\mathbf{V}_3(k_1, k_2, k_3) = Z_g + \tfrac{1}{2}\alpha\,\Gamma(\tfrac{\varepsilon}{2}) \int dF_3 \left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\varepsilon/2} + O(\alpha^2) , \tag{361}$$

(rest is just to complete the work, no need to write this part up :)

take the $\varepsilon \to 0$ limit. The result is

$$\mathbf{V}_3(k_1, k_2, k_3) = Z_g + \tfrac{1}{2}\alpha \left[\frac{2}{\varepsilon} + \int dF_3 \, \ln\left(\frac{4\pi\tilde{\mu}^2}{e^{\gamma} D} \right) \right] + O(\alpha^2) \,, \qquad (362)$$

where we have used $\int dF_3 = 1$. We use $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$, set

$$Z_g = 1 + C \,, \qquad (363)$$

and rearrange to get

$$\begin{aligned} \mathbf{V}_3(k_1, k_2, k_3) \;&=\; 1 + \Big\{ \alpha \Big[\tfrac{1}{\varepsilon} + \ln(\mu/m) \Big] + C \Big\} \\ &\quad - \tfrac{1}{2}\alpha \int dF_3 \, D \ln\big(D/m^2\big) \\ &\quad + O(\alpha^2) \,. \end{aligned} \qquad (364)$$

If we take C to have the form

$$C = -\alpha \Big[\tfrac{1}{\varepsilon} + \ln(\mu/m) + \kappa_C \Big] + O(\alpha^2) \,, \qquad (365)$$

where κ_C is a purely numerical constant, then we get

$$\boxed{\mathbf{V}_3(k_1, k_2, k_3) = 1 - \tfrac{1}{2}\alpha \int dF_3 \, \ln\big(D/m^2\big) - \kappa_C \alpha + O(\alpha^2) \,.} \qquad (366)$$

Thus this choice of C renders $\mathbf{V}_3(k_1, k_2, k_3)$ finite and independent of μ, as required.

We now need a condition, analogous to $\Pi(-m^2) = 0$ and $\Pi'(-m^2) = 0$, to fix the value of κ_C. These conditions on $\Pi(k^2)$ were mandated by known properties of the exact propagator, but there is nothing directly comparable for the vertex. Different choices of κ_C correspond to different definitions of

the coupling g. This is because, in order to measure g, we would measure a cross section that depends on g; these cross sections also depend on κ_C. Thus we can use any value for κ_C that we might fancy, as long as we all agree on that value when we compare our calculations with experimental measurements. It is then most convenient to simply set $\kappa_C = 0$. This corresponds to the condition

$$\mathbf{V}_3(0, 0, 0) = 1 \,. \qquad (367)$$

This condition can then be used to fix the higher-order (in g) terms in Z_g.

≈ have a good summer ≈