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Notes on Quantum Field Theory

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Notes for the third quarter of a QFT course, introducing gauge theories. Please send any comments or corrections to

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Part III: Spin One

54: Maxwell's Equations

Prerequisite: 3

The photon is the quintessential spin-one particle. The phenomenon of emission and absorbtion of photons by matter is a critical topic in many areas of physics, and so that is the context in which most physicists first encounter a serious treatment of photons. We will use a brief review of this subject (in this section and the next) as our entry point into the theory of quantum electrodynamics.

Let us begin with classical electrodynamics. Maxwell's equations are

$$\nabla\cdot\mathbf{E} = \rho\,, \tag{1}$$

$$\nabla\times\mathbf{B} - \dot{\mathbf{E}} = \mathbf{J}\,, \tag{2}$$

$$\nabla\times\mathbf{E} + \dot{\mathbf{B}} = 0\,, \tag{3}$$

$$\nabla\cdot\mathbf{B} = 0\,, \tag{4}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ is the charge density, and $\mathbf{J}$ is the current density. We have written Maxwell's equations in *Heaviside-Lorentz units*, and also set $c = 1$. In these units, the magnitude of the force between two charges of magnitude Q is $Q^2/4\pi r^2$.

Maxwell's equations must be supplemented by formulae that give us the dynamics of the charges and currents (such as the Lorentz force law for point particles). For now, however, we will treat the charges and currents as specified sources, and focus on the dynamics of the electromagnetic fields.

The last two of Maxwell's equations, the ones with no sources on the right-hand side, can be solved by writing the $\mathbf{E}$ and $\mathbf{B}$ fields in terms of a scalar potential φ and a vector potential $\mathbf{A}$,

$$\begin{aligned} \mathbf{E} &= -\nabla\varphi - \dot{\mathbf{A}} \;, && (5) \\ \mathbf{B} &= \nabla \times \mathbf{A} \;. && (6) \end{aligned}$$

Thus, the potentials uniquely determine the fields, but the fields do not uniquely determine the potentials. Given a particular φ and $\mathbf{A}$ that result in a particular $\mathbf{E}$ and $\mathbf{B}$, we will get the same $\mathbf{E}$ and $\mathbf{B}$ from any other potentials φ' and $\mathbf{A}'$ that are related by

$$\begin{aligned} \varphi' &= \varphi + \dot{\Gamma} \;, && (7) \\ \mathbf{A}' &= \mathbf{A} - \nabla\Gamma \;, && (8) \end{aligned}$$

where Γ is an arbitrary function of spacetime. A change of potentials that does not change the fields is called a *gauge transformation*. The fields are *gauge invariant*.

All this becomes more compact and elegant in a relativistic notation. Define the four-vector potential

$$A^\mu = (\varphi, \mathbf{A}) \;; \tag{9}$$

A^μ is also called the *gauge field*. We also define the *field strength*

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \;. \tag{10}$$

Obviously, $F^{\mu\nu}$ is antisymmetric: $F^{\mu\nu} = -F^{\nu\mu}$. Comparing eqs. (5), (9), and (10), we see that

$$F^{0i} = E^i \;. \tag{11}$$

Comparing eqs. (6), (9), and (10), we see that

$$F^{ij} = \varepsilon^{ijk} B_k \;. \tag{12}$$

The first two of Maxwell's equations can now be written as

$$\partial_\nu F^{\mu\nu} = J^\mu \;, \tag{13}$$

where

$$J^\mu = (\rho, \mathbf{J}) \tag{14}$$

is the charge-current density four-vector. If we take the four-divergence of eq. (13), we get $\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\mu J^\mu$. The left-hand side of this equation vanishes, because $\partial_\mu \partial_\nu$ is symmetric on exchange of μ and ν, while $F^{\mu\nu}$ is antisymmetric. We conclude that we must have

$$\partial_\mu J^\mu = 0 \; ; \tag{15}$$

that is, the electromagnetic current must be conserved.

The last two of Maxwell's equations can be written as

$$\varepsilon_{\mu\nu\rho\sigma} \partial^\rho F^{\mu\nu} = 0 \; . \tag{16}$$

Plugging in eq. (10), we see that eq. (16) is automatically satisfied, since the antisymmetric combination of two derivatives vanishes.

Eqs. (7) and (8) can be combined into

$$A'^\mu = A^\mu - \partial^\mu \Gamma \; . \tag{17}$$

Setting $F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$ and using eq. (17), we get

$$F'^{\mu\nu} = F^{\mu\nu} - (\partial^\mu \partial^\nu - \partial^\nu \partial^\mu)\Gamma \; . \tag{18}$$

The last term vanishes because derivatives commute; thus the field strength is gauge invariant,

$$F'^{\mu\nu} = F^{\mu\nu} \; . \tag{19}$$

Next we will find an action that results in Maxwell's equations as the equations of motion. We will treat the current as an external source. The action we seek should be Lorentz invariant, gauge invariant, parity and time-reversal invariant, and no more than second order in derivatives. The only candidate is $S = \int d^4x \, \mathcal{L}$, where

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \; . \tag{20}$$

The first term is obviously gauge invariant, because $F^{\mu\nu}$ is. After a gauge transformation, eq. (17), the second term becomes $J^\mu A'_\mu$, and the difference is

$$\begin{aligned} J^\mu(A'_\mu - A_\mu) &= -J^\mu \partial_\mu \Gamma \\ &= -(\partial_\mu J^\mu)\Gamma - \partial_\mu(J^\mu \Gamma) \ . \end{aligned} \tag{21}$$

The first term in eq. (21) vanishes because the current is conserved. The second term is a total divergence, and its integral over d^4x vanishes (assuming suitable boundary conditions at infinity). Thus the action specified by eq. (20) is gauge invariant.

Setting $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and multiplying out the terms, eq. (20) becomes

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \tfrac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu + J^\mu A_\mu \tag{22}$$

$$= +\tfrac{1}{2}A_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + J^\mu A_\mu - \partial^\mu K_\mu \ , \tag{23}$$

where $K_\mu = A^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu)$. The last term is a total divergence, and can be dropped. From eq. (23), we can see that varying A^μ while requiring S to be unchanged yields the equation of motion

$$(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + J^\mu = 0 \ . \tag{24}$$

Noting that $\partial_\nu F^{\mu\nu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial^\mu\partial^\nu - g^{\mu\nu}\partial^2)A_\nu$, we see that eq. (24) is equivalent to eq. (13).

55: Electrodynamics in Coulomb Gauge

Prerequisite: 54

Next we would like to construct the hamiltonian, and quantize the electromagnetic field.

There is an immediate difficulty, caused by the gauge invariance: we have too many degrees of freedom. This problem manifests itself in several ways. For example, the lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu \tag{25}$$

$$= -\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \tfrac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu + J^\mu A_\mu \tag{26}$$

does not contain the time derivative of A^0. Thus, this field has no canonically conjugate momentum and no dynamics.

Dealing with this complication generally requires a large amount of new formalism. In this section, we will instead proceed heuristically, with the goal of reaching a physically relevant answer as quickly as possible. We will revisit this issue when we take up path-integral quantization, where it is more easily resolved.

We begin by eliminating the gauge freedom. We do this by *choosing a gauge*. We choose a gauge by imposing a *gauge condition*. This is a condition which we require $A^\mu(x)$ to satisfy. The idea is that there should be only one $A^\mu(x)$ that results in a given $F^{\mu\nu}(x)$ and also satisfies the gauge condition.

One possible class of gauge conditions is $n^\mu A_\mu(x) = 0$, where n^μ is a constant four-vector. If n is spacelike ($n^2 > 0$), then we have chosen *axial gauge*; if n is lightlike, ($n^2 = 0$), it is *lightcone gauge*; and if n is timelike, ($n^2 < 0$), it is *temporal gauge*.

Another gauge is *Lorentz gauge*, where the condition is $\partial^\mu A_\mu = 0$. We will meet a family of closely related gauges in section ??.

In this section, we will pick *Coulomb gauge*, also known as *radiation gauge* or *transverse gauge*. The condition for Coulomb gauge is

$$\nabla\cdot\mathbf{A}(x) = 0\ . \tag{27}$$

Let us write out the lagrangian in terms of the scalar and vector potentials, $\varphi = A^0$ and A_i. Starting from eq. (26), we get

$$\begin{aligned}\mathcal{L} \;=\;& \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i \\ &+ \tfrac{1}{2}\nabla_i A_j \nabla_j A_i + \dot{A}_i \nabla_i \varphi \\ &+ \tfrac{1}{2}\nabla_i\varphi\nabla_i\varphi - \rho\varphi\ .\end{aligned} \tag{28}$$

In the second line of eq. (28), the ∇_i in each term can be integrated by parts; in the first term, we will then get a factor of $\nabla_j(\nabla_i A_i)$, and in the second term, we will get a factor of $\nabla_i\dot{A}_i$. Both of these vanish by virtue of the gauge condition $\nabla_i A_i = 0$, and so both of these terms can simply be dropped.

If we now vary φ (and require $S = \int d^4x\,\mathcal{L}$ to be stationary), we find that φ obeys Poisson's equation,

$$-\nabla^2\varphi = \rho\ . \tag{29}$$

The solution is

$$\varphi(\mathbf{x},t) = \int d^3y\ \frac{\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}\ . \tag{30}$$

This solution is unique if we impose the boundary conditions that φ and ρ both vanish at spatial infinity.

Eq. (30) tells us that $\varphi(\mathbf{x},t)$ is given entirely in terms of the charge density at the same time, and so has no dynamics of its own. It is therefore legitimate to plug eq. (30) back into the lagrangian. After an integration by parts to turn $\nabla_i\varphi\nabla_i\varphi$ into $-\varphi\nabla^2\varphi = \varphi\rho$, the result is

$$\mathcal{L} = \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i + \mathcal{L}_{\rm coul}\ , \tag{31}$$

where

$$\mathcal{L}_{\rm coul} = -\frac{1}{2}\int d^3y\ \frac{\rho(\mathbf{x},t)\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}\ . \tag{32}$$

We can now vary A_i, and find that each compenent obeys the massless Klein-Gordon equation, with J_i as a source,

$$-\partial^2 A_i = J_i \; . \tag{33}$$

For a free field ($J_i = 0$), the general solution is

$$\mathbf{A}(x) = \sum_{\lambda=\pm} \int \widetilde{dk} \left[\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) e^{ikx} + \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) a^\dagger_\lambda(\mathbf{k}) e^{-ikx} \right], \tag{34}$$

where $k^0 = \omega = |\mathbf{k}|$, $\widetilde{dk} = d^3k/(2\pi)^3 2\omega$, and $\boldsymbol{\varepsilon}_+(\mathbf{k})$ and $\boldsymbol{\varepsilon}_-(\mathbf{k})$ are polarization vectors. We choose them so that the helicity of the photon state $a^\dagger_\pm(\mathbf{k})|0\rangle$ is ± 1; that is, so that

$$\hat{\mathbf{k}}\cdot\mathbf{J} a^\dagger_+(\mathbf{k})|0\rangle = \pm a^\dagger_+(\mathbf{k})|0\rangle \; , \tag{35}$$

where $\mathbf{J}$ is the angular momentum operator. For $\mathbf{k}$ in the z direction, $\mathbf{k} = (0, 0, k)$,

$$\begin{aligned} \boldsymbol{\varepsilon}_+(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(1, -i, 0) \; , \\ \boldsymbol{\varepsilon}_-(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(1, +i, 0) \; , \end{aligned} \tag{36}$$

up to overall phase factors that are physically irrelevant. More generally, the two polarization vectors along with the unit vector in the $\mathbf{k}$ direction form an orthonormal and complete set,

$$\mathbf{k}\cdot\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) = 0 \; , \tag{37}$$

$$\boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k})\cdot\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) = \delta_{\lambda'\lambda} \; , \tag{38}$$

$$\sum_{\lambda=\pm} \varepsilon^*_{i\lambda}(\mathbf{k})\varepsilon_{j\lambda}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \; . \tag{39}$$

The coefficients $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$ will become operators after quantization, which is why we have used the dagger symbol for their conjugation.

In complete analogy with the procedure used for a scalar field in section 3, we can invert eq. (34) and its time derivative to get

$$a_\lambda(\mathbf{k}) = +i\, \boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\int d^3x \; e^{-ikx} \overleftrightarrow{\partial_0} \mathbf{A}(x) \; , \tag{40}$$

$$a^\dagger_\lambda(\mathbf{k}) = -i\, \boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})\cdot\int d^3x \; e^{+ikx} \overleftrightarrow{\partial_0} \mathbf{A}(x) \; , \tag{41}$$

where $f\overleftrightarrow{\partial_\mu}g = f(\partial_\mu g) - (\partial_\mu f)g$.

Now we can proceed to the hamiltonian formalism. First, we compute the canonically conjugate momentum to A_i,

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}_i \ . \tag{42}$$

The hamiltonian density is then

$$\begin{aligned} \mathcal{H} &= \Pi_i \dot{A}_i - \mathcal{L} \\ &= \tfrac{1}{2}\Pi_i\Pi_i + \nabla_j A_i \nabla_j A_i - J_i A_i + \mathcal{H}_{\rm coul} \ , \end{aligned} \tag{43}$$

where $\mathcal{H}_{\rm coul} = -\mathcal{L}_{\rm coul}$.

To quantize the field, we should apparently impose the canonical commutation relations

$$[A_i(\mathbf{x},t), \Pi_j(\mathbf{y},t)] = i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) \ . \tag{44}$$

This, however, is inconsistent with the gauge condition, since acting on the left-hand side with $\nabla_{\mathbf{x}i}$ (and summing over i) should give zero, while the right-hand side fails to vanish.

To understand this properly, we need the formalism for quantization of constrained systems. Instead of introducing it, we will proceed heuristically, and simply alter the right-hand side of eq. (44) to read

$$[A_i(\mathbf{x},t), \Pi_j(\mathbf{y},t)] = i\int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) . \tag{45}$$

This procedure projects the offending components out of the delta function.

The commutation relations of the $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$ operators follow from eqs. (40) and (41), along with eq. (45) and $[A_i, A_j] = [\Pi_i, \Pi_j] = 0$ (at equal times). The result is

$$[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{k}')] = 0 \ , \tag{46}$$

$$[a^\dagger_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] = 0 \ , \tag{47}$$

$$[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] = (2\pi)^3 2\omega\, \delta^3(\mathbf{k}'-\mathbf{k})\delta_{\lambda\lambda'} \ . \tag{48}$$

We interpret $a^\dagger_\lambda(\mathbf{k})$ and $a_\lambda(\mathbf{k})$ as creation and annihilation operators for photons of definite momentum and helicity, with positive helicity corresponding to left-circular polarization and negative helicity to right-circular polarization.

It is now straightfoward to write the hamiltonian in terms of these operators. We find

$$H = \sum_{\lambda=\pm} \int \widetilde{dk}\; \omega \; a^\dagger_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) + 2\mathcal{E}_0 V - \int d^3x \; \mathbf{J}(x)\cdot\mathbf{A}(x) + H_{\rm coul} \; , \tag{49}$$

where $\mathcal{E}_0 = \int d^3k\,\omega$ is the zero-point energy per unit volume for each oscillator, V is the volume of space,

$$H_{\rm coul} = \frac{1}{2}\int d^3x\, d^3y\; \frac{\rho(\mathbf{x},t)\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|} \; , \tag{50}$$

and we use eq. (34) to express $A_i(x)$ in terms of $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$ at any one particular time (say, $t = 0$). This is sufficient, because H itself is time independent.

This form of the hamiltonian of electrodynamics is often used as the starting point for calculations of atomic transition rates, with the charges and currents treated via the nonrelativistic Schrödinger equation. The Coulomb interaction appears explicitly, and the $\mathbf{J}\cdot\mathbf{A}$ term allows for the creation and annihilation of photons of definite polarization.

56: LSZ Reduction for Photons

Prerequisite: 5, 55

In section 55, we found that the creation and annihilation operators for free photons could be written as

$$a^\dagger_\lambda(\mathbf{k}) = -i\,\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})\cdot\int d^3x\; e^{+ikx}\,\overleftrightarrow{\partial_0}\,\mathbf{A}(x)\;, \tag{51}$$

$$a_\lambda(\mathbf{k}) = +i\,\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\int d^3x\; e^{-ikx}\,\overleftrightarrow{\partial_0}\,\mathbf{A}(x)\;, \tag{52}$$

where $\boldsymbol{\varepsilon}_\lambda(\mathbf{k})$ is a polarization vector. From here, we can follow the analysis of section 5 line by line to deduce the LSZ reduction formula for photons. The result is that the creation operator for each incoming photon should be replaced by

$$a^\dagger_\lambda(\mathbf{k})_{\rm in} \to i\,\varepsilon^{\mu*}_\lambda(\mathbf{k})\int d^4x\; e^{+ikx}(-\partial^2)A_\mu(x)\;, \tag{53}$$

and the destruction operator for each outgoing photon should be replaced by

$$a_\lambda(\mathbf{k})_{\rm out} \to i\,\varepsilon^{\mu}_\lambda(\mathbf{k})\int d^4x\; e^{-ikx}(-\partial^2)A_\mu(x)\;, \tag{54}$$

and then we should take the vacuum expectation value of the time-ordered product. Note that, in writing eqs. (53) and (54), we have made them look nicer by introducing $\varepsilon^0_\lambda(\mathbf{k}) \equiv 0$, and then using four-vector dot products rather than three-vector dot products.

The LSZ formula is valid provided the field is normalized according to the free-field formulae

$$\langle 0|A^i(x)|0\rangle = 0\;, \tag{55}$$

$$\langle k,\lambda|A^i(x)|0\rangle = \varepsilon^i_\lambda(\mathbf{k})e^{ikx}\;. \tag{56}$$

The zero on the right-hand side of eq. (55) is required by rotation invariance, and only the overall scale of the right-hand side of eq. (56) might be different in an interacting theory.

The renormalization of A_i necessitates including appropriate Z factors in the lagrangian:

$$\mathcal{L} = -\tfrac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + Z_1 J^\mu A_\mu \ . \tag{57}$$

Here Z_3 and Z_1 are the traditional names of these factors. (We will meet Z_2 in section ??.) We must choose Z_3 so that eq. (56) holds; Z_1 will be fixed by requiring some 1PI vertex function to take on a certain value for certain external momenta.

Next we must compute the correlation functions $\langle 0|\mathrm{T} A_i(x) \ldots |0\rangle$. As usual, we will begin by working with free field theory. The analysis is again almost identical to the case of a scalar field; see problem 8.3. We find that, in free field theory,

$$\langle 0|\mathrm{T} A^i(x) A^j(y)|0\rangle = \tfrac{1}{i} \Delta^{ij}(x-y) \ , \tag{58}$$

where the propagator is

$$\Delta^{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4} \, \frac{e^{ik(x-y)}}{k^2 - i\epsilon} \, \textstyle\sum_{\lambda=\pm} \varepsilon^{i*}_\lambda(\mathbf{k}) \varepsilon^j_\lambda(\mathbf{k}) \ . \tag{59}$$

As with a free scalar field, correlations of an odd number of fields vanish, and correlations of an even number of fields are given in terms of the propagator by Wick's theorem; see section 8.

We would now like to evaluate the path integral for the free electromagnetic field

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}A \ e^{i \int d^4x \, [-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu]} \ . \tag{60}$$

Here we treat the current $J_\mu(x)$ as an external source.

We will evaluate $Z_0(J)$ in Coulomb gauge. This means that we will integrate over only those field configurations that satisfy $\nabla\cdot\mathbf{A} = 0$.

We begin by integrating over A^0. Because the action is quadratic in A^μ, this is equivalent to solving the variational equation for A^0, and then substituting the solution back into the lagrangian. The result is that we have the Coulomb term in the action,

$$S_{\rm coul} = -\frac{1}{2}\int d^4x\; d^4y\; \delta(x^0{-}y^0)\; \frac{J^0(x)J^0(y)}{4\pi|\mathbf{x}{-}\mathbf{y}|}\;. \tag{61}$$

Since this term does not depend on the vector potential, we simply get a factor of $\exp(iS_{\rm coul})$ in front of the remaining path integral over A_i. We perform this integral formally (as we did with fermion fields in section 43) by requiring it to yield correct results for the correlation functions of A_i when we take functional derivatives of it with respect to J_i.

Putting all of this together, we get

$$Z_0(J) = \exp\biggl[iS_{\rm coul} + \frac{i}{2}\int d^4x\,d^4y\;J_i(x)\Delta^{ij}(x-y)J_j(y)\biggr]. \tag{62}$$

We can make $Z_0(J)$ look prettier by writing it as

$$Z_0(J) = \exp\biggl[\frac{i}{2}\int d^4x\,d^4y\;J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\biggr], \tag{63}$$

where we have defined

$$\Delta^{\mu\nu}(x-y) \;\equiv\; \int \frac{d^4k}{(2\pi)^4}\;e^{ik(x-y)}\;\tilde\Delta^{\mu\nu}(k)\;, \tag{64}$$

$$\tilde\Delta^{\mu\nu}(k) \;\equiv\; -\frac{1}{\mathbf{k}^2}\,\delta^{\mu0}\delta^{\nu0} + \frac{1}{k^2-i\epsilon}\textstyle\sum_{\lambda=\pm}\varepsilon^{\mu*}_\lambda(\mathbf{k})\varepsilon^{\nu}_\lambda(\mathbf{k})\;. \tag{65}$$

The first term on the right-hand side of eq. (65) reproduces the Coulomb term in eq. (62) by virtue of the facts that

$$\int_{-\infty}^{+\infty}\frac{dk^0}{2\pi}\;e^{-ik^0(x^0-y^0)} \;=\; \delta(x^0-y^0)\;, \tag{66}$$

$$\int\frac{d^3k}{(2\pi)^3}\;\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2} \;=\; \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\;. \tag{67}$$

The second term on the right-hand side of eq. (65) reproduces the second term in eq. (62) by virtue of the fact that $\varepsilon^0_\lambda(\mathbf{k})=0$.

Next we will simplify eq. (65). We begin by introducing a unit vector in the time direction

$$\hat t^\mu = (1,\mathbf{0})\;. \tag{68}$$

Next we need a unit vector in the $\mathbf{k}$ direction, which we will call $\hat{z}^\mu$. We first note that $\hat{t}\cdot k = -k^0$, and so we can write

$$(0, \mathbf{k}) = k^\mu + (\hat{t}\cdot k)\hat{t}^\mu \ . \tag{69}$$

The square of this four-vector is

$$\mathbf{k}^2 = k^2 + (\hat{t}\cdot k)^2 \ , \tag{70}$$

where we have used $\hat{t}^2 = -1$. Thus the unit vector that we want is

$$\hat{z}^\mu = \frac{k^\mu + (\hat{t}\cdot k)\hat{t}^\mu}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}} \ . \tag{71}$$

Now we recall from section 55 that

$$\sum_{\lambda=\pm} \varepsilon_\lambda^{i*}(\mathbf{k})\varepsilon_\lambda^{j}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \ . \tag{72}$$

This can be extended to $i \to \mu$ and $j \to \nu$ by writing

$$\sum_{\lambda=\pm} \varepsilon_\lambda^{\mu*}(\mathbf{k})\varepsilon_\lambda^{\nu}(\mathbf{k}) = g^{\mu\nu} + \hat{t}^\mu\hat{t}^\nu - \hat{z}^\mu\hat{z}^\nu \ . \tag{73}$$

It is not hard to check that the right-hand side of eq. (73) vanishes if $\mu = 0$ or $\nu = 0$, and agrees with eq. (72) for $\mu = i$ and $\nu = j$. Putting all this together, we can now write eq. (65) as

$$\tilde{\Delta}^{\mu\nu}(k) = -\frac{\hat{t}^\mu\hat{t}^\nu}{k^2 + (\hat{t}\cdot k)^2} + \frac{g^{\mu\nu} + \hat{t}^\mu\hat{t}^\nu - \hat{z}^\mu\hat{z}^\nu}{k^2 - i\epsilon} \ . \tag{74}$$

The next step is to consider the terms in this expression that contain factors of k^μ or k^ν; from eq. (71), we see that these will arise from the $\hat{z}^\mu\hat{z}^\nu$ term. In eq. (64), a factor of k^μ can be written as a derivative with respect to x^μ acting on $e^{ik(x-y)}$. This derivative can then be integrated by parts in eq. (63) to give a factor of $\partial^\mu J_\mu(x)$. But $\partial^\mu J_\mu(x)$ vanishes, because the current must be conserved. Similarly, a factor of k^ν can be turned into $\partial^\nu J_\nu(y)$, and also leads to a vanishing contribution. Therefore, *we can ignore any terms in $\tilde{\Delta}^{\mu\nu}(k)$ that contain factors of k^μ or k^ν.*

From eq. (71), we see that this means we can make the substitution

$$\hat{z}^\mu \to \frac{(\hat{t}\cdot k)\hat{t}^\mu}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}} \ . \tag{75}$$

Then eq. (74) becomes

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left[g^{\mu\nu} + \left(-\frac{k^2}{k^2 + (\hat{t}\cdot k)^2} + 1 - \frac{(\hat{t}\cdot k)^2}{k^2 + (\hat{t}\cdot k)^2}\right)\hat{t}^\mu\hat{t}^\nu\right], \tag{76}$$

where the three coefficients of $\hat{t}^\mu\hat{t}^\nu$ come from the Coulomb term, the $\hat{t}^\mu\hat{t}^\nu$ term in the polarization sum, and the $\hat{z}^\mu\hat{z}^\nu$ term, respectively. A bit of algebra now reveals that the net coefficient of $\hat{t}^\mu\hat{t}^\nu$ vanishes, leaving us with the elegant expression

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - i\epsilon} \ . \tag{77}$$

Written in this way, the photon propagator is said to be in *Feynman gauge*. [It would still be in Coulomb gauge if we had retained the k^μ and k^ν terms in eq. (74).]

In the next section, we will rederive eq. (77) from a more explicit path-integral point of view.

57: The Path Integral for Photons

Prerequisite: 8, 56

In this section we will attempt to evaluate the path integral directly, using the methods of section 8. We begin with

$$Z_0(J) = \int \mathcal{D}A \, e^{iS_0} \,, \tag{78}$$

$$S_0 = \int d^4x \left[-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \right] \,. \tag{79}$$

Following section 8, we Fourier-transform to momentum space, where we find

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Big[-\widetilde{A}_\mu(k) \Big(k^2 g^{\mu\nu} - k^\mu k^\nu \Big) \widetilde{A}_\nu(-k) + \widetilde{J}^\mu(k) \widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k) \widetilde{A}_\mu(k) \Big]. \tag{80}$$

The next step is to shift the integration variable $\widetilde{A}$ so as to "complete the square". This involves inverting the 4×4 matrix $k^2 g^{\mu\nu} - k^\mu k^\nu$. However, this matrix has a zero eigenvalue, and cannot be inverted.

To better understand what is going on, it is convenient to note that the matrix of interest can be written as $k^2 P^{\mu\nu}(k)$, where we define

$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu / k^2 \,. \tag{81}$$

This matrix is a projection matrix because, as is easily checked,

$$P^{\mu\nu}(k) P_\nu{}^\lambda(k) = P^{\mu\lambda}(k) \,. \tag{82}$$

Thus the only allowed eigenvalues of P are one and zero. There is at least one zero eigenvalue, because

$$P^{\mu\nu}(k) k_\nu = 0 \,. \tag{83}$$

On the other hand, the sum of the eigenvalues is given by the trace

$$g_{\mu\nu}P^{\mu\nu}(k) = 3\ . \tag{84}$$

Thus the remaining three eigenvalues must all be one.

Now let us imagine carrying out the path integral of eq. (78), with S_0 given by eq. (80). Let us decompose the field $\widetilde{A}_\mu(k)$ into components aligned along a set of linearly independent four-vectors, one of which is k_μ. (It will not matter whether or not this basis set is orthonormal.) Because the term quadratic in $\widetilde{A}_\mu$ involves the matrix $k^2P^{\mu\nu}(k)$, and $P^{\mu\nu}(k)k_\nu = 0$, the component of $\widetilde{A}_\mu(k)$ that lies along k_μ does not contribute to this quadratic term. Furthermore, it does not contribute to the linear term either, because $\partial^\mu J_\mu(x) = 0$ implies $k^\mu \widetilde{J}_\mu(k) = 0$. Thus this component does not appear in the path integal at all! It then makes no sense to integrate over it. We therefore define $\int \mathcal{D}A$ to mean integration over only those components that are spanned by the remaining three basis vectors, and therefore satisfy $k^\mu \widetilde{A}_\mu(k) = 0$. This is equivalent to imposing Lorentz gauge, $\partial^\mu A_\mu(x) = 0$.

The matrix $P^{\mu\nu}(k)$ is simply the matrix that projects a four-vector into the subspace orthogonal to k^μ. Within the subspace, $P^{\mu\nu}(k)$ is equivalent to the identity matrix. Therefore, within the subspace, the inverse of $k^2P^{\mu\nu}(k)$ is $(1/k^2)P^{\mu\nu}(k)$. Employing the ϵ trick to pick out vacuum boundary conditions replaces k^2 with $k^2 - i\epsilon$.

We can now continue following the procedure of section 8, with the result that

$$\begin{aligned} Z_0(J) &= \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\, \widetilde{J}_\mu(k)\, \frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}\, \widetilde{J}_\nu(-k)\right] \\ &= \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right], \end{aligned} \tag{85}$$

where

$$\Delta^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \frac{P^{\mu\nu}(k)}{k^2 - i\epsilon} \tag{86}$$

is the photon propagator in *Lorentz gauge* or *Landau gauge*. Of course, because the current is conserved, the $k^\mu k^\nu$ term in $P^{\mu\nu}(k)$ does not contribute, and so the result is equivalent to that of Feynman gauge, where $P^{\mu\nu}(k)$ is replaced by $g^{\mu\nu}$.

58: The Feynman Rules for Quantum Electrodynamics

Prerequisite: 45, 57

In the section, we will allow the photons to interact with the electrons and positrons of a Dirac field. We do this by taking the electromagnetic current $j^\mu(x)$ to be proportional to the Noether current corresponding to the U(1) symmetry of a Dirac field (see section 36). Specifically,

$$j^\mu(x) = e\overline{\Psi}(x)\gamma^\mu\Psi(x)\ . \tag{87}$$

Here $e = -0.302822$ is the charge of the electron in Heaviside-Lorentz units, with $\hbar = c = 1$. (We will rely on context to distinguish this e from the base of natural logarithms.) In these units, the the fine-structure constant is $\alpha = e^2/4\pi = 1/137.036$. With the normalization of eq. (87), $Q = \int d^3x\, j^0(x)$ is the electric charge operator.

Of course, when we specify a number in quantum field theory, we must always have a renormalization scheme in mind; $e = -0.302822$ corresponds to on-shell renormalization. The value of e will be different in other renormalization schemes, such as $\overline{\rm MS}$, as we will see in section ??.

The complete lagrangian of our theory is thus

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi + e\overline{\Psi}\gamma^\mu\Psi A_\mu\ . \tag{88}$$

In this section, we will be concerned with tree-level processes only, and so we omit renormalizing Z factors.

We have a problem, though. A Noether current is conserved only when the fields obey the equations of motion, or, equivalently, only at points in field space where the action is stationary. On the other hand, in our development of photon path integrals in sections 56 and 57, we assumed that the current was *always* conserved.

This issue is resolved by enlarging the definition of a gauge transformation to include a transformation on the Dirac field as well as the electromagnetic field. Specifically, we define a gauge transformation to consist of

$$A^\mu(x) \rightarrow A^\mu(x) - \partial^\mu \Gamma(x) \ , \tag{89}$$

$$\Psi(x) \rightarrow \exp[-ie\Gamma(x)]\Psi(x) \ , \tag{90}$$

$$\overline{\Psi}(x) \rightarrow \exp[+ie\Gamma(x)]\overline{\Psi}(x) \ . \tag{91}$$

It is not hard to check that $\mathcal{L}(x)$ is *invariant* under this transformation, whether or not the fields obey their equations of motion. To perform this check most easily, we first rewrite $\mathcal{L}$ as

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\not{D}\Psi - m\overline{\Psi}\Psi \ , \tag{92}$$

where we have defined the *gauge covariant derivative* (or just *covariant derivative* for short)

$$D_\mu \equiv \partial_\mu - ieA_\mu \ . \tag{93}$$

In the last section, we found that $F^{\mu\nu}$ is invariant under eq. (89), and so the FF term in $\mathcal{L}$ is obviously invariant as well. It is also obvious that the $m\overline{\Psi}\Psi$ term in $\mathcal{L}$ is invariant under eqs. (90) and (91). This leaves the $\overline{\Psi}\not{D}\Psi$ term. This term will also be invariant if, under the gauge transformation, the covariant derivative of Ψ transforms as

$$D_\mu\Psi(x) \rightarrow \exp[-ie\Gamma(x)]D_\mu\Psi(x) \ . \tag{94}$$

To see if this is true, we note that

$$\begin{aligned} D_\mu\Psi &\rightarrow \Big(\partial_\mu - ie[A_\mu - \partial_\mu\Gamma]\Big)\Big(\exp[-ie\Gamma]\Psi\Big) \\ &= \exp[-ie\Gamma]\Big(\partial_\mu\Psi - ie(\partial_\mu\Gamma)\Psi - ie[A_\mu - \partial_\mu\Gamma]\Psi\Big) \\ &= \exp[-ie\Gamma]\Big(\partial_\mu - ieA_\mu\Big)\Psi \\ &= \exp[-ie\Gamma]D_\mu\Psi \ . \end{aligned} \tag{95}$$

So eq. (94) holds, and $\overline{\Psi}\not{D}\Psi$ is gauge invariant.

We can also write the transformation rule for D_μ a little more abstractly as

$$D_\mu \to e^{-ie\Gamma} D_\mu \, e^{+ie\Gamma} \; , \tag{96}$$

where the ordinary derivative in D_μ is defined to act on anything to its right, including any fields that are left unwritten in eq. (96). Thus we have

$$\begin{aligned} D_\mu \Psi \;&\to\; \Big(e^{-ie\Gamma} D_\mu e^{+ie\Gamma}\Big)\Big(e^{-ie\Gamma}\Psi\Big) \\ &=\; e^{-ie\Gamma} D_\mu \Psi \; , \end{aligned} \tag{97}$$

which is, of course, the same as eq. (95). We can also express the field strength in terms of the covariant derivative by noting that

$$[D^\mu, D^\nu]\Psi(x) = -ieF^{\mu\nu}(x)\Psi(x) \; . \tag{98}$$

We can write this more abstractly as

$$F^{\mu\nu} = \frac{i}{e}[D^\mu, D^\nu] \; , \tag{99}$$

where, again, the ordinary derivative in each covariant derivative acts on anything to its right. From eqs. (96) and (99), we see that, under a gauge transformation,

$$\begin{aligned} F^{\mu\nu} \;&\to\; \frac{i}{e}\Big[e^{-ie\Gamma} D^\mu \, e^{+ie\Gamma}, e^{-ie\Gamma} D^\nu \, e^{+ie\Gamma}\Big] \\ &=\; e^{-ie\Gamma}\Big(\frac{i}{e}[D^\mu, D^\nu]\Big)e^{+ie\Gamma} \\ &=\; e^{-ie\Gamma} F^{\mu\nu} e^{+ie\Gamma} \\ &=\; F^{\mu\nu} \; . \end{aligned} \tag{100}$$

In the last line, we are able to cancel the $e^{\pm ie\Gamma}$ factors against each other because no derivatives act on them. Eq. (100) shows us that (as we already knew) $F^{\mu\nu}$ is gauge invariant.

It is interesting to note that the gauge transformation on the fermion fields, eqs. (90–91), is a generalization of the U(1) transformation

$$\Psi \;\to\; e^{-i\alpha}\Psi \; , \tag{101}$$

$$\overline{\Psi} \;\to\; e^{+i\alpha}\overline{\Psi} \; , \tag{102}$$

that is a symmetry of the free Dirac lagrangian. The difference is that, in the gauge transformation, the phase factor is allowed to be a function of spacetime, rather than a constant that is the same everywhere. Thus, the gauge transformation is also called a *local* U(1) transformation, while eqs. (101–102) correspond to a *global* U(1) transformation. We say that, in a gauge theory, the global U(1) symmetry is promoted to a local U(1) symmetry, or that we have *gauged* the U(1) symmetry.

In section 57, we argued that the path integral over A_μ should be restricted to those components of $\widetilde{A}_\mu(k)$ that are orthogonal to k_μ, because the component parallel to k_μ did not appear in the integrand. Now we must make a slightly more subtle argument. We argue that the path integral over the parallel component is redundant, because the fermionic path integral over Ψ and $\overline{\Psi}$ already includes all possible values of $\Gamma(x)$. Therefore, as in section 57, we should not integrate over the parallel component. (We will make a more precise and careful version of this argument when we discuss nonabelian gauge theories in section ??.)

By the standard procedure, this leads us to the following form of the path integral for quantum electrodynamics:

$$
\begin{aligned}
Z(\overline{\eta},\eta,J) \;\propto\; & \exp\left[ie\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)(\gamma^\mu)_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right)\right] \\
& \times Z_0(\overline{\eta},\eta)Z_0(J)\;, \qquad (103)
\end{aligned}
$$

where

$$
Z_0(\overline{\eta},\eta) = \exp\left[i\int d^4x\,d^4y\;\overline{\eta}(x)S(x-y)\eta(y)\right], \qquad (104)
$$

$$
Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\,d^4y\;J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right], \qquad (105)
$$

and

$$
S(x-y) = \int\frac{d^4p}{(2\pi)^4}\,\frac{(-\not{p}+m)}{p^2+m^2-i\epsilon}\,e^{ip(x-y)}\;, \qquad (106)
$$

$$
\Delta_{\mu\nu}(x-y) = \int\frac{d^4k}{(2\pi)^4}\,\frac{g_{\mu\nu}}{k^2-i\epsilon}\,e^{ik(x-y)} \qquad (107)
$$

are the appropriate Feynman propagators for the corresponding free fields. We impose the normalization $Z(0,0,0) = 1$, and write

$$Z(\overline{\eta},\eta,J) = \exp[W(\overline{\eta},\eta,J)] \; . \tag{108}$$

Then $W(\overline{\eta},\eta,J)$ can be expressed as a series of connected Feynman diagrams with sources.

The rules for internal and external Dirac fermions were worked out in the context of Yukawa theory in section 45, and they follow here with no change. For external photons, the LSZ analysis of section 56 implies that each external photon line carries a factor of the polarization vector $\varepsilon^{\mu}(\mathbf{k})$. Putting everything together, we get the following set of Feynman rules for tree-level processes in quantum electrodynamics.

1) For each *incoming electron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with the electron's four-momentum, p_i.

2) For each *outgoing electron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with the electron's four-momentum, p_i'.

3) For each *incoming positron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with *minus* the positron's four-momentum, $-p_i$.

4) For each *outgoing positron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with *minus* the positron's four-momentum, $-p_i'$.

5) For each *incoming photon*, draw a wavy line with an arrow pointed *towards* the vertex, and label it with the photon's four-momentum, k_i. (Wavy lines for photons is a standard convention.)

6) For each *outgoing photon*, draw a wavy line with an arrow pointed *away* from the vertex, and label it with the photon's four-momentum, k_i'.

7) The only allowed vertex joins two solid lines, one with an arrow pointing towards it and one with an arrow pointing away from it, and one wavy line (whose arrow can point in either direction). Using this vertex, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are *topologically inequivalent*.

8) Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each

vertex. For a tree diagram, this fixes the momenta on all the internal lines.

9) The value of a diagram consists of the following factors:

for each incoming photon, $\varepsilon^{\mu*}_{\lambda_i}(\mathbf{k}_i)$;

for each outgoing photon, $\varepsilon^{\mu}_{\lambda'_i}(\mathbf{k}'_i)$;

for each incoming electron, $u_{s_i}(\mathbf{p}_i)$;

for each outgoing electron, $\overline{u}_{s'_i}(\mathbf{p}'_i)$;

for each incoming positron, $\overline{v}_{s_i}(\mathbf{p}_i)$;

for each outgoing positron, $v_{s'_i}(\mathbf{p}'_i)$;

for each vertex, $ie\gamma^\mu$;

for each internal photon line, $-ig^{\mu\nu}/(k^2 - i\epsilon)$,

where k is the four-momentum of that line;

for each internal fermion line, $-i(-\not{p} + m)/(p^2 + m^2 - i\epsilon)$,

where p is the four-momentum of that line.

10) Spinor indices are contracted by starting at one end of a fermion line: specifically, the end that has the arrow pointing away from the vertex. The factor associated with the external line is either $\overline{u}$ or $\overline{v}$. Go along the complete fermion line, following the arrows backwards, and write down (in order from left to right) the factors associated with the vertices and propagators that you encounter. The last factor is either a u or v. Repeat this procedure for the other fermion lines, if any. The vector index on each vertex is contracted with the vector index on either the photon propagator (if the attached photon line is internal) or the photon polarization vector (if the attached photon line is external).

11) Two diagrams that are identical *except for the momentum and spin labels on two external fermion lines* that *have their arrows pointing in the same direction* (either both towards or both away from the vertex) have a relative minus sign.

12) The value of $i\mathcal{T}$ (at tree level) is given by a sum over the values of all the contributing diagrams.

In the next section, we will do a sample calculation.

59: Tree-Level Scattering in QED

Prerequisite: 48, 58

In the last section we wrote down the Feynman rules for quantum electrodynamics. In this section, we will compute the scattering amplitude (and its spin-averaged square) at tree level for the process of electron-positron annihilation into a pair of photons, $e^+e^- \to \gamma\gamma$.

The contributing diagrams are shown in fig. (1), and the associated expression for the scattering amplitude is

$$\mathcal{T}_{e^+e^-\to\gamma\gamma} = e^2\,\varepsilon^{\mu}_{1'}\varepsilon^{\nu}_{2'}\,\overline{v}_2\left[\gamma_\mu\left(\frac{-\not{p}_1+\not{k}'_1+m}{-t+m^2}\right)\gamma_\nu+\gamma_\nu\left(\frac{-\not{p}_1+\not{k}'_2+m}{-u+m^2}\right)\gamma_\mu\right]u_1\;, \tag{109}$$

where $\varepsilon^{\mu}_{1'}$ is shorthand for $\varepsilon^{\mu}_{\lambda'_1}(\mathbf{k}'_1)$, $\overline{v}_2$ is shorthand for $\overline{v}_{s_2}(\mathbf{p}_2)$, and so on. The Mandelstam variables are

$$\begin{aligned} s &= -(p_1+p_2)^2 = -(k'_1+k'_2)^2\;,\\ t &= -(p_1-k'_1)^2 = -(p_2-k'_2)^2\;,\\ u &= -(p_1-k'_2)^2 = -(p_2-k'_1)^2\;, \end{aligned} \tag{110}$$

and they obey $s+t+u=2m^2$.

Following the procedure of section 46, we write eq. (109) as

$$\mathcal{T} = \varepsilon^{\mu}_{1'}\varepsilon^{\nu}_{2'}\,\overline{v}_2 A_{\mu\nu}u_1\;, \tag{111}$$

where

$$A_{\mu\nu} \equiv e^2\left[\gamma_\mu\left(\frac{-\not{p}_1+\not{k}'_1+m}{-t+m^2}\right)\gamma_\nu+\gamma_\nu\left(\frac{-\not{p}_1+\not{k}'_2+m}{-u+m^2}\right)\gamma_\mu\right]. \tag{112}$$

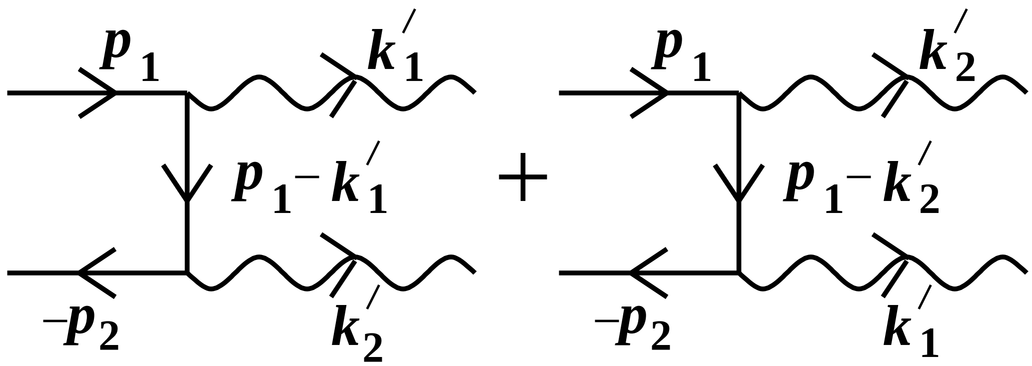

Figure 1: Diagrams for $e^+e^- \to \gamma\gamma$, corresponding to eq. (109).

We also have

$$\mathcal{T}^* = \overline{\mathcal{T}} = \varepsilon_{1'}^{\rho*}\varepsilon_{2'}^{\sigma*}\,\overline{u}_1\,\overline{A_{\rho\sigma}}\,v_2\ . \tag{113}$$

Using $\overline{\not a \not b \ldots} = \ldots \not b \not a$, we see from eq. (112) that

$$\overline{A_{\rho\sigma}} = A_{\sigma\rho}\ . \tag{114}$$

Thus we have

$$|\mathcal{T}|^2 = \varepsilon_{1'}^{\mu}\varepsilon_{2'}^{\nu}\,\varepsilon_{1'}^{\rho*}\varepsilon_{2'}^{\sigma*}\,(\overline{v}_2 A_{\mu\nu} u_1)(\overline{u}_1 A_{\sigma\rho} v_2)\ . \tag{115}$$

Next, we will average over the initial electron and positron spins, using the technology of section 46; the result is

$$\tfrac{1}{4}\sum_{s_1,s_2}|\mathcal{T}|^2 = \tfrac{1}{4}\varepsilon_{1'}^{\mu}\varepsilon_{2'}^{\nu}\varepsilon_{1'}^{\rho*}\varepsilon_{2'}^{\sigma*}\,\mathrm{Tr}\Big[A_{\mu\nu}(-\not p_1+m)A_{\sigma\rho}(-\not p_2-m)\Big]\ . \tag{116}$$

We would also like to sum over the final photon polarizations. From eq. (116), we see that we must evaluate

$$\sum_{\lambda=\pm}\varepsilon_{\lambda}^{\mu}(\mathbf{k})\varepsilon_{\lambda}^{\rho*}(\mathbf{k})\ . \tag{117}$$

We did this polarization sum in Coulomb gauge in section 56, with the result that

$$\sum_{\lambda=\pm}\varepsilon_{\lambda}^{\mu}(\mathbf{k})\varepsilon_{\lambda}^{\rho*}(\mathbf{k}) = g^{\mu\rho} + \hat{t}^{\mu}\hat{t}^{\rho} - \hat{z}^{\mu}\hat{z}^{\rho}\ , \tag{118}$$

where $\hat{t}^\mu$ is a unit vector in the time direction, and $\hat{z}^\mu$ is a unit vector in the $\mathbf{k}$ direction that can be expressed as

$$\hat{z}^\mu = \frac{k^\mu + (\hat{t}\cdot k)\hat{t}^\mu}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}} \ . \tag{119}$$

It is tempting to drop the k^μ and k^ρ terms in eq. (118), on the grounds that the photons couple to a conserved current, and so these terms should not contribute. (We indeed used this argument to drop the analogous terms in the photon propagator.) This also follows from the notion that the scattering amplitude should be invariant under a gauge transformation, as represented by a transformation of the external polarization vectors of the form

$$\varepsilon^\mu_\lambda(\mathbf{k}) \to \varepsilon^\mu_\lambda(\mathbf{k}) - i\tilde{\Gamma}(k)k^\mu \ . \tag{120}$$

Thus, if we write a scattering amplitude $\mathcal{T}$ for a process that includes a particular outgoing photon with four-momentum k^μ as

$$\mathcal{T} = \varepsilon^\mu_\lambda(\mathbf{k})\mathcal{M}_\mu \ , \tag{121}$$

or a particular incoming photon with four-momentum k^μ as

$$\mathcal{T} = \varepsilon^{\mu*}_\lambda(\mathbf{k})\mathcal{M}_\mu \ , \tag{122}$$

then in either case we should have

$$k^\mu \mathcal{M}_\mu = 0 \ . \tag{123}$$

Eq. (123) is in fact valid; we will give a proof of it, based on the Ward identity for the electromagnetic current, in section ??. For now, we will take eq. (123) as given, and so drop the k^μ and k^ρ terms in eq. (118).

This leaves us with

$$\sum_{\lambda=\pm} \varepsilon^\mu_\lambda(\mathbf{k})\varepsilon^{\rho*}_\lambda(\mathbf{k}) \to g^{\mu\rho} + \hat{t}^\mu\hat{t}^\rho - \frac{(\hat{t}\cdot k)^2}{k^2 + (\hat{t}\cdot k)^2}\,\hat{t}^\mu\hat{t}^\rho \ . \tag{124}$$

But, for an external photon, $k^2 = 0$. Thus the second and third terms in eq. (124) cancel, leaving us with the beautifully simple substitution rule

$$\sum_{\lambda=\pm} \varepsilon^\mu_\lambda(\mathbf{k})\varepsilon^{\rho*}_\lambda(\mathbf{k}) \to g^{\mu\rho} \ . \tag{125}$$

Using eq. (125), we can sum $|\mathcal{T}|^2$ over the polarizations of the outgoing photons, in addition to averaging over the spins of the incoming fermions; the result is

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &\equiv \tfrac{1}{4}\sum_{\lambda_1',\lambda_2'}\sum_{s_1,s_2}|\mathcal{T}|^2 \\ &= \tfrac{1}{4}\mathrm{Tr}\Big[A_{\mu\nu}(-\not{p}_1+m)A^{\nu\mu}(-\not{p}_2-m)\Big] \\ &= e^4\left[\frac{\langle\Phi_{tt}\rangle}{(m^2-t)^2}+\frac{\langle\Phi_{tu}\rangle+\langle\Phi_{ut}\rangle}{(m^2-t)(m^2-u)}+\frac{\langle\Phi_{uu}\rangle}{(m^2-u)^2}\right], \qquad (126)\end{aligned}$$

where

$$\langle\Phi_{tt}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\mu(-\not{p}_1+\not{k}_1'+m)\gamma_\nu(-\not{p}_1+m)\gamma^\nu(-\not{p}_1+\not{k}_1'+m)\gamma^\mu(-\not{p}_2-m)\Big], \qquad (127)$$

$$\langle\Phi_{uu}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\nu(-\not{p}_1+\not{k}_2'+m)\gamma_\mu(-\not{p}_1+m)\gamma^\mu(-\not{p}_1+\not{k}_2'+m)\gamma^\nu(-\not{p}_2-m)\Big], \qquad (128)$$

$$\langle\Phi_{tu}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\mu(-\not{p}_1+\not{k}_1'+m)\gamma_\nu(-\not{p}_1+m)\gamma^\mu(-\not{p}_1+\not{k}_2'+m)\gamma^\nu(-\not{p}_2-m)\Big], \qquad (129)$$

$$\langle\Phi_{ut}\rangle = \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\nu(-\not{p}_1+\not{k}_2'+m)\gamma_\mu(-\not{p}_1+m)\gamma^\nu(-\not{p}_1+\not{k}_1'+m)\gamma^\mu(-\not{p}_2-m)\Big]. \qquad (130)$$

Examinging eqs. (127) and (128), we see that $\langle\Phi_{tt}\rangle$ and $\langle\Phi_{uu}\rangle$ are transformed into each other by $k_1' \leftrightarrow k_2'$, which is equivalent to $t \leftrightarrow u$. The same is true of eqs. (129) and (130). Thus we need only compute $\langle\Phi_{tt}\rangle$ and $\langle\Phi_{tu}\rangle$, and then take $t \leftrightarrow u$ to get $\langle\Phi_{uu}\rangle$ and $\langle\Phi_{ut}\rangle$.

Now we can apply the gamma-matrix technology of section 47. In particular, we will need the $d = 4$ relations

$$\gamma^\mu\gamma_\mu = -4\,, \qquad (131)$$

$$\gamma^\mu\not{a}\gamma_\mu = 2\not{a}\,, \qquad (132)$$

$$\gamma^\mu\not{a}\not{b}\gamma_\mu = 4(ab)\,, \qquad (133)$$

$$\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = 2\not{c}\not{b}\not{a}\,, \qquad (134)$$

in addition to the trace formulae. We also need

$$\begin{aligned}p_1p_2 &= -\tfrac{1}{2}(s-2m^2)\,, \\ k_1'k_2' &= -\tfrac{1}{2}s\,, \\ p_1k_1' = p_2k_2' &= +\tfrac{1}{2}(t-m^2)\,, \\ p_1k_2' = p_2k_1' &= +\tfrac{1}{2}(u-m^2)\,. \qquad (135)\end{aligned}$$

which follow from eq. (110) plus the mass-shell conditions $p_1^2 = p_2^2 = -m^2$ and $k_1'^2 = k_2'^2 = 0$. After a lengthy and tedious calculation, we find

$$\langle\Phi_{tt}\rangle = 2[tu - m^2(3t+u) - m^4] \; , \tag{136}$$

$$\langle\Phi_{tu}\rangle = 2m^2(s - 4m^2) \; , \tag{137}$$

which then implies

$$\langle\Phi_{uu}\rangle = 2[tu - m^2(3u+t) - m^4] \; , \tag{138}$$

$$\langle\Phi_{ut}\rangle = 2m^2(s - 4m^2) \; . \tag{139}$$

This completes our calculation.

Other tree-level scattering processes in QED pose no new calculational difficulties, and are left to the problems.

In the high-energy limit, where the electron can be treated as massless, we can use the method of *spinor helicity*, which was introduced in section 50. We take this up in the next section.

60: Spinor Helicity for QED

Prerequisite: 48

In section 50, we introduced a special notation for u and v spinors of definite helicity for *massless* electrons and positrons. This notation greatly simplifies calculations in the high-energy limit (s, $|t|$, and $|u|$ all much greater than m^2).

We define the *twistors*

$$
\begin{aligned}
|p] &\equiv u_-(p) = v_+(p) \,, \\
|p\rangle &\equiv u_+(p) = v_-(p) \,, \\
[p| &\equiv \overline{u}_+(p) = \overline{v}_-(p) \,, \\
\langle p| &\equiv \overline{u}_-(p) = \overline{v}_+(p) \,.
\end{aligned} \tag{140}
$$

We then have

$$
\begin{aligned}
[k|\,|p] &= [k\,p] \,, \\
\langle k|\,|p\rangle &= \langle k\,p\rangle \,, \\
[k|\,|p\rangle &= 0 \,, \\
\langle k|\,|p] &= 0 \,,
\end{aligned} \tag{141}
$$

where the *twistor products* $[k\,p]$ and $\langle k\,p\rangle$ are antisymmetric,

$$
\begin{aligned}
[k\,p] &= -[p\,k] \,, \\
\langle k\,p\rangle &= -\langle p\,k\rangle \,,
\end{aligned} \tag{142}
$$

and related by complex conjugation, $\langle p\,k\rangle^* = [k\,p]$. They can be expressed explicitly in terms of the components of the massless four-momenta k and p. However, more useful are the relations

$$\langle p\,q\rangle\,[q\,r]\,\langle r\,s\rangle\,[s\,p] = \mathrm{Tr}\,\tfrac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s} \tag{143}$$

and

$$\begin{aligned}\langle k\,p\rangle\,[p\,k] &= \mathrm{Tr}\,\tfrac{1}{2}(1-\gamma_5)\not{k}\not{p} \\ &= -2k\cdot p \\ &= -(k+p)^2\ .\end{aligned} \tag{144}$$

Finally, for any massless four-momentum p we can write

$$-\not{p} = |p\rangle[p| + |p]\langle p|\ . \tag{145}$$

We will quote other results from section 50 as we need them.

To apply this formalism to quantum electrodynamics, we need to write photon polarization vectors in terms of twistors. The formulae we need are

$$\varepsilon^\mu_+(k) = \frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\,\langle q\,k\rangle}\ , \tag{146}$$

$$\varepsilon^\mu_-(k) = \frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}\,[q\,k]}\ , \tag{147}$$

where q is an arbitrary *reference momentum.*

The simplest way to verify eqs. (146) and (147) is to do so for a a specific choice of k, and then rely on the Lorentz transformation properties of twistors to conclude that the result must hold in any frame, and therefore for any massless four-momentum k. So, we will choose $k^\mu = (\omega, \omega\hat{\mathbf{z}}) = \omega(1,0,0,1)$. Then, the most general form of $\varepsilon^\mu_+(k)$ is

$$\varepsilon^\mu_+(k) = e^{i\phi}\tfrac{1}{\sqrt{2}}(0,1,-i,0) + Ck^\mu\ . \tag{148}$$

Here $e^{i\phi}$ is an arbitrary phase factor, and C is an arbitrary complex number; the freedom to add a multiple of k comes from the underlying gauge invariance.

To verify that eq. (146) reproduces eq. (148), we need the explicit form of the twistors $|k]$ and $|k\rangle$ when the three-momentum is in the z direction. Using results in section 50 we find

$$|k] = \sqrt{2\omega}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad |k\rangle = \sqrt{2\omega}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{149}$$

The most general form for $\langle q|$ is

$$\langle q| = (0,\,0,\,\alpha,\,\beta)\;, \tag{150}$$

where α and β are arbitrary complex numbers. Plugging eqs. (149) and (150) into eq. (146), and using

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix} \tag{151}$$

along with $\sigma^\mu = (I,\vec\sigma)$ and $\bar\sigma^\mu = (I,-\vec\sigma)$, we find that we reproduce eq. (148) with $e^{i\phi} = -1$ and $C = \beta/\alpha\omega$. There is now no need to check eq. (147), because $\varepsilon^\mu_-(k) = -[\varepsilon^\mu_+(k)]^*$, as can be seen by using $\langle q\,k\rangle^* = -[q\,k]$ along with another result from section 50, $\langle q|\gamma^\mu|k]^* = \langle k|\gamma^\mu|q]$.

In quantum electodynamics, the vector index on a photon polarization vector is always contracted with the vector index on a gamma matrix. We can get a convenient formula for $\not\varepsilon_\pm(k)$ by using the Fierz identities

$$-\tfrac{1}{2}\gamma^\mu\langle q|\gamma_\mu|k] \;=\; |k]\langle q| + |q\rangle[k|\;, \tag{152}$$

$$-\tfrac{1}{2}\gamma^\mu[q|\gamma_\mu|k\rangle \;=\; |k\rangle[q| + |q]\langle k|\;. \tag{153}$$

We then have

$$\not\varepsilon_+(k;q) \;=\; -\frac{\sqrt{2}}{\langle q\,k\rangle}\Big(|k]\langle q| + |q\rangle[k|\Big), \tag{154}$$

$$\not\varepsilon_-(k;q) \;=\; -\frac{\sqrt{2}}{[q\,k]}\Big(|k\rangle[q| + |q]\langle k|\Big), \tag{155}$$

where we have added the reference momentum as an explicit argument on the left-hand sides. Since the phase of $\not\varepsilon_\pm(k;q)$ is arbitrary, the minus signs on the right-hand sides of eqs. (154) and (155) can be dropped.

Now we have all the tools we need for doing calculations. However, we can simplify things even further by making maximal use of crossing symmetry.

Note from eq. (140) that u_- (which is the factor associated with an incoming electron) and v_+ (an outgoing positron) are both represented by the twistor $|p]$, while $\overline{u}_+$ (an outgoing electron) and $\overline{v}_-$ (an incoming positron) are both represented by $[p|$. Thus the square-bracket twistors correspond to outgoing fermions with positive helicity, and incoming fermions with negative helicity. Similarly, the angle-bracket twistors correspond to outgoing fermions with negative helicity, and incoming fermions with positive helicity.

Let us adopt a convention in which all particles are assigned four-momenta that are treated as outgoing. A particle that has an assigned four-momentum p then has physical four-momentum $\epsilon_p p$, where $\epsilon_p = \mathrm{sign}(p^0) = +1$ if the particle is physically outgoing, and $\epsilon_p = \mathrm{sign}(p^0) = -1$ if the particle is physically incoming.

Since the physical three-momentum of an incoming particle is opposite to its assigned three-momentum, a particle with negative helicity relative to its physical three-momentum has positive helicity relative to its assigned three-momentum. From now on, we will refer to the helicity of a particle relative to its assigned momentum. Thus a particle that we say has "positive helicity" actually has negative physical helicity if it is incoming, and positive physical helicity if it is outgoing.

With this convention, the square-bracket twistors $|p]$ and $[p|$ represent positive-helicity fermions, and the angle-bracket twistors $|p\rangle$ and $\langle p|$ represent negative-helicity fermions. When $\epsilon_p = \mathrm{sign}(p^0) = -1$, we analytically continue the twistors by replacing each $\omega^{1/2}$ in eq. (149) with $i|\omega|^{1/2}$. Then all of our formulae for twistors and polarizations hold without change, with the exception of the rule for complex conjugation of a twistor product, which becomes

$$\langle p\, k\rangle^* = \epsilon_p \epsilon_k [k\, p] \ . \qquad (156)$$

Now we are ready to calculate some amplitudes. Consider first the process of fermion-fermion scattering. The contributing tree-level diagrams are shown in fig. (2).

The first thing to notice is that a diagram is zero if two external fermion lines that meet at a vertex have the same helicity. This is because (as shown

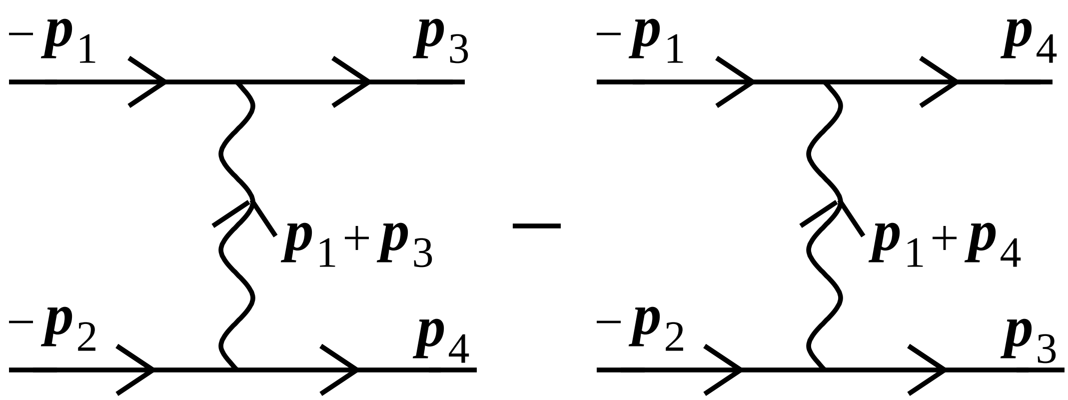

Figure 2: Diagrams for fermion-fermion scattering, with all momenta treated as outgoing.

in section 50) we get zero if we sandwich the product of an odd number of gamma matrices between two twistors of the same helicity. In particular, we have $\langle p|\gamma^\mu|k\rangle = 0$ and $[p|\gamma^\mu|k] = 0$. Thus, we will get a nonzero result for the tree-level amplitude only if two of the helicities are positive, and two are negative. This means that, of the $2^4 = 16$ possible combinations of helicities, only six give a nonzero tree-level amplitude: $\mathcal{T}_{++--}$, $\mathcal{T}_{+-+-}$, $\mathcal{T}_{+--+}$, $\mathcal{T}_{--++}$, $\mathcal{T}_{-+-+}$, and $\mathcal{T}_{-++-}$, where the notation is $\mathcal{T}_{s_1s_2s_3s_4}$. Furthermore, the last three of these are related to the first three by complex conjugation, so we only have three amplitudes to compute.

Let us begin with $\mathcal{T}_{+--+}$. Only the first diagram of fig. (2) contributes, because the second has two postive-helicity lines meeting at a vertex. To evaluate the first diagram, we note that the two vertices contribute a factor of $(ie)^2 = -e^2$, and the internal photon line contributes a factor of $ig_{\mu\nu}/s_{13}$, where we have defined the Mandelstam variable

$$s_{ij} \equiv -(p_i + p_j)^2 \ . \tag{157}$$

Following the charge arrows backwards on each fermion line, and dividing by i to get $\mathcal{T}$ (rather than $i\mathcal{T}$), we find

$$\begin{aligned} \mathcal{T}_{+--+} &= -e^2 \, \langle 3|\gamma^\mu|1] \, [4|\gamma_\mu|2\rangle \, /s_{13} \\ &= +2e^2 \, [1\,4] \, \langle 2\,3\rangle \, /s_{13} \ , \end{aligned} \tag{158}$$

where $\langle 3|$ is short for $\langle p_3|$, etc, and we have used yet another form of the Fierz identity to get the second line.

The computation of $\mathcal{T}_{+-+-}$ is exactly analogous, except that now it is only the second diagram of fig. (2) that contributes. According to the Feynman rules, this diagram comes with a relative minus sign, and so we have

$$\mathcal{T}_{+-+-} = -2e^2\,[1\,3]\,\langle 2\,4\rangle\,/s_{14}\,. \tag{159}$$

Finally, we turn to $\mathcal{T}_{++--}$. Now both diagrams contribute, and we have

$$\begin{aligned}\mathcal{T}_{++--} &= -e^2\left(\frac{\langle 3|\gamma^\mu|1]\,\langle 4|\gamma_\mu|2]}{s_{13}} - \frac{\langle 4|\gamma^\mu|1]\,\langle 3|\gamma_\mu|2]}{s_{14}}\right) \\ &= -2e^2\,[1\,2]\,\langle 3\,4\rangle\left(\frac{1}{s_{13}}+\frac{1}{s_{14}}\right) \\ &= +2e^2\,[1\,2]\,\langle 3\,4\rangle\left(\frac{s_{12}}{s_{13}s_{14}}\right)\,,\end{aligned} \tag{160}$$

where we used the Mandelstam relation $s_{12}+s_{13}+s_{14}=0$ to get the last line.

To get the cross section for a particular set of helicities, we must take the absolute squares of the amplitudes. These follow from eqs. (144) and (156):

$$|\langle 1\,2\rangle|^2 = |[1\,2]|^2 = \epsilon_1\epsilon_2 s_{12}\,. \tag{161}$$

Then taking the absolute square of any of eqs. (158–160) then yields an overall factor of $\epsilon_1\epsilon_2\epsilon_3\epsilon_4$. Since there are always two incoming and two outgoing particles, this factor equals one.

We can compute the spin-averaged cross section by summing the absolute squares of eqs. (158–160), multiplying by two to account for the processes in which all helicities are opposite (and which have amplitudes that are related by complex conjugation), and then dividing by four to average over the initial helicities. The result is

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= 2e^4\left(\frac{s_{14}^2}{s_{13}^2}+\frac{s_{13}^2}{s_{14}^2}+\frac{s_{12}^2 s_{34}^2}{s_{13}^2 s_{14}^2}\right) \\ &= 2e^4\left(\frac{s_{12}^4+s_{13}^4+s_{14}^4}{s_{13}^2 s_{14}^2}\right).\end{aligned} \tag{162}$$

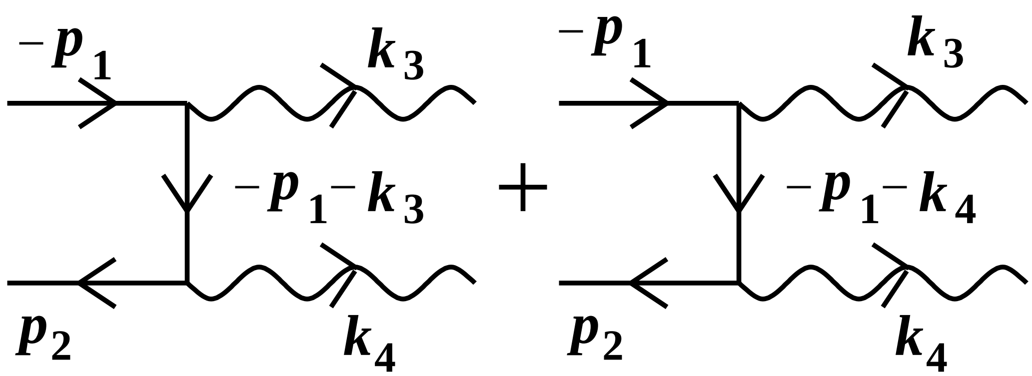

Figure 3: Diagrams for fermion-photon scattering, with all momenta treated as outgoing.

We used $s_{34} = s_{12}$ to get the second line.

For the processes of $e^-e^- \to e^-e^-$ and $e^+e^+ \to e^+e^+$, we have $s_{12} = s$, $s_{13} = t$, and $s_{14} = u$; for $e^+e^- \to e^+e^-$, we have $s_{13} = s$, $s_{14} = t$, and $s_{12} = u$.

Now we turn to processes with two external fermions and two external photons, as shown in fig. (3). The first thing to notice is that a diagram is zero if the two external fermion lines have the same helicity. This is because the corresponding twistors sandwich an odd number of gamma matrices: one from each vertex, and one from the massless fermion propagator $\tilde{S}(p) = -\not{p}/p^2$. Thus we need only compute $\mathcal{T}_{+-\lambda_3\lambda_4}$ since $\mathcal{T}_{-+\lambda_3\lambda_4}$ is related by complex conjugation.

Next we use eqs (154–155) and (141–142) to get

$$\not{\varepsilon}_-(k;p)|p] = 0 \,, \tag{163}$$

$$[p|\not{\varepsilon}_-(k;p) = 0 \,. \tag{164}$$

$$\not{\varepsilon}_+(k;p)|p\rangle = 0 \,, \tag{165}$$

$$\langle p|\not{\varepsilon}_+(k;p) = 0 \,, \tag{166}$$

Thus we can get some amplitudes to vanish with appropriate choices of the reference momenta in the photon polarizations.

So, let us consider

$$\mathcal{T}_{+-\lambda_3\lambda_4} = +\,e^2\,\langle 2|\not{\varepsilon}_{\lambda_4}(k_4;q_4)(\not{p}_1 + \not{k}_3)\not{\varepsilon}_{\lambda_3}(k_3;q_3)|1]\,/s_{13}$$

$$+\,e^2\,\langle 2|\not\epsilon_{\lambda_3}(k_3;q_3)(\not p_1+\not k_4)\not\epsilon_{\lambda_4}(k_4;q_4)|1]\,/s_{14}\;. \qquad (167)$$

If we take $\lambda_3 = \lambda_4 = -$, then we can get both terms in eq. (167) to vanish by choosing $q_3 = q_4 = p_1$, and using eq. (163). If we take $\lambda_3 = \lambda_4 = +$, then we can get both terms in eq. (167) to vanish by choosing $q_3 = q_4 = p_2$, and using eq. (166).

Thus, we need only compute $\mathcal{T}_{+--+}$ and $\mathcal{T}_{+-+-}$. For $\mathcal{T}_{+-+-}$, we can get the second term in eq. (167) to vanish by choosing $q_3 = p_2$, and using eq. (166). Then we have

$$\begin{aligned}\mathcal{T}_{+-+-} &= e^2\,\langle 2|\not\epsilon_-(k_4;q_4)(\not p_1+\not k_3)\not\epsilon_+(k_3;p_2)|1]\,/s_{13} \\ &= e^2\,\frac{\sqrt{2}}{[q_4 4]}\,\langle 2\,4\rangle\,[q_4|(\not p_1+\not k_3)|2\rangle\,[3\,1]\,\frac{\sqrt{2}}{\langle 2\,3\rangle}\,\frac{1}{s_{13}}\;. \end{aligned} \qquad (168)$$

Next we note that $[p|\not p = 0$, and so it is useful to choose either $q_4 = p_1$ or $q_4 = k_3$. There is no obvious advantage in one choice over the other, and they must give equivalent results, so let us take $q_4 = p_1$. Then, using eq. (145) for $\not k_3$, we get

$$\mathcal{T}_{+-+-} = -2e^2\,\frac{\langle 2\,4\rangle\,[1\,3]\,\langle 3\,2\rangle\,[3\,1]}{[1\,4]\,\langle 2\,3\rangle\,s_{13}} \qquad (169)$$

Now we use $s_{13} = -\langle 2\,4\rangle\,[2\,4]$, and antisymmetry of the twistor products, to get

$$\mathcal{T}_{+-+-} = 2e^2\,\frac{[1\,3]^2}{[1\,4]\,[2\,4]}\;. \qquad (170)$$

We can now get $\mathcal{T}_{+--+}$ simply by exchanging the labels 3 and 4,

$$\mathcal{T}_{+--+} = 2e^2\,\frac{[1\,4]^2}{[1\,3]\,[2\,3]}\;. \qquad (171)$$

We can compute the spin-averaged cross section by summing the absolute squares of eqs. (170) and (171), multiplying by two to account for the processes in which all helicities are opposite (and which have amplitudes that are related by complex conjugation), and then dividing by four to average over the initial helicities. The result is

$$\langle|\mathcal{T}|^2\rangle = 2e^4\,\epsilon_3\epsilon_4\left(\frac{s_{13}}{s_{14}}+\frac{s_{14}}{s_{13}}\right). \qquad (172)$$

We used $|\langle 1,3\rangle|^2 = \epsilon_1\epsilon_3 s_{13}$, $s_{24} = s_{13}$, etc, as well as $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$, to put the result in this form. The role of the ϵ's is to ensure that each term is positive.

For the processes of $e^-\gamma \to e^-\gamma$ and $e^+\gamma \to e^+\gamma$, we have $s_{13} = s$, $s_{12} = t$, $s_{14} = u$, and $\epsilon_3\epsilon_4 = -1$; for $e^+e^- \to \gamma\gamma$ and $\gamma\gamma \to e^+e^-$ we have $s_{12} = s$, $s_{13} = t$, $s_{14} = u$, and $\epsilon_3\epsilon_4 = +1$.

Problems

60.1) Show that $k_\mu \varepsilon^\mu_\pm(k;q) = 0$ (as required by gauge invariance) and that $q_\mu \varepsilon^\mu_\pm(k;q) = 0$ as well.

60.2) For a process with n external particles and all momenta treated as outgoing, show that

$$\sum_{j=1}^{n} \langle i\, j\rangle\, [j\, k] = 0\ . \tag{173}$$

60.3) Use various identities to show that eq. (171) can also be written as

$$\mathcal{T}_{+-+-} = -2e^2\, \frac{\langle 2\, 4\rangle^2}{\langle 1\, 3\rangle\, \langle 2\, 3\rangle}\ . \tag{174}$$

60.4a) Show explicitly that you would get the same result as eq. (170) if you set $q_4 = k_3$ in eq. (168).

b) Show explicitly that you would get the same result as eq. (170) if you set $q_4 = p_2$ in eq. (168).

60.5) Show that the tree-level scattering amplitude for two or more photons that all have the same helicity, plus any number of fermions with arbitrary helicities, vanishes.

60.6a) Consider the scattering of two fermions and three photons. Which tree-level helicity amplitudes are zero?

b) Compute the nonzero tree amplitudes.

61: Scalar Electrodynamics

Prerequisite: 58

In this section, we will consider how charged spin-zero particles interact with photons. We begin with the lagrangian for a free complex scalar field φ,

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi \ . \tag{175}$$

The lagrangian is obviously invariant under the global U(1) symmetry

$$\begin{aligned} \varphi(x) &\rightarrow e^{-i\alpha}\varphi(x) \ , \\ \varphi^\dagger(x) &\rightarrow e^{+i\alpha}\varphi^\dagger(x) \ . \end{aligned} \tag{176}$$

We would like to promote this to a local U(1) symmetry,

$$\varphi(x) \rightarrow \exp[-ie\Gamma(x)]\varphi(x) \ , \tag{177}$$

$$\varphi^\dagger(x) \rightarrow \exp[+ie\Gamma(x)]\varphi^\dagger(x) \ . \tag{178}$$

In order to do so, we must replace each ordinary derivative in eq. (175) with a covariant derivative

$$D_\mu \equiv \partial_\mu - ieA_\mu \ , \tag{179}$$

where A_μ transforms as

$$A^\mu(x) \rightarrow A^\mu(x) - \partial^\mu \Gamma(x) \ , \tag{180}$$

which implies that D_μ transforms as

$$D_\mu \rightarrow \exp[-ie\Gamma(x)] D_\mu \exp[+ie\Gamma(x)] \ . \tag{181}$$

Our complete lagrangian for *scalar electrodynamics* is then

$$\mathcal{L} = -(D^\mu\varphi)^\dagger D_\mu\varphi - m^2\varphi^\dagger\varphi - \tfrac{1}{4}\lambda(\varphi^\dagger\varphi)^2 - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\ . \tag{182}$$

We have added the usual gauge-invariant kinetic term for the gauge field. We have also added a gauge-invariant quartic coupling for the scalar field; this turns out to be necessary for renormalizability, as we will see in section ??. For now, we omit the renormalizing Z factors.

Of course, eq. (182) is invariant under a *global* U(1) transformation as well as a *local* U(1) transformation: we simply set $\Gamma(x)$ to a constant. Then we can find the conserved Noether current corresponding to this symmetry, following the procedure of section 22. In the case of QED (by which we mean quantum electrodynamics with a Dirac field), this current is same as it is in the case of a free Dirac field, $j^\mu = \overline{\Psi}\gamma^\mu\Psi$. In the case of a complex scalar field, we find

$$j^\mu = \mathrm{Im}\Big(\varphi^\dagger \overleftrightarrow{D^\mu}\varphi\Big)\ , \tag{183}$$

where $A\overleftrightarrow{D^\mu}B \equiv AD^\mu B - (D^\mu A)B$. We see that this current depends on the gauge field. Furthermore, with a factor of e, this current is indeed the electromagnetic current, which is usefully defined in general as

$$j^\mu_{\text{EM}}(x) \equiv \frac{\partial\mathcal{L}}{\partial A_\mu(x)}\ . \tag{184}$$

We had not previously contemplated the notion that the electromagnetic current could involve the gauge field itself, but in scalar electrodynamics this arises naturally, and is essential for gauge invariance.

It also poses no special problem in the quantum theory. We will make the same assumption that we did in the case of QED: namely, that the correct procedure is to omit integration over the component of $\tilde{A}_\mu(k)$ that is parallel to k_μ, on the grounds that this integration is redundant. This leads to the same Feynman rules for internal and external photons as in QED. The Feyman rules for internal and external scalars are the same as those of problem ??. We will call the spin-zero particle with electric charge $+e$ a *scalar electron* or *selectron* (recall that our convention is that e is negative), and the spin-zero particle with electric charge $-e$ a *scalar positron* or *spositron.*

Scalar lines (traditionally drawn as dashed in scalar electrodynamics) carry a charge arrow whose direction must be preserved when lines are joined by vertices.

To determine the kinds of vertices we have, we first write out the interaction terms in the lagrangian of eq. (182):

$$\mathcal{L}_{\rm int} = ieA^\mu[(\partial_\mu\varphi^\dagger)\varphi - \varphi^\dagger\partial_\mu\varphi] - e^2A^\mu A_\mu\varphi^\dagger\varphi - \tfrac{1}{4}\lambda(\varphi^\dagger\varphi)^2 \ . \qquad (185)$$

This leads to the vertices shown in fig. (4). The vertex factors associated with the last two terms are $-2ie^2g_{\mu\nu}$ and $-i\lambda$. The vertex factor for the first term is slightly tricky, because we have to translate the derivatives into momenta while keeping the signs right; this is done in problem 61.1. The result is that the vertex factor is $ie(k_1+k_2)_\mu$, where the scalar four-momenta are as shown in fig. (4).

Putting everything together, we get the following set of Feynman rules for tree-level processes in scalar electrodynamics.

1) For each *incoming selectron*, draw a dashed line with an arrow pointed *towards* the vertex, and label it with the selectron's four-momentum, k_i.

2) For each *outgoing selectron*, draw a dashed line with an arrow pointed *away* from the vertex, and label it with the selectron's four-momentum, k_i'.

3) For each *incoming spositron*, draw a dashed line with an arrow pointed *away* from the vertex, and label it with *minus* the spositron's four-momentum, $-k_i$.

4) For each *outgoing spositron*, draw a dashed line with an arrow pointed *towards* the vertex, and label it with *minus* the spositron's four-momentum, $-k_i'$.

5) For each *incoming photon*, draw a wavy line with an arrow pointed *towards* the vertex, and label it with the photon's four-momentum, k_i.

6) For each *outgoing photon*, draw a wavy line with an arrow pointed *away* from the vertex, and label it with the photon's four-momentum, k_i'.

7) There are three allowed vertices, shown in fig. (4). Using these vertices, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are *topologically inequivalent.*

8) Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each

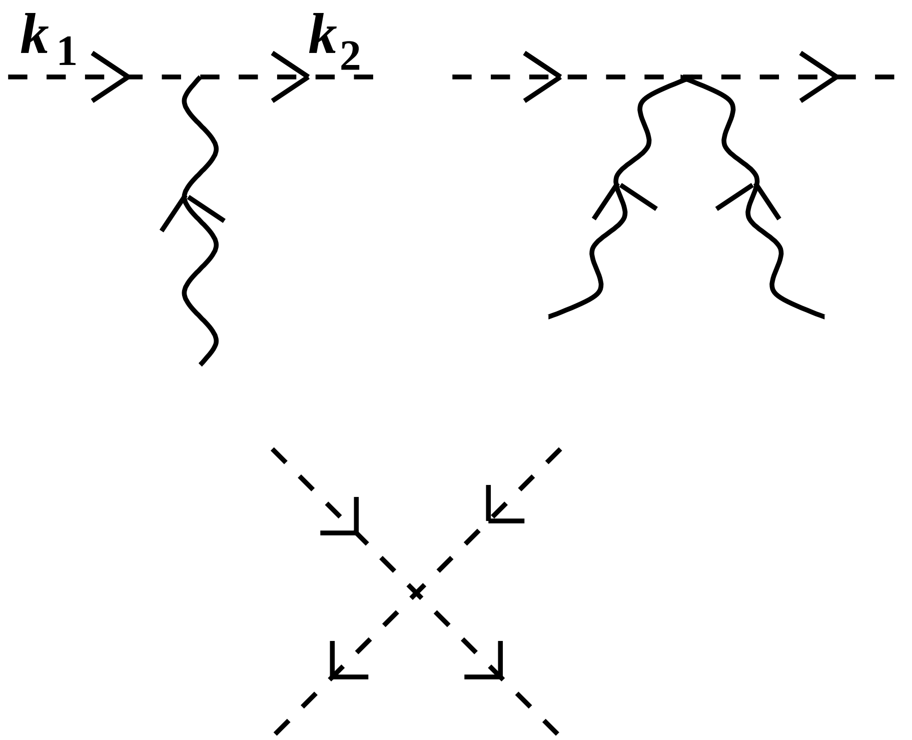

Figure 4: The three vertices of scalar electrodynamics; the corresponding vertex factors are $ie(k_1 + k_2)_\mu$, $-2ie^2 g_{\mu\nu}$, and $-i\lambda$.

vertex. For a tree diagram, this fixes the momenta on all the internal lines.

9) The value of a diagram consists of the following factors:

for each incoming photon, $\varepsilon^{\mu *}_{\lambda_i}(\mathbf{k}_i)$;

for each outgoing photon, $\varepsilon^{\mu}_{\lambda_i}(\mathbf{k}_i)$;

for each incoming or outgoing selectron or spositron, 1;

for each vertex, $ie(k_1+k_2)_\mu$, $-2ie^2 g_{\mu\nu}$, or $-i\lambda$,
according to the type of vertex;

for each internal photon line, $-ig^{\mu\nu}/(k^2-i\epsilon)$,
where k is the four-momentum of that line;

for each internal scalar , $-i/(k^2+m^2-i\epsilon)$,
where k is the four-momentum of that line.

10) The vector index on each vertex is contracted with the vector index on either the photon propagator (if the attached photon line is internal) or the photon polarization vector (if the attached photon line is external).

11) The value of $i\mathcal{T}$ (at tree level) is given by a sum over the values of all the contributing diagrams.

Let us compute the scattering amplitude for a particular process, $\widetilde{e}^+\widetilde{e}^- \to \gamma\gamma$, where $\widetilde{e}^-$ denotes a selectron. We have the diagrams of fig. (5).

The amplitude is

$$\begin{aligned} \mathcal{T} &= (ie)^2\,\frac{1}{i}\,\frac{(2k_1-k_3')_\mu \varepsilon^{\mu}_{3'}(k_1-k_3'-k_2)_\nu \varepsilon^{\nu}_{4'}}{M^2-t} + (3\leftrightarrow 4) \\ &\quad - 2ie^2 g_{\mu\nu}\varepsilon^{\mu}_{3'}\varepsilon^{\nu}_{4'}\;, \end{aligned} \tag{186}$$

where $t=-(k_1-k_3')^2$ and $u=-(k_1-k_4')^2$. This expression can be simplified by noting that $k_1-k_3'-k_2 = k_4'-2k_2$, and that $k_i\cdot\epsilon_i = 0$. Then we have

$$\mathcal{T} = -ie^2\left[\frac{4(k_1\cdot\varepsilon_{3'})(k_2\cdot\varepsilon_{4'})}{M^2-t} + \frac{4(k_1\cdot\varepsilon_{4'})(k_2\cdot\varepsilon_{3'})}{M^2-u} + 2(\varepsilon_{3'}\cdot\varepsilon_{4'})\right]. \tag{187}$$

To get the polarization-summed cross section, we take the absolute square of eq. (187), and use the substitution rule

$$\sum_{\lambda=\pm}\varepsilon^{\mu}_{\lambda}(\mathbf{k})\varepsilon^{\rho *}_{\lambda}(\mathbf{k}) \to g^{\mu\rho}\;. \tag{188}$$

This is a straightforward calculation.

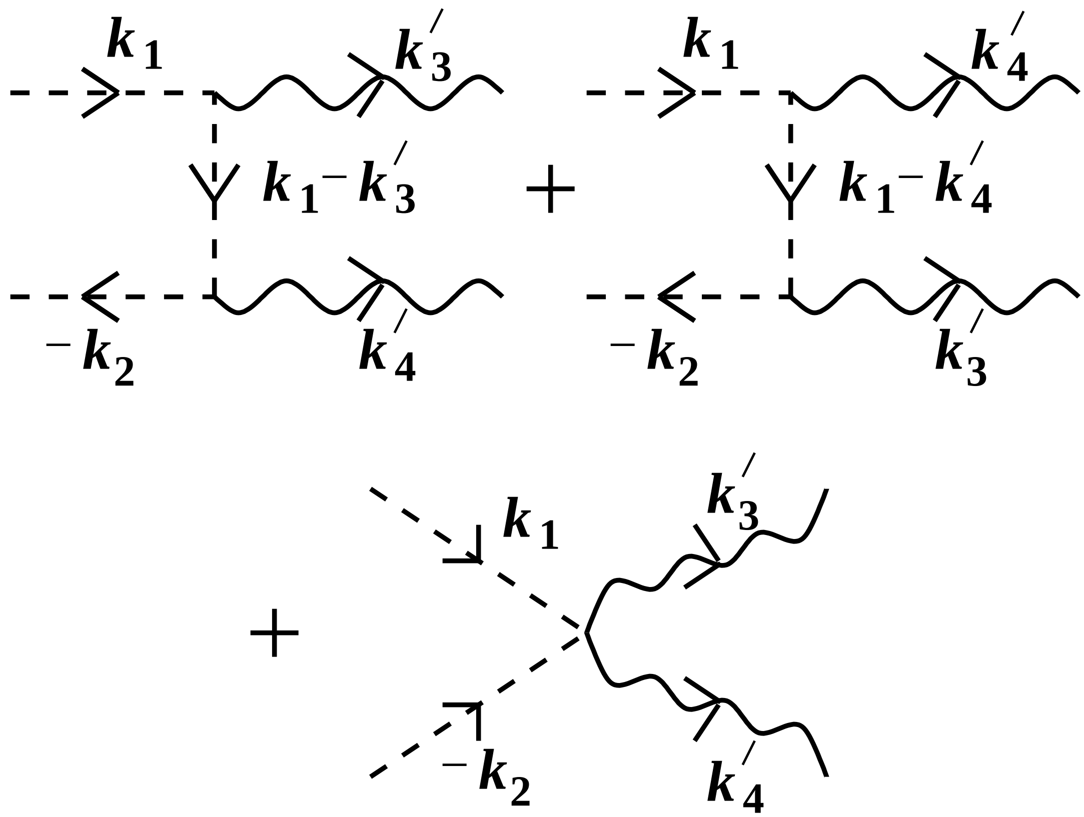

Figure 5: Diagrams for $\tilde{e}^+\tilde{e}^- \to \gamma\gamma$.

Problems

61.1) Compute the polarization-summed squared amplitude $\langle|\mathcal{T}|^2\rangle$ for eq. (187), and express your answer in terms of the Mandelstam variables.

61.2) Compute the scattering amplitude $\mathcal{T}$ and polarization averaged squared amplitude $\langle|\mathcal{T}|^2\rangle$ for the process $\widetilde{e}^-\gamma \to \widetilde{e}^-\gamma$.

62: Loop Corrections in Quantum Electrodynamics

Prerequisite: 51, 59

In this section we will compute the one-loop corrections in quantum electrodynamics of electrons and positrons, represented by a Dirac field.

First let us note that the general discussion of sections 18 and 29 leads us to expect that we will need to add to the lagrangian all possible terms whose coefficients have positive or zero mass dimension, and that respect the symmetries of the original lagrangian. These include Lorentz symmetry, the U(1) gauge symmetry, and the discrete symmetries of parity, time reversal, and charge conjugation.

The mass dimensions of the fields (in four spacetime dimensions) are $[A^\mu] = 1$ and $[\Psi] = \frac{3}{2}$. Gauge invariance requires that A^μ appear only in the form of a covariant derivative D^μ. (Recall that the field strength $F^{\mu\nu}$ can be expressed as the commutator of two covariant derivatives.) The only possible term we can write down that does not involve the Ψ field, and that has mass dimension four or less, is $\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$. This term, however, is odd under parity and time reversal. Similarly, there are no terms meeting all the requirements that involve Ψ: the only candidates contain either γ_5 (e.g., $i\overline{\Psi}\gamma_5\Psi$) and are forbidden by parity, or $\mathcal{C}$ (e.g, $\Psi^{\mathrm{T}}\mathcal{C}\Psi$) and are forbidden by the U(1) symmetry.

Therefore, the theory we will consider is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \;, \tag{189}$$

$$\mathcal{L}_0 = i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \;, \tag{190}$$

$$\mathcal{L}_1 = iZ_1 e\overline{\Psi}A\!\!\!/\Psi + \mathcal{L}_{\rm ct} \;, \tag{191}$$

$$\mathcal{L}_{\rm ct} = i(Z_2{-}1)\overline{\Psi}\partial\!\!\!/\Psi - (Z_m{-}1)m\overline{\Psi}\Psi - \tfrac{1}{4}(Z_3{-}1)F^{\mu\nu}F_{\mu\nu} \;. \tag{192}$$

We will use an on-shell renormalization scheme: the lagrangian parameter m is the actual mass of the electron, $\alpha = e^2/4\pi$ is the coefficient of $1/r^2$ in Coulomb's Law (as determined by doing electron-electron scattering at very low energy), and the fields are normalized according to the requirements of the LSZ formula.

We can write the exact photon propagator (in momentum space) as a geometric series of the form

$$\tilde{\mathbf{\Delta}}_{\mu\nu}(k) = \tilde{\Delta}_{\mu\nu}(k) + \tilde{\Delta}_{\mu\rho}(k)\Pi^{\rho\sigma}(k)\tilde{\Delta}_{\sigma\nu}(k) + \ldots \; , \tag{193}$$

where $i\Pi^{\mu\nu}(k)$ is given by a sum of 1PI diagrams with two external photon lines (and the external propagators removed), and $\tilde{\Delta}_{\mu\nu}(k)$ is the free photon propagator,

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1{-}\xi)\frac{k_\mu k_\nu}{k^2}\right) . \tag{194}$$

Here we have allowed ourselves some freedom of choice for the gauge by including the arbitrary parameter ξ multiplying a $k_\mu k_\nu$ term; observable squared amplitudes should not depend on ξ.

This suggests that $\Pi^{\mu\nu}(k)$ should be *transverse*,

$$k_\mu \Pi^{\mu\nu}(k) = k_\nu \Pi^{\mu\nu}(k) = 0 \; , \tag{195}$$

so that the $k_\mu k_\nu$ terms in $\tilde{\Delta}_{\mu\nu}(k)$ vanish when attached to the fermion lines in $\Pi^{\mu\nu}(k)$. Eq. (195) is in fact valid; we will give a proof of it, based on the Ward identity for the electromagnetic current, in section ??. For now, we will take eq. (195) as given. This implies that we can write

$$\Pi^{\mu\nu}(k) = \Pi(k^2)\Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big) \tag{196}$$

$$= k^2\,\Pi(k^2)P^{\mu\nu}(k) \; , \tag{197}$$

where $\Pi(k^2)$ is a scalar function, and $P^{\mu\nu}(k) = g^{\mu\nu} - k^\mu k^\nu/k^2$ is the projection matrix introduced in section 57.

Note that we can also write

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(P_{\mu\nu}(k) + \xi\,\frac{k_\mu k_\nu}{k^2}\right) . \tag{198}$$

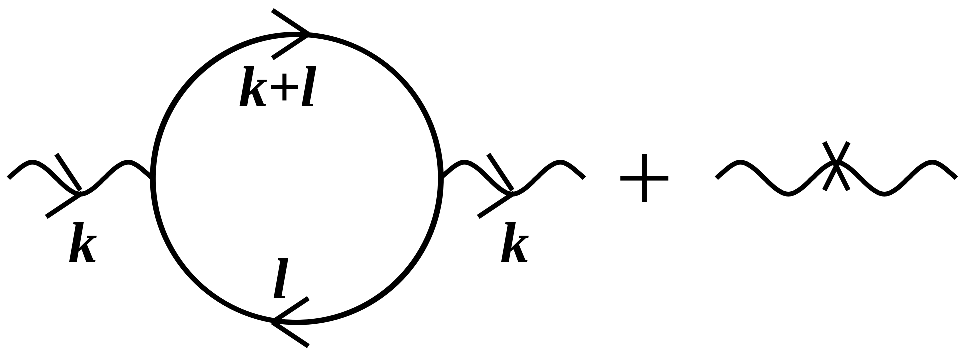

Figure 6: The one-loop and counterterm corrections to the photon propagator in QED.

Then, using eqs. (197) and (198) in eq. (193), and summing the geometric series, we find

$$\tilde{\mathbf{\Delta}}_{\mu\nu}(k) = \frac{P_{\mu\nu}(k)}{k^2[1-\Pi(k^2)]-i\epsilon} + \xi\,\frac{k_\mu k_\nu/k^2}{k^2-i\epsilon}\;. \tag{199}$$

The ξ dependent term should be physically irrelevant (and can be set to zero by the gauge choice $\xi = 0$, corresponding to Lorentz gauge). The remaining term has a pole at $k^2 = 0$ with residue $P_{\mu\nu}(k)/[1-\Pi(0)]$. In our on-shell renormalization scheme, we should have $\Pi(0) = 0$; this corresponds to the field normalization that is needed for the validity of the LSZ formula. (This is most easily checked in Coulomb gauge.)

Let us now turn to the calculation of $\Pi^{\mu\nu}(k)$. The one-loop and counterterm contributions are shown in fig. (6). We have

$$\begin{aligned} i\Pi^{\mu\nu}(k) \;\;&=\;\; (-1)(iZ_1 e)^2\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\Big[\tilde{S}(\not\ell+\not k)\gamma^\mu\tilde{S}(\not\ell)\gamma^\nu\Big] \\ &\qquad -\,i(Z_3{-}1)(k^2 g^{\mu\nu}-k^\mu k^\mu)+O(e^4)\;, \end{aligned} \tag{200}$$

where the factor of minus one is for the closed fermion loop, and $\tilde{S}(\not p) = (-\not p{+}m)/(p^2{+}m^2{-}i\epsilon)$ is the free fermion propagator in momentum space. Anticipating that $Z_1 = 1 + O(e^2)$, we can set $Z_1 = 1$ in the first term.

We can write

$$\mathrm{Tr}\Big[\tilde{S}(\not{\ell}+\not{k})\gamma^{\mu}\tilde{S}(\not{\ell})\gamma^{\nu}\Big] = \int_0^1 dx\, \frac{4N}{(q^2+D)^2}\ , \tag{201}$$

where we have combined denominators in the usual way: $q = \ell + xk$ and

$$D = x(1{-}x)k^2 + m^2 - i\epsilon\ . \tag{202}$$

The numerator is

$$4N = \mathrm{Tr}\Big[\tilde{S}(-\not{\ell}-\not{k}+m)\gamma^{\mu}\tilde{S}(-\not{\ell}+m)\gamma^{\nu}\Big] \tag{203}$$

Completing the trace, we get

$$N = (\ell{+}k)^{\mu}k^{\nu} + k^{\mu}(\ell{+}k)^{\nu} - [\ell(\ell{+}k) + m^2]g^{\mu\nu}\ . \tag{204}$$

Setting $\ell = q{-}xk$ and and dropping terms linear in q (because they integrate to zero), we find

$$N \to 2q^{\mu}q^{\nu} - 2x(1{-}x)k^{\mu}k^{\nu} - [q^2 - x(1{-}x)k^2 + m^2]g^{\mu\nu}\ . \tag{205}$$

The integrals diverge, and so we analytically continue to $d = 4{-}\varepsilon$ dimensions, and replace e with $e\tilde{\mu}^{\varepsilon/2}$ (so that e remains dimensionless for any d).

Next we recall a result from section 31:

$$\int d^dq\; q^{\mu}q^{\nu}f(q^2) = \frac{1}{d}\,g^{\mu\nu}\int d^dq\; q^2 f(q^2)\ . \tag{206}$$

This allows the replacement

$$N \to -2x(1{-}x)k^{\mu}k^{\nu} + \Big[\Big(\tfrac{2}{d}-1\Big)q^2 + x(1{-}x)k^2 - m^2\Big]\,g^{\mu\nu}\ . \tag{207}$$

Using the results of section 14, along with a little manipulation of gamma functions, we can show that

$$\Big(\frac{2}{d}-1\Big)\int \frac{d^dq}{(2\pi)^d}\,\frac{q^2}{(q^2+D)^2} = 2D\int \frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+D)^2}\ . \tag{208}$$

Thus we can make the replacement $(2/d-1)q^2 \to 2D$ in eq. (**??**), and we find

$$N \to 2x(1{-}x)(k^2g^{\mu\nu} - k^{\mu}k^{\nu})\ . \tag{209}$$

This guarantees that the one-loop contribution to $\Pi^{\mu\nu}(k)$ is transverse (as we expected) in any number of spacetime dimensions.

Now we evaluate the integral over q, using

$$\tilde\mu^{\varepsilon}\int\frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+D)^2} \;=\; \frac{i}{16\pi^2}\,\Gamma(\tfrac{2}{\varepsilon})\Big(4\pi\tilde\mu^2/D\Big)^{\varepsilon}$$
$$\;=\; \frac{i}{8\pi^2}\left[\frac{1}{\varepsilon}-\tfrac12\ln(D/\mu^2)\right]\,, \qquad (210)$$

where $\mu^2 = 4\pi e^{-\gamma}\tilde\mu^2$, and we have dropped terms of order ε in the last line. Combining eqs. (196), (200), (201), (209), and (210), we get

$$\Pi(k^2) = -\frac{e^2}{\pi^2}\int_0^1 dx\,x(1{-}x)\left[\frac{1}{\varepsilon}-\tfrac12\ln(D/\mu^2)\right]-(Z_3{-}1)+O(e^4)\,. \qquad (211)$$

Imposing $\Pi(0)=0$ fixes

$$Z_3 = 1-\frac{e^2}{6\pi^2}\left(\frac{1}{\varepsilon}+\text{finite}\right)+O(e^4) \qquad (212)$$

and

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\,x(1{-}x)\ln(D/m^2)+O(e^4)\,. \qquad (213)$$

Next we turn to the fermion propagator. The exact propagator can be written in Lehmann-Källén form as

$$\tilde{\mathbf{S}}(\not p) = \frac{1}{\not p+m-i\epsilon}+\int_{m^2_{\rm th}}^{\infty}ds\,\frac{\rho_\Psi(s)}{\not p+\sqrt{s}-i\epsilon}\,. \qquad (214)$$

We see that the first term has a pole at $\not p = -m$ with residue one. This residue corresponds to the field normalization that is needed for the validity of the LSZ formula.

There is a problem, however: for QED, the threshold mass $m_{\rm th}$ is m, corresponding to the contribution of a fermion and a zero-energy photon. Thus the second term has a branch point at $\not p = -m$. The pole in the first term is therefore not isolated, and its residue is ill defined.

This is a reflection of an underlying infrared divergence, associated with the massless photon. To sidestep it, we will have to impose an infrared cutoff that moves the branch point away from the pole. The simplest method is to change the denominator of the photon propagator from k^2 to $k^2 + \lambda^2$, where λ plays the role of a fictitious photon mass. Ultimately, as in section 25, we must deal with this issue by computing cross-sections that take into account detector inefficiencies. In the case of QED, we must specify the lowest photon energy $\omega_{\rm min}$ that can be detected. Only after computing cross sections with extra undectable photons, and then summing over them, is it safe to take the limit $\lambda \to 0$.

An alternative is to use dimensional regularization for the infrared divergences as well as the ultraviolet ones. As discussed in section 25, there are no soft-particle infrared divergences for $d > 4$ (and no colinear divergences at all in QED with massive electrons). In practice, infrared-divergent integrals are finite away from even-integer dimensions, just like ultraviolet-divergent integrals. Thus we simply keep $d = 4 - \varepsilon$ all the way through to the very end, taking the $\varepsilon \to 0$ limit only after summing over cross sections with extra undetectable photons, all computed in $4 - \varepsilon$ dimensions. This method is calculationally the simplest, but requires careful bookkeeping to segregate the infrared and ultraviolet singularities. For that reason, we will not pursue it further.

We can write the exact fermion propagator in the form

$$\tilde{\mathbf{S}}(\not p)^{-1} = \not p + m - i\epsilon - \Sigma(\not p) \; , \tag{215}$$

where $i\Sigma(\not p)$ is given by the sum of 1PI diagrams with two external fermion lines (and the external propagators removed). The fact that $\tilde{\mathbf{S}}(\not p)$ has a pole at $\not p = -m$ with residue one implies that $\Sigma(-m) = 0$ and $\Sigma'(-m) = 0$; this fixes the coefficients Z_2 and Z_m. As we will see, we must have an infrared cutoff in place in order to have a finite value for $\Sigma'(-m)$.

Let us now turn to the calculation of $\Sigma(\not p)$. The one-loop and counterterm contributions are shown in fig. (7). We have

$$\begin{aligned} i\Sigma(\not p) \;=\; & (ie)^2\Bigl(\tfrac{1}{i}\Bigr)^2 \int \frac{d^4\ell}{(2\pi)^4} \Bigl[\gamma^\nu \tilde{S}(\not p + \not \ell)\gamma^\mu\Bigr]\tilde{\Delta}_{\mu\nu}(\ell) \\ & - (Z_2{-}1)\not p - (Z_m{-}1)m + O(e^4) \; . \end{aligned} \tag{216}$$

It is simplest to work in Feynman gauge, where we take

$$\tilde{\Delta}_{\mu\nu}(\ell) = \frac{g_{\mu\nu}}{\ell^2 + \lambda^2 - i\epsilon} \; ; \tag{217}$$

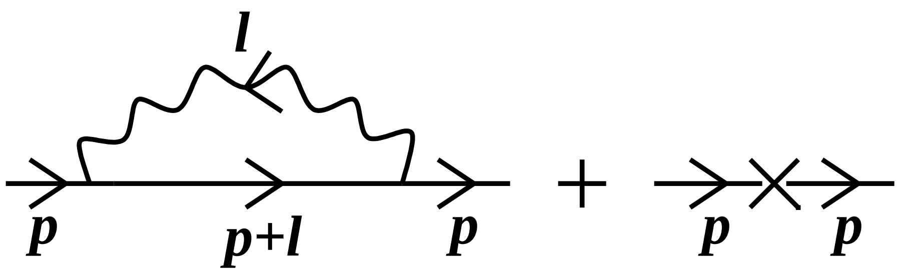

Figure 7: The one-loop and counterterm corrections to the fermion propagator in QED.

here we have included the fictitious photon mass λ as an infrared cutoff.

We now apply the usual bag of tricks to get

$$\begin{aligned} i\Sigma(\not{p}) &= -e^2\tilde{\mu}^{\varepsilon}\int_0^1 dx \int \frac{d^dq}{(2\pi)^d}\,\frac{N}{(q^2+D)^2} \\ &\quad -(Z_2{-}1)\not{p}-(Z_m{-}1)m+O(e^4)\;, \end{aligned} \tag{218}$$

where $q = \ell + xk$ and

$$D = x(1{-}x)p^2 + xm^2 + (1{-}x)\lambda^2\;, \tag{219}$$

$$\begin{aligned} N &= \gamma_\mu(-\not{p}-\not{\ell}+m)\gamma^\mu \\ &= -(d{-}2)(\not{p}+\not{\ell}) - dm \\ &= -(d{-}2)[\not{q}+(1{-}x)\not{p}] - dm\;, \end{aligned} \tag{220}$$

where we have used (from section 47) $\gamma_\mu\gamma^\mu = -d$ and $\gamma_\mu\not{p}\gamma^\mu = (d{-}2)\not{p}$. The term linear in q integrates to zero, and then, using eq. (210), we get

$$\begin{aligned} \Sigma(\not{p}) &= -\frac{e^2}{8\pi^2}\int_0^1 dx\Big((2{-}\varepsilon)(1{-}x)\not{p}+(4{-}\varepsilon)m\Big)\left[\frac{1}{\varepsilon}-\tfrac{1}{2}\ln(D/\mu^2)\right] \\ &\quad -(Z_2{-}1)\not{p}-(Z_m{-}1)m+O(e^4)\;. \end{aligned} \tag{221}$$

We see that finiteness of $\Sigma(\not{p})$ requires

$$Z_2 = 1-\frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon}+\text{finite}\right)+O(e^4)\;, \tag{222}$$

$$Z_m = 1-\frac{e^2}{2\pi^2}\left(\frac{1}{\varepsilon}+\text{finite}\right)+O(e^4)\;. \tag{223}$$

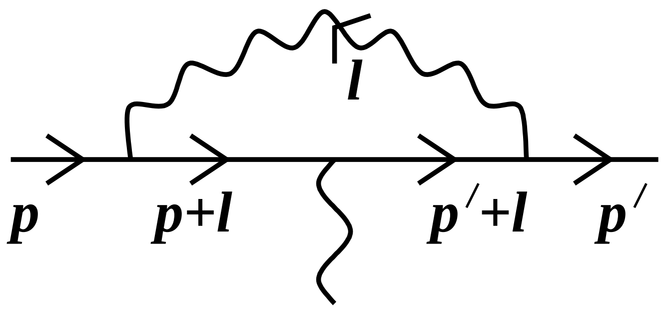

Figure 8: The one-loop correction to the photon-fermion-fermion vertex in QED.

We can impose $\Sigma(-m) = 0$ by writing

$$\Sigma(\not{p}) = \frac{e^2}{8\pi^2}\left[\int_0^1 dx\,\Big((1-x)\not{p} + 2m\Big)\ln(D/D_0) + \kappa_2(\not{p}+m)\right], \tag{224}$$

where D_0 is D evaluated at $p^2 = -m^2$,

$$D_0 = x^2m^2 + (1-x)\lambda^2\;, \tag{225}$$

and κ_2 is a constant to be determined. We fix κ_2 by imposing $\Sigma'(-m) = 0$. In differentiating with respect to $\not{p}$, we take the p^2 in D, eq. (219), to be $-\not{p}^2$; we find

$$\begin{aligned}\kappa_2 &= -2\int_0^1 dx\,x(1-x^2)m^2/D_0 \\ &= -2\ln(m/\lambda) + 1\;,\end{aligned} \tag{226}$$

where we have dropped terms that go to zero with the infrared cutoff λ.

Next we turn to the loop correction to the vertex. We define the vertex function $i\mathbf{V}^\mu(p',p)$ as the sum of one-particle irreducible diagrams with one incoming fermion with momentum p, one outgoing fermion with momentum p', and one incoming photon with momentum $k = p' - p$. The original vertex $iZ_1e\gamma^\mu$ is the first term in this sum, and the diagram of fig. (8) is the second. Thus we have

$$i\mathbf{V}^\mu(p',p) = iZ_1e\gamma^\mu + i\mathbf{V}^\mu_{1\,\mathrm{loop}}(p',p) + O(e^5)\;, \tag{227}$$

where

$$i\mathbf{V}^{\mu}_{1\,\text{loop}}(p',p) = (ie)^3\left(\tfrac{1}{i}\right)^3 \int \frac{d^d\ell}{(2\pi)^d} \left[\gamma^\rho \tilde{S}(\not{p}'+\not{\ell})\gamma^\mu \tilde{S}(\not{p}+\not{\ell})\gamma^\nu\right]\tilde{\Delta}_{\nu\rho}(\ell) \; . \tag{228}$$

We again use eq. (217) for the photon propagator, and combine denominators in the usual way. We then get

$$i\mathbf{V}^{\mu}_{1\,\text{loop}}(p',p)/e = e^2 \int dF_3 \int \frac{d^4q}{(2\pi)^4} \, \frac{N^\mu}{(q^2+D)^3} \; , \tag{229}$$

where the integral over Feynman parameters is

$$\int dF_3 \equiv 2\int_0^1 dx_1 dx_2 dx_3 \, \delta(x_1{+}x_2{+}x_3{-}1) \; , \tag{230}$$

and

$$\begin{aligned} q &= \ell + x_1 p + x_2 p' \; , & (231) \\ D &= x_1(1{-}x_1)p^2 + x_2(1{-}x_2)p'^2 - 2x_1x_2 p{\cdot}p' + (x_1{+}x_2)m^2 + x_3\lambda^2 & (232) \\ N^\mu &= \gamma_\nu(-\not{p}' - \not{\ell} + m)\gamma^\mu(-\not{p} - \not{\ell} + m)\gamma^\nu & \\ &= \gamma_\nu[-\not{q} + x_1\not{p} - (1{-}x_2)\not{p}' + m]\gamma^\mu[-\not{q} - (1{-}x_1)\not{p} + x_2\not{p}' + m]\gamma^\nu & \\ &= \gamma_\nu\not{q}\gamma^\mu\not{q}\gamma^\nu + \widetilde{N}^\mu + (\text{linear in } q) \; , & (233) \end{aligned}$$

where

$$\widetilde{N}^\mu = \gamma_\nu[x_1\not{p} - (1{-}x_2)\not{p}' + m]\gamma^\mu[-(1{-}x_1)\not{p} + x_2\not{p}' + m]\gamma^\nu \; . \tag{234}$$

The terms linear in q in eq. (233) integrate to zero, and only the first term is divergent. After continuing to d dimensions, we can use eq. (206) to make the replacement

$$\gamma_\nu\not{q}\gamma^\mu\not{q}\gamma^\nu \to \frac{1}{d}\, q^2\, \gamma_\nu\gamma_\rho\gamma^\mu\gamma^\rho\gamma^\nu \; . \tag{235}$$

Then we use $\gamma_\rho\gamma^\mu\gamma^\rho = (d{-}2)\gamma^\mu$ twice to get

$$\gamma_\nu\not{q}\gamma^\mu\not{q}\gamma^\nu \to \frac{(d{-}2)^2}{d}\, q^2\, \gamma^\mu \; . \tag{236}$$

Performing the usual manipulations, we find

$$\mathbf{V}^{\mu}_{1\,\mathrm{loop}}(p',p)/e = \frac{e^2}{8\pi^2}\left[\left(\frac{1}{\varepsilon} - 1 - \tfrac{1}{2}\int dF_3\;\ln(D/\mu^2)\right)\gamma^\mu + \tfrac{1}{4}\int dF_3\,\frac{\widetilde{N}^\mu}{D}\right]. \tag{237}$$

From eq. (227), we see that finiteness of $\mathbf{V}^\mu(p',p)$ requires

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4)\;. \tag{238}$$

To completely fix $\mathbf{V}^\mu(p',p)$, we need a suitable condition to impose on it. We take this up in the next section.

Problems

63: The Vertex Function in Quantum Electrodynamics

Prerequisite: 62

In the last section, we computed the one-loop contribution to the vertex function $\mathbf{V}^\mu(p',p)$ in quantum electrodynamics, where p is the four-momentum of an incoming electron (or outgoing positron), and p' is the four-momentum of an outgoing electron (or incoming positron). We left open the issue of the renormalization condition we wish to impose on $\mathbf{V}^\mu(p',p)$.

For the theories we have studied previously, we have usually made the mathematically convenient (but physically obscure) choice to define the coupling constant as the value of the vertex function when all external four-momenta are set to zero. However, in the case of quantum electrodynamics, the masslessness of the photon gives us the opportunity to do something more physically meaningful: we can define the coupling constant as the value of the vertex function when all three particles are on shell: $p^2 = p'^2 = -m^2$, and $q^2 = 0$, where $q \equiv p' - p$ is the photon four-momentum. Because the photon is massless, these three on-shell conditions are compatible with momentum conservation.

Of course, the vertex function $\mathbf{V}^\mu(p',p)$ is a four-vector of 4×4 matrices, so we are speaking schematically when we talk of its value. To be more precise, let us sandwich $\mathbf{V}^\mu(p',p)$ between the spinor factors that are appropriate for an incoming electron with momentum p and an outgoing electron with momentum p', impose the on-shell conditions, and define the electron charge e via

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^\mu(p',p)u_s(\mathbf{p})\Big|_{\substack{p^2=p'^2=-m^2\\(p'-p)^2=0}} = e\,\overline{u}_{s'}(\mathbf{p}')\gamma^\mu u_s(\mathbf{p})\Big|_{\substack{p^2=p'^2=-m^2\\(p'-p)^2=0}}\,. \qquad (239)$$

This definition is in accord with the usual one provided by Coulomb's law. To see why, consider the process of electron-electron scattering, computed with the exact propagators and vertices of the quantum action. The contributing Feynman diagrams are the usual ones at tree-level, shown in fig. (**??**), but with each vertex representing the exact vertex fuction $\mathbf{V}^\mu(p',p)$, and the wavy internal line representing the exact photon propagator $\tilde{\mathbf{\Delta}}_{\mu\nu}(q)$. (There is also a contribution, not shown, from a four-point vertex connecting all four lines, but this vertex does not have the $1/q^2$ from the photon propagator, and so does not contribute to the Coulomb force.) In the last section, we renormalized the photon propagator so that it approaches its tree-level value $\tilde{\Delta}_{\mu\nu}(q)$ when $q^2 \to 0$. And we have just chosen to renormalize the vertex function by requiring it to approach its tree-level value when $q^2 \to 0$, and when sandwiched between external spinors for on-shell incoming and outgoing electrons. Therefore, as $q^2 \to 0$, the exact electron-electron scattering amplitude approaches what we get from the tree diagrams, with the electron charge equal to e. Physically, $q^2 \to 0$ means that the electron's momentum changes very little during the scattering. Measuring a slight deflection in the trajectory of one charged particle (due to the presence of another) is how we measure the coefficient in Coulomb's law. Thus, eq. (239) corresponds to this traditional definition of the charge of the electron.

We can simplify eq. (239) by noting that the on-shell conditions actually enforce $p' = p$. So we can rewrite eq. (239) as

$$\begin{aligned} \overline{u}_s(\mathbf{p})\mathbf{V}^\mu(p,p)u_s(\mathbf{p}) &= e\,\overline{u}_s(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) \\ &= 2ep^\mu \ , \end{aligned} \tag{240}$$

where $p^2 = -m^2$ is implicit. We have taken $s' = s$, because otherwise the right-hand side vanishes (and hence does not specify a value for e).

Now we can use eq. (240) to completely determine $\mathbf{V}^\mu(p',p)$. Using the freedom to choose the finite part of Z_1, we first write it as

$$\mathbf{V}^\mu(p',p) = e\gamma^\mu - \frac{e^3}{16\pi^2}\int dF_3\Big[\Big(\ln(D/D_0) + \kappa_1\Big)\gamma^\mu - \frac{N^\mu}{2D}\Big] + O(e^5) \ , \tag{241}$$

where

$$\begin{aligned} D \;&=\; x_1(1{-}x_1)p^2 + x_2(1{-}x_2)p'^2 - 2x_1x_2 p{\cdot}p' \\ &\quad + (x_1{+}x_2)m^2 + x_3\lambda^2 \ , \end{aligned} \tag{242}$$

D_0 is D evaluated at $p' = p$ and $p^2 = -m^2$,

$$\begin{aligned} D_0 &= (x_1{+}x_2)^2 m^2 + x_3\lambda^2 \\ &= (1{-}x_3)^2 m^2 + x_3\lambda^2 \;, \end{aligned} \tag{243}$$

and

$$N^\mu = \gamma_\nu [x_1 \not{p} - (1{-}x_2)\not{p}' + m]\gamma^\mu[-(1{-}x_1)\not{p} + x_2\not{p}' m]\gamma^\nu \;; \tag{244}$$

N^μ was called $\widetilde{N}^\mu$ in section 62, but we have dropped the tilde for notational convenience.

We fix the constant κ_1 in eq. (241) by imposing eq. (240). This yields

$$2\kappa_1 p^\mu = \int dF_3 \, \frac{\overline{u}_s(\mathbf{p}) N_0^\mu u_s(\mathbf{p})}{2D_0} \;, \tag{245}$$

where N_0^μ is N^μ with $p' = p$ and $p^2 = p'^2 = -m^2$.

So now we must evaluate $\overline{u} N_0^\mu u$. To do so, we first write

$$N^\mu = \gamma_\nu(\not{a}_1{+}m)\gamma^\mu(\not{a}_2{+}m)\gamma^\nu \;, \tag{246}$$

where

$$\begin{aligned} a_1 &= x_1 p - (1{-}x_2)p' \;, \\ a_2 &= x_2 p' - (1{-}x_1)p \;. \end{aligned} \tag{247}$$

Now we use the gamma matrix contraction identities to get

$$N^\mu = 2\not{a}_2\gamma^\mu\not{a}_1 + 4m(a_1{+}a_2)^\mu + 2m^2\gamma^\mu \;. \tag{248}$$

Here we have set $d = 4$, because we have already removed the divergence and taken the limit $\varepsilon \to 0$. Setting $p' = p$, and using $\not{p}u = -mu$ and $\overline{u}\not{p} = -m\overline{u}$, along with $\overline{u}\gamma^\mu u = 2p^\mu$ and $\overline{u}u = 2m$, and recalling that $x_1{+}x_2{+}x_3 = 1$, we find

$$\overline{u} N_0^\mu u = 4(1{-}4x_3{+}x_3^2)m^2 p^\mu \;. \tag{249}$$

Using eqs. (243), (245), and (249), we get

$$\begin{aligned}\kappa_1 &= \int dF_3\,\frac{1-4x_3+x_3^2}{(1-x_3)^2+x_3\lambda^2/m^2} \\ &= 2\int_0^1 dx_3\,(1-x_3)\,\frac{1-4x_3+x_3^2}{(1-x_3)^2+x_3\lambda^2/m^2} \\ &= -4\ln(m/\lambda)+5 \end{aligned} \tag{250}$$

in the limit of $\lambda \to 0$. We see that an infrared regulator is necessary for the vertex function as well as the fermion propagator.

Now that we have $\mathbf{V}^\mu(p',p)$, we can extract some physics from it. Consider again the process of electron-electron scattering, shown in fig. (**??**). In order to compute the contributions of these diagrams, we must evaluate

$$\overline{u}'\,V^\mu\,u \equiv \overline{u}_{s'}(\mathbf{p}')\mathbf{V}^\mu(p',p)u_s(\mathbf{p})\;, \tag{251}$$

with $p^2 = -p'^2 = -m^2$, but with $q^2 = (p'-p)^2$ arbitrary.

To evaluate $\overline{u}'N^\mu u$, we first use the anticommutation relations of the gamma matrices to move all the $\not{p}$'s in N^μ to the far right, where we can use $\not{p}u = -mu$, and all the $\not{p}'$'s to the far left, where we can use $\overline{u}'\not{p}' = -m\overline{u}'$. This results in

$$\begin{aligned} N^\mu \;\to\; &[4(1-x_1-x_2+x_1x_2)p\cdot p' + 2(2x_1-x_1^2+2x_2-x_2^2)m^2]\gamma^\mu \\ &+ 4m(x_1^2-x_2+x_1x_2)p^\mu + 4m(x_2^2-x_1+x_1x_2)p'^\mu\;. \end{aligned} \tag{252}$$

Next, replace $p\cdot p'$ with $-\frac{1}{2}q^2-m^2$, group the p^μ and p'^μ terms into $p'+p$ and $p'-p$ combinations, and make use of $x_1+x_2+x_3 = 1$ to simplify some coefficients. The result is

$$\begin{aligned} N^\mu \;\to\; &2[(1-2x_3-x_3^2)m^2 - (x_3+x_1x_2)q^2]\gamma^\mu \\ &- 2m(x_3-x_3^2)(p'+p)^\mu \\ &- 2m[(x_1+x_1^2)-(x_2+x_2^2)](p'-p)^\mu\;. \end{aligned} \tag{253}$$

In the denominator, set $p^2 = p'^2 = -m^2$ and $p\cdot p' = -\frac{1}{2}q^2-m^2$ to get

$$D \to x_1x_2q^2 + (1-x_3)^2m^2 + x_3\lambda^2\;. \tag{254}$$

Now we note that the right-hand side of eq. (254) is symmetric under $x_1 \leftrightarrow x_2$. Thus the last line of eq. (253) will vanish when we integrate $\overline{u}'N^\mu u/D$ over the Feynman parameters. Finally, we use the Gordon identity from section 38,

$$\overline{u}'(p'+p)^\mu u = \overline{u}'[2m\gamma^\mu + 2iS^{\mu\nu}q_\nu]u \; , \qquad (255)$$

where $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, to get

$$\begin{aligned} N^\mu \;\; &\to \;\; 2[(1-4x_3+x_3^2)m^2 - (x_3+x_1x_2)q^2]\gamma^\mu \\ &\qquad - 4im(x_3-x_3^2)S^{\mu\nu}q_\nu \; . \end{aligned} \qquad (256)$$

So now we have

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^\mu(p',p)u_s(\mathbf{p}) = e\overline{u}'\Big[F_1(q^2)\gamma^\mu + \frac{i}{m}F_2(q^2)S^{\mu\nu}q_\nu\Big]u \; , \qquad (257)$$

where we have defined the *form factors*

$$\begin{aligned} F_1(q^2) \;\; &= \;\; 1 - \frac{e^2}{16\pi^2}\int dF_3 \left[\ln\!\left(1 + \frac{x_1x_2q^2/m^2}{(1-x_3)^2}\right) + \frac{1-4x_3+x_3^2}{(1-x_3)^2 + x_3\lambda^2/m^2} \right. \\ &\qquad \left. + \; \frac{(x_3+x_1x_2)q^2/m^2 - (1-4x_3+x_3^2)}{x_1x_2q^2/m^2 + (1-x_3)^2 + x_3\lambda^2/m^2}\right] + O(e^4) \; , \end{aligned} \qquad (258)$$

$$F_2(q^2) \;\; = \;\; \frac{e^2}{8\pi^2}\int dF_3 \, \frac{x_3-x_3^2}{x_1x_2q^2/m^2 + (1-x_3)^2} + O(e^4) \; . \qquad (259)$$

We have set $\lambda = 0$ in eq. (259), and in the logarithm term in eq. (258), because these terms do not suffer from infrared divergences.

We can simplify $F_2(q^2)$ by using the delta function in dF_3 to do the integral over x_2 (which replaces x_2 with $1-x_3-x_1$), making the change of variable $x_1 = y(1-x_3)$, and performing the integral over x_3 from zero to one; the result is

$$F_2(q^2) = \frac{e^2}{8\pi^2}\int_0^1 \frac{dy}{1 - y(1-y)q^2/m^2} + O(e^4) \; . \qquad (260)$$

This last integral can also be done in closed form, but we will be mostly interested in its value at $q^2 = 0$, corresponding to an on-shell photon:

$$F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2) \; , \qquad (261)$$

where $\alpha = e^2/4\pi = 1/137.036$ is the fine-structure constant. We will explore the physical consequences of eq. (261) in the next section.

Problems

64: The Magnetic Moment of the Electron

Prerequisite: 63

In the last section, we computed the one-loop contribution to the vertex function $\mathbf{V}^\mu(p',p)$ in quantum electrodynamics, where p is the four-momentum of an incoming electron, and p' is the four-momentum of an outgoing electron. We found

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^\mu(p',p)u_s(\mathbf{p}) = e\overline{u}'\Big[F_1(q^2)\gamma^\mu + \frac{i}{m}F_2(q^2)S^{\mu\nu}q_\nu\Big]u \;, \tag{262}$$

where $q = p' - p$ is the four-momentum of the photon (treated as incoming), and with complicated expressions for the *form factors* $F_1(q^2)$ and $F_2(q^2)$. For our purposes in this section, all we will need to know is that

$$\begin{aligned} F_1(0) &= 1 \quad \text{exactly,} \\ F_2(0) &= \frac{\alpha}{2\pi} + O(\alpha^2) \; . \end{aligned} \tag{263}$$

Eq. (262) corresponds to terms in the quantum action of the form

$$\Gamma = \int d^4x \; \Big[eF_1(0)\overline{\Psi}\not{A}\Psi + \frac{1}{2m}F_2(0)F_{\mu\nu}\overline{\Psi}S^{\mu\nu}\Psi + \ldots\Big] , \tag{264}$$

where the ellipses stand for terms with more derivatives. Applied to Γ, the usual procedure for extracting the Feynman rules from an action yield a vertex factor that corresponds to eq. (262) with $q^2 = 0$. To see this, recall that an incoming photon translates into a factor of $A_\mu \sim \varepsilon^*_\mu e^{iqx}$, and therefore of $F_{\mu\nu} \sim i(q_\mu\varepsilon^*_\nu - q_\nu\varepsilon^*_\mu)e^{iqx}$; the two terms in $F_{\mu\nu}$ cancel the extra factor of one half in the second term in eq. (264).

Now we will see what eq. (264) predicts for the *magnetic moment* of the electron. We define the magnetic moment by the following procedure. We take the photon field A^μ be a classical field that corresponds to a constant magnetic field in the z direction: $A^0 = 0$ and $\mathbf{A} = (0, Bx, 0)$; this yields $F_{12} = -F_{21} = B$, with all other components of $F_{\mu\nu}$ vanishing. Then we define a normalized state of an electron at rest, with spin up along the z axis:

$$|e\rangle \equiv \int \widetilde{dp}\, f(\mathbf{p}) b_+^\dagger(\mathbf{p})|0\rangle \;, \tag{265}$$

where the wave packet is rotationally invariant (so that there is no orbital angular momentum) and sharply peaked at $\mathbf{p} = 0$, something like

$$f(\mathbf{p}) \sim \exp(-a^2\mathbf{p}^2/2) \tag{266}$$

with $a \ll 1/m$. We normalize the wave packet by $\int \widetilde{dp}\, |f(\mathbf{p})|^2 = 1$; then we have $\langle e|e\rangle = 1$.

Now we define the interaction hamiltonian as what we get from the two displayed terms in eq. (264), using our specified field A^μ, and with the form-factor values of eq. (263):

$$H_1 \equiv -eB \int d^3x\, \overline{\Psi} \Big[x\gamma^2 + \frac{\alpha}{2\pi m} S^{12} \Big] \Psi \;. \tag{267}$$

Then the electron's magnetic moment μ is specified by

$$\mu B \equiv -\langle e|H_1|e\rangle \;. \tag{268}$$

Eq. (268) is the standard definition of the magnetic moment of a normalized quantum state with angular momentum in the positive z direction.

Now we turn to the computation. We need to evaluate $\langle e|\overline{\Psi}_\alpha(x)\Psi_\beta(x)|e\rangle$. Using the usual plane-wave expansions, we have

$$\langle 0|b_+(\mathbf{p}')\overline{\Psi}_\alpha(x)\Psi_\beta(x)b_+^\dagger(\mathbf{p})|0\rangle = \overline{u}_+(\mathbf{p}')_\alpha u_+(\mathbf{p})_\beta\, e^{i(p-p')x} \;. \tag{269}$$

Thus we get

$$\begin{aligned} \langle e|H_1|e\rangle &= -eB \int \widetilde{dp}\; \widetilde{dp}'\; d^3x\; e^{i(p-p')x} \\ &\qquad \times f^*(\mathbf{p}')\overline{u}_+(\mathbf{p}') \Big[x\gamma^2 + \frac{\alpha}{2\pi m} S^{12} \Big] u_+(\mathbf{p}) f(\mathbf{p}) \;. \end{aligned} \tag{270}$$

We can write the factor of x as $-i\partial_{p_1}$ acting on $e^{i(p-p')x}$, and integrate by parts to put this derivative onto $u_+(\mathbf{p})f(\mathbf{p})$; the wave packets kill any surface terms.

Then we can complete the integral over d^3x to get a factor of $(2\pi)^3\delta^3(\mathbf{p}'-\mathbf{p})$, and do the integral over $\widetilde{dp}'$. The result is

$$\langle e|H_1|e\rangle = -eB\int \frac{\widetilde{dp}}{2\omega}\, f^*(\mathbf{p})\overline{u}_+(\mathbf{p})\Big[\, i\gamma^2\partial_{p_1} + \frac{\alpha}{2\pi m}S^{12}\Big]u_+(\mathbf{p})f(\mathbf{p})\ . \qquad (271)$$

Suppose the ∂_{p_1} acts on $f(\mathbf{p})$. Since $f(\mathbf{p})$ is rotationally invariant, the result is odd in p_1. We then use $\overline{u}_+(\mathbf{p})\gamma^i u_+(\mathbf{p}) = 2p^i$ to conclude that this term is odd in both p_1 and p_2, and hence integrates to zero.

The remaining contribution from the first term has the ∂_{p_1} acting on $u_+(\mathbf{p})$. Recall from section 38 that

$$u_s(\mathbf{p}) = \exp(i\eta\,\hat{\mathbf{p}}\!\cdot\!\mathbf{K})u_s(\mathbf{0})\ , \qquad (272)$$

where $K^j = S^{j0} = \frac{i}{2}\gamma^j\gamma^0$ is the boost matrix, $\hat{\mathbf{p}}$ is a unit vector in the $\mathbf{p}$ direction, and $\eta = \sinh^{-1}(|\mathbf{p}|/m)$ is the rapidity. Since the wave packet is sharply peaked at $\mathbf{p} = 0$, we can expand eq. (272) to linear order in $\mathbf{p}$, take the derivative with respect to p_1, and then set $\mathbf{p} = 0$; the result is

$$\begin{aligned} \partial_{p_1}u_+(\mathbf{p})\Big|_{\mathbf{p}=0} &= \frac{i}{m}K^1u_+(\mathbf{0}) \\ &= -\frac{1}{2m}\,\gamma^1\gamma^0u_+(\mathbf{0}) \\ &= -\frac{1}{2m}\,\gamma^1u_+(\mathbf{0})\ , \end{aligned} \qquad (273)$$

where we used $\gamma^0u_s(\mathbf{0}) = u_s(\mathbf{0})$ to get the last line. Then we have

$$\begin{aligned} \overline{u}_+(\mathbf{p})i\gamma^2\partial_{p_1}u_+(\mathbf{p})\Big|_{\mathbf{p}=0} &= \overline{u}_+(\mathbf{0})\frac{-i}{2m}\gamma^2\gamma^1u_+(\mathbf{0}) \\ &= \frac{1}{m}\,\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0}) \end{aligned} \qquad (274)$$

Plugging this into eq. (271) yields

$$\begin{aligned} \langle e|H_1|e\rangle &= -eB\int \frac{\widetilde{dp}}{2\omega}\,|f(\mathbf{p})|^2\,\Big(1+\frac{\alpha}{2\pi}\Big)\,\frac{1}{m}\,\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0})\ . \\ &= -\frac{eB}{2m^2}\Big(1+\frac{\alpha}{2\pi}\Big)\,\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0})\ . \end{aligned} \qquad (275)$$

Next we use $S^{12}u_\pm(\mathbf{0}) = \pm\frac{1}{2}u_\pm(\mathbf{0})$ and $\overline{u}_\pm(\mathbf{0})u_\pm(\mathbf{0}) = 2m$ to get

$$\langle e|H_1|e\rangle = -\frac{eB}{2m}\Big(1+\frac{\alpha}{2\pi}\Big)\ . \qquad (276)$$

Comparing with eq. (268), we see that the magnetic moment of the electron is

$$\mu = g \, \frac{1}{2} \, \frac{eB}{2m} \; , \tag{277}$$

where $eB/2m$ is the *Bohr magneton*, the extra factor of $1/2$ is for the electron's spin (a classical spinning ball of charge would have a magnetic moment equal to the Bohr magneton times its angular momemtum), and g is the *Landé g factor*, given by

$$g = 2\Big(1 + \tfrac{\alpha}{2\pi} + O(\alpha^2)\Big) \; . \tag{278}$$

Since g can be measured to high precision, calculations of μ provide a stringent test of quantum electrodynamics. Corrections up through the α^4 term have been computed; the result is currently in good agreement with experiment.

Problems