Appendix S

Solutions

Chapter 1. Overture

Solution 1.1: 3-disk symbolic dynamics. *There are 2^k topologically different k-step trajectories starting from each disk, and the 3-disk pinball has $3 \cdot 2^{n-1}$ periodic points with length n itineraries composed of disk labels $\{1, 2, 3\}$.*

As explained in sect. 1.4, each orbit segment can be characterized by either of the two symbols 0 and 1, differentiating topologically bouncing back or going onto the third disk.

Prime cycles in the 3-disk space (prime cycles in fundamental domain, respectively) are

- *Of length 2: 12,13,32 (or 0).*
- *Of length 3: 123,321 (or 1).*
- *Of length 4: 1213,1232,1323 (or 01).*
- *Of length 5: 12123,12132,12313,12323,13132,13232 (or 00111).*

Some of the cycles are listed in table **??** *and drawn in figure 23.2.*

(Yueheng Lan)

Solution 1.1: 3-disk symbolic dynamics. *Starting from a disk we cannot end up at the same disk in the next step, see figure S.1. We have 3 choices for the first disk and 2 choices for the next disk at each step, hence at most $3 \cdot 2^{n-1}$ itineraries of length n*

Thus, it remains to show that any symbol sequence with the only constraint of no two identical consecutive symbols is realized. The most convenient way to do so is to work with the phase space representation of the pinball machine. Parametrize the state right after a reflection by the label of the disk, the arc length parameter corresponding to the point of reflection, and the $\sin\phi$ with ϕ being the angle of reflection relative to the normal vector, see figure S.1. Thus the Poincarśection consists of three cylinders, withthe arc length parameter is cyclic on each disk, as shown in figure S.2.

Consider disk "1" as the starting point. Fixing the angle of reflection, by varying the position all the way around the disk we first escape, then hit disk "3" , then escape,

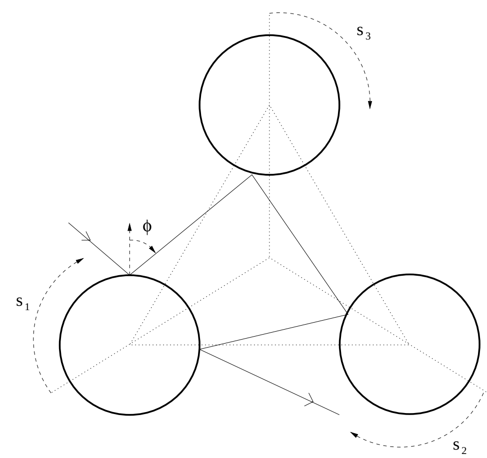

Figure S.1: Geometry of the 3-disk pinball.

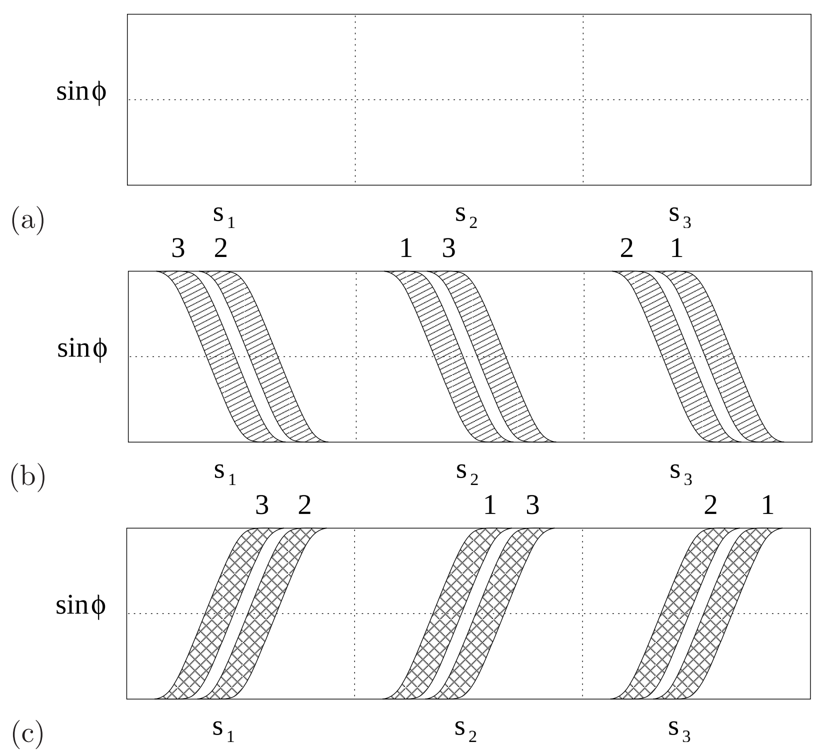

Figure S.2: (a) The phase space of the 3-disk pinball. (b) The part of phase space which remains on the table for one more iterate. (c) The images of the disks in one iteration.

then hit disk "2" , and then escape again, when increasing the arc length parameter in the manner indicated in figure S.1 (a). Thus–if the disks are sufficiently well separated–there are two strips of initial conditions which do not escape. By symmetry this yields figure S.1 (b) where the numbers indicate onto which disk these initial trajectories are going to end up on. By time reversal Figure S.1 (c) shows the strips labeled by disk where the pinball came from.

Combining figure S.1 (b) and (c) we obtain three sections, which are the same except for the labeling of the disks. One of such section is shown in figure S.3.

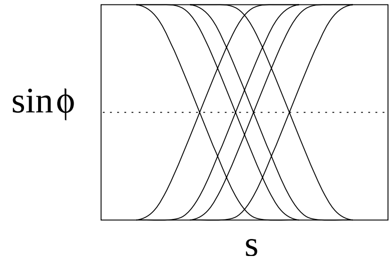

Figure S.3: The intersection of one iterate images and preimages.

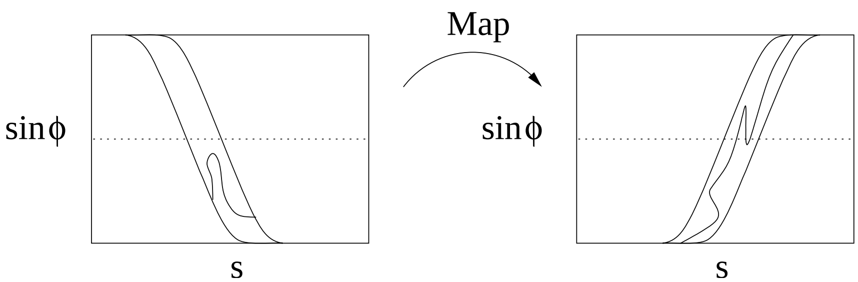

Figure S.4: Monotonicity of the billiard map.

The billiard map enjoys a certain monotonicity, as depicted in figure S.4, which easily verified by inspecting figure S.1. It says that any curve connecting the two boundaries of one of the strips gets mapped to a curve within the image of that strip running all the way across from top to bottom.

This, in particular, means that the intersections of the image of the previous disk and the initial conditions to land onto the next disk, see figure S.3, will map onto (thin) strips running across from to to bottom, as shown in figure S.5.

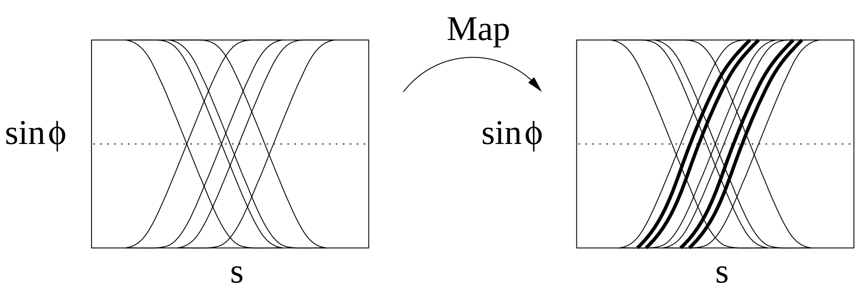

Figure S.5: Images in the second iterate. This is, of course, schematically, because we dropped the labels of the disks; in fact, the two intersection regions get mapped onto two different disks.

Finally, since the images of the intersection regions run all the way across in the vertical direction, we can iterate the argument. Every time the number of strips doubles, and we find regions of states which can go to either of the two neighboring disks at every step. Hence any symbol sequence with no repeat of consecutive symbols can be realized.

Sensitivity!initial conditions

The itineraries of periodic points of period 2, 3, 4, 5 *are*

n	*all periodic cycles*
2	**12 13** *21* **23** *31 32*
3	**123 132** *213 231 312 321*
4	*1212* **1213 1232** *1312 1313* **1323** *2121 2123 2131 2313 2321 2323 3121 3131 3132 3212 3231 3232*
5	**12123 12132** *12312* **12313 12323** *13123* **13132** *13212 13213* **13232** *21213 21231 21313 21321 21323 23121 23123 23131 23213 23231 31212 31231 31232 31312 31321 32121 32131 32132 32312 32321* .

The prime cycles (lexically lowest cycle point itinerary within a non-repeating cycle) are indicated in bold, and the ones given in the exercise are sketched in figure S.6.

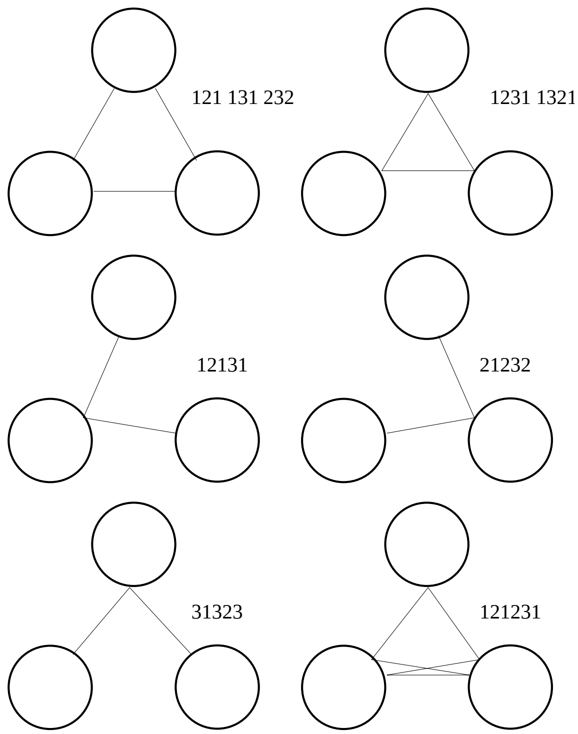

Figure S.6**:** Sketch of the indicated prime cycles.

(Alexander Grigo)

Solution 1.2: Sensitivity to initial conditions. *To estimate the pinball sensitivity we consider a narrow beam of point particles bouncing between two disks, figure S.7 (a). Or if you find this easier to visualize, think of a narrow ray of light. We assume that the ray of light is focused along the axis between the two points. This is where the least unstable periodic orbit lies, so its stability should give us an upper bound on the number of bounces we can expect to achieve. To estimate the stability we assume that the ray of light has a width* $w(t)$ *and a "dispersion angle"* $\theta(t)$ *(we assume both are small), figure S.7 (b). Between bounces the dispersion angle stays constant while the width increases as*

$$w(t) \approx w(t') + (t - t')\theta$$

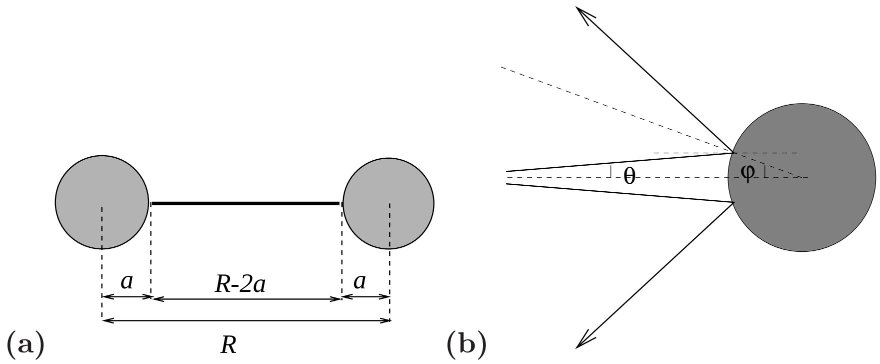

Figure S.7: The 2-disk pinball (a) geometry, (b) defocusing of scattered rays.

At each bounce the width stays constant while the angle increases by

$$\theta_{n+1} = \theta_n + 2\phi \approx \theta_n + w(t)/a.$$

where θ_n denotes the angle after bounce n. Denoting the width of the ray at the nth bounce by w_n then we obtain the pair of coupled equations

$$w_{n+1} = w_n + \left(R - 2a\right)\theta_n \tag{S.1}$$

$$\theta_n = \theta_{n-1} + \frac{w_n}{a} \tag{S.2}$$

where we ignore corrections of order w_n^2 and θ_n^2. Solving for θ_n we find

$$\theta_n = \theta_0 + \frac{1}{a}\sum_{j=1}^{n} w_n.$$

Assuming $\theta_0 = 0$ then

$$w_{n+1} = w_n + \frac{R-2a}{a}\sum_{j=1}^{n} w_n$$

Plugging in the values in the question we find the width at each bounce in Ångstrøms grows as 1, 5, 29, 169, 985, etc. To find the asymptotic behavior for a large number of bounces we try an solution of the form $w_n = ax^n$. Substituting this into the equation above and ignoring terms that do not grow exponentially we find solutions

$$w_n \approx a w_n^{asym} = a(3 \pm 2\sqrt{2})^n$$

The solution with the positive sign will clearly dominate. The constant a we cannot determine by this local analysis although it is clearly proportional to w_0. However, the asymptotic solution is a good approximation even for quite a small number of bounces.

To find an estimate of a we see that w_n/w_n^{asym} very rapidly converges to 0.146447, thus

$$w_n \approx 0.146447 w_0 (3+2\sqrt{2})^n \approx 0.1 \times w_0 \times 5.83^n$$

The outside edges of the ray of light will miss the disk when the width of the ray exceeds 2 cm; this occurs after 11 bounces.

(Adam Prügel-Bennett)

Solution 1.2: Sensitivity to initial conditions, another try. *Adam's estimate is not very good - do you have a better one? The first problem with it is that the instability is very underestimated. As we shall check in exercise 9.3, the exact formula for the 2-cycle stability is $\Lambda = R - 1 + R\sqrt{1-2/R}$. For $R = 6$, $a = 1$ this yields $w_n/w_0 \approx (5+2\sqrt{6})^n = 9.898979^n$, so if that were the whole story, the pinball would be not likely to make it much beyond 8 bounces.*

The second problem is that local instability overestimates the escape rate from an enclosure; trajectories are reinjected by scatterers. In the 3-disk pinball the particle leaving a disk can be reinjected by hitting either of other 2 disks, hence $w_n/w_0 \approx (9.9/2)^n$. This interplay between local instability and global reinjection will be cast into the exact formula (Q.1) involving "Lyapunov exponent" and "Kolmogorov entropy".
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In order to relate this estimate to our best continuous time escape rate estimate $\gamma = 0.4103\ldots$ (see table 18.2), we will have to also compute the mean free flight time (18.21). As a crude estimate, we take the shortest disk-to-disk distance, $\langle T\rangle = R-2 = 4$. The continuous time escape rate result implies that $w_n/w_0 \approx e^{(R-2)\gamma n} = (5.16)^n$, in the same ballpark as the above expansion-reinjection estimate.

(Predrag Cvitanović)