How well can one resolve the state space of a chaotic flow?

Predrag Cvitanović and Domenico Lippolis

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Outline

1. **Dynamicist's view of noise**
   - intuition
   - idea #1: partition by periodic points
   - strategy
2. **Fokker-Planck evolution**
   - idea #2: evolve densities, not Langevin trajectories
   - idea #3: for unstable directions, look back
3. **optimal partition hypothesis**
   - idea #4: finite-dimensional Fokker-Planck matrices
4. **conclusions, open questions**
   - literature

dynamical theory of turbulence?

dynamics of high-dimensional flows - open questions

is the dynamics like what we know from low dimensional systems?

describe the attracting 'inertial manifold' for Navier-Stokes?

Ruelle / Gutzwiller periodic orbit theory?

deterministic chaos vs. noise

any physical system:

noise limits the resolution that can be attained in partitioning

noisy orbits

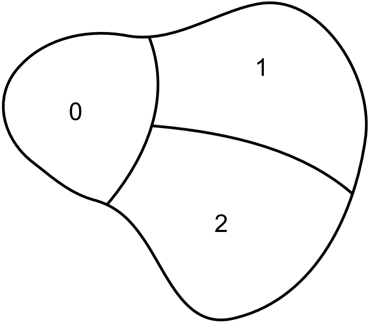
probabilistic densities smeared out by the noise:
a finite # fits into the attractor

goal: determine

the **finest attainable** partition

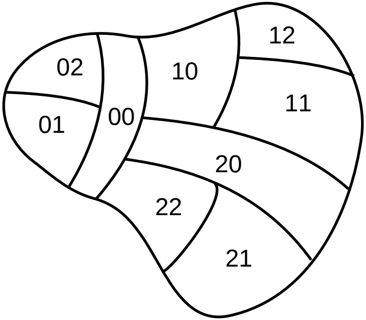
deterministic partition

state space coarse partition



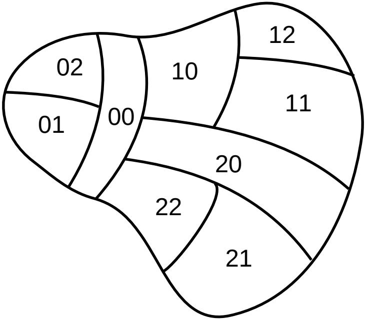
$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$
ternary alphabet
$\mathcal{A} = \{1, 2, 3\}$.

1-step memory refinement

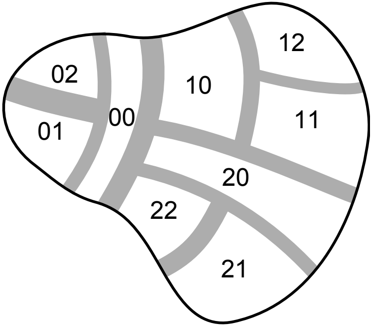


$\mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2}$
labeled by nine 'words'
$\{00, 01, 02, \cdots, 21, 22\}$.

deterministic vs. noisy partitions



deterministic partition



noise blurs the boundaries

can be refined
ad infinitum

when overlapping, no further refinement of partition

periodic points instead of boundaries

- mhm, do not know how to compute boundaries...
- however, each partition contains a short periodic point smeared into a ‘cigar’ by noise

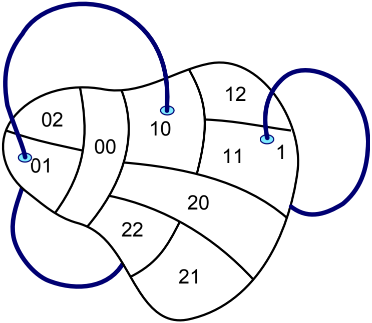
periodic points instead of boundaries

- each partition contains a short periodic point smeared into a 'cigar' by noise

compute the size of a noisy periodic point neighborhood!

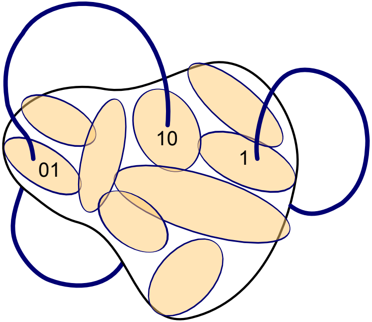
periodic orbit partition

deterministic partition



some short periodic points:
fixed point $\overline{1} = \{x_1\}$
two-cycle $\overline{01} = \{x_{01}, x_{10}\}$

noisy partition



periodic points blurred by the Langevin noise into cigar-shaped densities

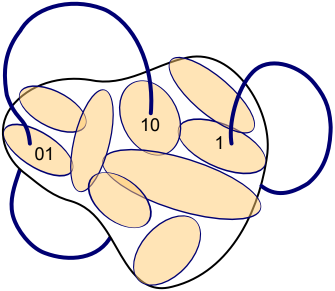
- successive refinements of a deterministic partition: exponentially shrinking neighborhoods
- as the periods of periodic orbits increase, the diffusion always wins:

partition stops at the finest attainable partition, beyond which the diffusive smearing exceeds the size of any deterministic subpartition.

- the local diffusion rate differs from a trajectory to a trajectory, as different neighborhoods merge at different times, so

there is *no one single time* beyond which noise takes over

noisy periodic orbit partition



optimal partition hypothesis

optimal partition:
the maximal set of resolvable periodic point neighborhoods

why care?

if the high-dimensional flow has only a few unstable directions, the overlapping stochastic 'cigars' provide a *finite cover* of the noisy chaotic attractor, embedded in a state space of arbitrarily high dimension

strategy

- use periodic orbits to partition state space
- compute local eigenfunctions of the Fokker-Planck operator to determine their neighborhoods
- done once neighborhoods overlap

roll your own cigar

next, derive **the** (well known) **key formula**

composition law for the covariance matrix Q_a

of a linearly evolved Gaussian density,

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

density covariance matrix at time a: Q_a
Langevin noise covariance matrix: Δ_a
Jacobian matrix of linearized flow: M_a

derivation

deterministic velocity field (or 'drift'): $v(x)$
additive noise, uncorrelated in time: $\hat{\xi}(t)$

d-dimensional stochastic, Langevin flow

$$\frac{dx}{dt} = v(x) + \hat{\xi}(t)\,,$$

derivation

d-dimensional stochastic, Langevin flow

$$\frac{dx}{dt} = v(x) + \hat{\xi}(t)\,,$$

keep things simple: illustrate by

d-dimensional *discrete time* stochastic flow

$$x' = f(x) + \xi_a$$

uncorrelated in time

$$\langle \xi_a \rangle = 0\,, \qquad \langle \xi_a \cdot \xi_b \rangle = 2\,d\,D\,\delta_{ab}$$

[all results apply both to the continuous and discrete time flows]

standard normal (Gaussian) probability distribution

d-dimensional *discrete time* stochastic flow

$$x' = f(x) + \xi_a$$

1-time step evolution = probability of reaching x' given Gaussian distributed $\xi_a = x' - f(x)$

$$\frac{1}{\sqrt{4\pi D}} \exp\left(-\frac{\xi_a^2}{4D}\right)$$

variance $2D$, standard deviation $\sqrt{2D}$

local Fokker-Planck operator

let

$$\{\dots, x_{-1}, x_0, x_1, x_2, \dots\}$$

be a deterministic trajectory

$$x_{a+1} = f(x_a)$$

noisy trajectory is centered on the deterministic trajectory

$$x = x_a + z_a \,, \qquad f_a(z_a) = f(x_a + z_a) - x_{a+1}$$

local Fokker-Planck operator:

$$\mathcal{L}_a(z_{a+1}, z_a) = \frac{1}{\sqrt{4\pi D}} \exp\left[-\frac{(z_{a+1} - f_a(z_a))^2}{4D}\right]$$

Fokker-Planck formulation replaces individual noisy trajectories by evolution of their densities

$$\mathcal{L}^k(z_k, z_0) = \int [dx]\, e^{-\frac{1}{2}\sum_a (z_{a+1} - f_a(z_a))^T \frac{1}{\Delta}(z_{a+1} - f_a(z_a))}$$

evolution to time k is given by the d-dimensional path integral over the $k-1$ intermediate noisy trajectory points

$$\mathcal{L}^k(z_k, z_0) = \int [dx]\, e^{-\frac{1}{2}\sum_a (z_{a+1}-f_a(z_a))^T \frac{1}{\Delta}(z_{a+1}-f_a(z_a))}$$

zero mean and covariance matrix (diffusion tensor)

$$\left\langle \xi_j(t_a) \right\rangle = 0\,, \qquad \left\langle \xi_{a,i}\, \xi_{a,j}^T \right\rangle = \Delta_{ij}\,,$$

where $\langle \cdots \rangle$ stands for ensemble average over many realizations of the noise map $f(x_a)$ is nonlinear. Taylor expand

$$f_a(z_a) = M_a z_a + \cdots$$

approximate the noisy map by its linearized action,

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approximate the noisy map by its linearized action,

$$z_{a+1} = M_a z_a + \xi_a \,,$$

where M_a is the Jacobian matrix, $(M_a)_{ij} = \partial f(x_a)_i / \partial x_j$

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linearized Fokker-Planck operator

$$\mathcal{L}_a(z_{a+1}, z_a) = \frac{1}{N} \, e^{-\frac{1}{2}(z_{a+1} - M_a z_a)^T \frac{1}{\Delta} (z_{a+1} - M_a z_a)}$$

[WKB, semiclassical, saddlepoint, ... approximation] linearized evolution operator maps the cigar-shaped Gaussian density distribution with covariance matrix Q_a

$$\rho_a(z_a) = \frac{1}{C_a} e^{-\frac{1}{2} z_a^T \frac{1}{Q_a} z_a}$$

into cigar $\rho_{a+1}(z_{a+1})$ one time step later

convolution of a Gaussian with a Gaussian is again a Gaussian

rolled your own cigar

convolution of a Gaussian with a Gaussian is again a Gaussian: the covariance of the transported packet is given by

composition law for the covariance matrix Q_a

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

rolled your own cigar

composition law for the covariance matrix Q_a

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

in one time step a Gaussian density distribution with covariance matrix Q_a is smeared into a Gaussian 'cigar' whose widths and orientation are given by the eigenvalues and eigenvectors of Q_{a+1}

rolled your own cigar

composition law for the covariance matrix Q_a

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

(1) deterministically transported and deformed local density covariance matrix $Q \to MQM^T$, and

(2) and noise covariance matrix Δ

add up as sums of squares

This covariance is an interplay of the Brownian noise and the deterministic nonlinear contraction/amplification

local noise is *never* uniform over the state space

The diffusive dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow induces a history dependent effective noise.

the variance Q_a is built up from the deterministically propagated $M_a^n Q_{a-n} M_a^{nT}$ initial distribution, and the sum of noise kicks at intervening times, $M_a^k \Delta_{a-k} M_a^{kT}$, also propagated deterministically.

If M is contracting, over time the memory of the covariance Q_{a-n} of the starting density is lost, with iteration leading to the limit distribution

$$Q_a = \Delta_a + M_{a-1} \Delta_{a-1} M_{a-1}^T + M_{a-2}^2 \Delta_{a-2} (M_{a-2}^2)^T + \cdots .$$

If all eigenvalues of M are strictly contracting, any initial compact measure converges to the unique invariant Gaussian measure $\rho_0(z)$ whose covariance matrix satisfies the fixed point condition

$$Q = MQM^T + \Delta$$

The eigenvectors of Q are orthogonal and have orientations distinct from the left/right eigenvectors of the non-normal Jacobian matrix M.

Ornstein-Uhlenbeck process

contracting noisy 1-dimensional map

$$z_{n+1} = \Lambda z_n + \xi_n \,, \qquad |\Lambda| < 1$$

width of the natural measure concentrated at the deterministic fixed point $z = 0$

$$Q = \frac{2D}{1 - |\Lambda|^2} \,, \qquad \rho_0(z) = \frac{1}{\sqrt{2\pi\, Q}} \exp\left(-\frac{z^2}{2\, Q}\right) \,,$$

- is balance between contraction by Λ and diffusive smearing by $2D$ at each time step

Ornstein-Uhlenbeck process

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- for strongly contracting Λ, the width is due to the noise only
- As $|\Lambda| \to 1$ the width diverges: the trajectories are no longer confined, but diffuse by Brownian motion

things fall apart, centre cannot hold

but what if M has *expanding* Floquet multipliers?
both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future

Fokker-Planck operator is non-selfadjoint

If right eigenvector is peaked (attracting fixed point)
the left eigenvector is flat (probability conservation)

adjoint Fokker-Planck operator

to estimate the size of a noisy neighborhood of a trajectory point x_a along its *unstable* directions, we need to determine the effect of noise on the points *preceding* x_a

this is described by the *adjoint Fokker-Planck operator*

$$\begin{aligned} \tilde{\rho}(y, k-1) &= \mathcal{L}^\dagger \circ \tilde{\rho}(y, k) \\ &= \int [dy] \, \exp\left\{ -\frac{1}{2}(y - f(x))^T \frac{1}{\Delta}(y - f(x)) \right\} \tilde{\rho}(y, k) \,, \end{aligned}$$

carries a density concentrated around the previous point x_{n-1} to a density concentrated around x_n.

case of *repelling* fixed point

if M has only *expanding* Floquet multipliers, both deterministic dynamics and noise tend to smear densities away from the fixed point

balance between the two is described by the *adjoint Fokker-Planck operator*.
The evolution of the covariance matrix Q is now given by

$$Q_a + \Delta = M_a Q_{a+1} M_a^T \,,$$

optimal partition challenge

finally in position to address our challenge:

Determine the finest possible partition for a given noise.

does it work?

evaluation of these Gaussian densities requires no Fokker-Planck PDE formalism

width of a Gaussian packet centered on a trajectory is fully specified by a deterministic computation that is already a pre-computed byproduct of the periodic orbit computations: the deterministic orbit and its linear stability

resolution of a one-dimensional chaotic repeller

As an illustration of the method, consider the chaotic repeller on the unit interval

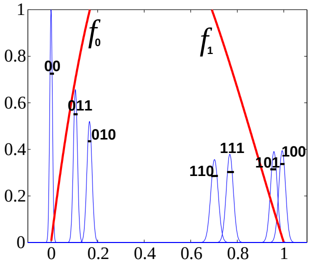
$$x_{n+1} = \Lambda_0 \, x_n(1 - x_n)(1 - b x_n) + \xi_n \,, \quad \Lambda_0 = 8, \; b = 0.6 \,,$$

with noise strength $2D = 0.002$.

optimal partition, 1 dimensional map

f_0, f_1: branches of deterministic map

a deterministic orbit itinerary is given by the $\{f_0, f_1\}$ branches visitation sequence



[symbolic dynamics, however, is not a prerequisite for implementing the method]

'the best possible of all partitions' hypothesis formulated as an algorithm

- calculate the local adjoint Fokker-Planck operator eigenfunction width Q_a for every unstable periodic point x_a
- assigned one-standard deviation neighborhood $[x_a - Q_a, x_a + Q_a]$ to every unstable periodic point x_a
- cover the state space with neighborhoods of orbit points of higher and higher period n_p
- stop refining the local resolution whenever the adjacent neighborhoods of x_a and x_b overlap:

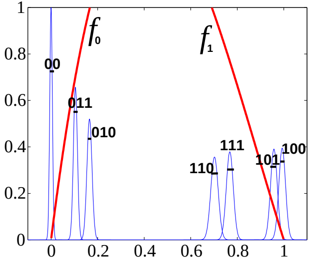
$$|x_a - x_b| < Q_a + Q_b$$

optimal partition, 1 dimensional map

f_0, f_1: branches of deterministic map

The local eigenfunctions $\tilde{\rho}_a$ partition state space by neighborhoods of periodic points of period 3.

The neighborhoods $\mathcal{M}_{000}$ and $\mathcal{M}_{001}$ overlap, so $\mathcal{M}_{00}$ cannot be resolved further.



all neighborhoods $\{\mathcal{M}_{0101}, \mathcal{M}_{0100}, \cdots\}$ of period $n_p = 4$ cycle points overlap, so

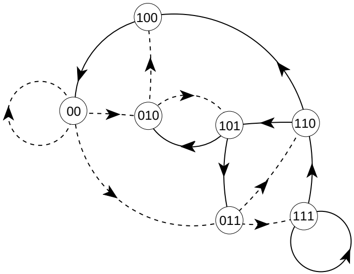
state space can be resolved into 7 neighborhoods

$$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \mathcal{M}_{110}, \mathcal{M}_{111}, \mathcal{M}_{101}, \mathcal{M}_{100}\}$$

Markov partition

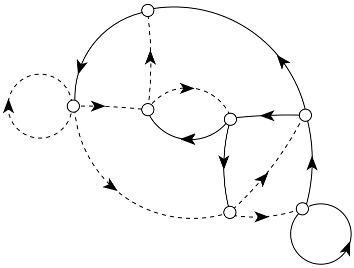
evolution in time maps intervals
$\mathcal{M}_{011} \to \{\mathcal{M}_{110}, \mathcal{M}_{111}\}$
$\mathcal{M}_{00} \to \{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}\}$, etc..

summarized by the transition graph (links correspond to elements of transition matrix T_{ba}): the regions b that can be reached from the region a in one time step



transition graph

7 nodes = 7 regions of the optimal partition
dotted links = symbol 0 (next region reached by f_0)
full links = symbol 1 (next region reached by f_1)



region labels in the nodes can be omitted, with links keeping track of the symbolic dynamics.

(1) deterministic dynamics is full binary shift, but

(2) noise dynamics nontrivial and *finite*

predictions

escape rate and the Lyapunov exponent of the repeller

are given by the leading eigenvalue of this $[7 \times 7]$ graph / transition matrix

numerical results are consistent with the full Fokker-Planck PDE simulations

what is novel?

- we have shown how to compute the **locally optimal partition**, for a given dynamical system and given noise, in terms of local eigenfunctions of the forward-backward actions of the Fokker-Planck operator and its adjoint
- **A handsome reward:** as the optimal partition is always finite, the dynamics on this 'best possible of all partitions' is encoded by a finite transition graph of finite memory, and the Fokker-Planck operator can be represented by a finite matrix

the payback

claim:

optimal partition hypothesis

- the best of all possible state space partitions
- optimal for the given noise

the payback

claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices
- finite matrix calculations, finite cycle expansions $\Rightarrow$ optimal estimates of long-time observables (escape rates, Lyapunov exponents, etc.)

questions

- how to combine Fokker-Planck and adjoint Fokker-Planck operators to describe hyperbolic periodic points (saddles)
  Hint: H. H. Rugh (1992)? combined deterministic evolution

  operator and adjoint operators to describe hyperbolic periodic points (saddles)

questions

- apply to Navier-Stokes turbulence

  computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes, is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating orbits of a given hyperbolic flow?

references

- D. Lippolis and P. Cvitanović, *How well can one resolve the state space of a chaotic map?*, Phys. Rev. Lett. 104, 014101 (2010); arXiv.org:0902.4269
- D. Lippolis and P. Cvitanović, *Optimal resolution of the state space of a chaotic flow in presence of noise* (in preparation)

brief history of noise

The literature on stochastic dynamical systems is vast, starts with the Laplace 1810 memoir.

all of this literature assumes uniform / bounded hyperbolicity and seeks to define a single, globally averaged diffusion induced average resolution (Heisenberg time, in the context of semi-classical quantization).

brief history of noise
cost function

appears to have been first introduced by Wiener as the exact solution for a purely diffusive Wiener-Lévy process in one dimension.
Onsager and Machlup use it in their variational principle to study thermodynamic fluctuations in a neighborhood of single, linearly attractive equilibrium point (i.e., without any dynamics).

brief history of noise

The dynamical 'action' Lagrangian, and symplectic noise Hamiltonian were first written down by Freidlin and Wentzell (1970's), whose formulation of the 'large deviation principle' was inspired by the Feynman quantum path integral (1940's). Feynman, in turn, followed Dirac (1933's) who was the first to discover that in the short-time limit the quantum propagator (imaginary time, quantum sibling of the Wiener stochastic distribution) is exact. Gaspard: 'pseudo-energy of the Onsager-Machlup-Freidlin-Wentzell scheme.' Roncadelli: the 'Wiener-Onsager-Machlup Lagrangian.'