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On the mode-locking universality for critical circle maps

Predrag Cvitanović†§, Gemunu H Gunaratne‡ and Michael J Vinson‡
†Niels Bohr Institute, Blegdamsveg 17, DK-2100 Copenhagen Ø, Denmark
‡James Franck Institute, The University of Chicago, 5640 South Ellis Avenue, Chicago IL 60637, USA

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Abstract. The conjectured universality of the Hausdorff dimension of the fractal set formed by the set of the irrational winding parameter values for critical circle maps is shown to follow from the universal scalings for quadratic irrational winding numbers.

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1. Introduction

One of the most common and experimentally well explored routes to chaos is the two-frequency mode-locking route. A typical example is a dynamical system which possesses a natural frequency ω_1 and is in addition driven by an external frequency ω_2; as the ratio ω_1/ω_2 is varied, the system sweeps through infinitely many mode-locked states. If the mode-locked states overlap, chaos sets in. Both quantitatively and qualitatively this behaviour is often well described by one-dimensional circle maps $f(x+1)=f(x)+1$ restricted to the circle, such as the sine map

$$x_{n+1}=x_n+\Omega-\frac{k}{2\pi}\sin(2\pi x_n) \quad \bmod 1. \tag{1}$$

Here k parametrizes the strength of the mode–mode interaction, and Ω parametrises the ω_1/ω_2 frequency ratio. For $k=0$, the map is a simple rotation, and Ω is just the winding number $W(k,\Omega)=\lim_{n\to\infty} x_n/n$. For $0\leqslant k<1$, the map is invertible. Circle maps with zero slope at the inflection point ($k=1$ in (1)) are called critical: they delineate the borderline of chaos in this scenario. As Ω is varied from 0 to 1, the iterates of a circle map either mode-lock, with the winding number given by a rational number $P/Q \in (0,1)$, or wind irrationally. The complement of the set of parameter values Ω for which the map mode-locks is called the set of irrational

§ Carlsberg fellow.

windings. The measure of mode-locked intervals increases as k increases and at the critical line (here $k = 1$) the intervals fill up the entire Ω axis†, squeezing the set of irrational windings into a Cantor set of zero measure.

Jensen *et al* [1] have estimated that the Hausdorff dimension of this set is $D_H = 0.870\ldots$. What is more remarkable is that their numerical work indicates that this dimension is *universal* for all critical circle maps with the same order of the inflection point (for the generic, physically relevant case the inflection is cubic). This universality, which applies to the entire critical line in the parameter space, at first glance appears to be very different from the earlier examples of universality in transitions to chaos [4–6], all of which describe only infinitesimal regions in the parameter space.

The main point of this paper is that the universality of this Hausdorff dimension actually follows from the Shenker *et al* [4–6] universality. We establish this by relating D_H to the circle map renormalisation transformation operators; more precisely, to the universal scaling numbers for quadratic irrationals. Such relations have been proposed before [7, 8], but their implementation requires both renormalisation group computations of many quadratic irrational scaling numbers and careful control of convergence, undertaken for the first time here.

Our construction and the numerical results support the hyperbolicity conjecture advocated (in various guises) in [9–14]. We derive an elegant formula for the Jensen *et al* dimension, equation (14), that is of interest in itself, both as an insight into the dynamical systems theory, and in the light of the fact that the universality of this dimension has not yet been rigorously established. So far several mechanisms have been proposed. The first class of proposals assumes that the infinitesimal neighbourhood of the golden mean winding number can be mapped by a smooth transformation onto the entire critical line; as the unstable manifold of the golden mean is universal, so should be the 'thermodynamic' averages over the associated Cantor set [10, 12]. This amounts to picking out from the class of all circle maps with cubic inflection a circle map which happens to be universal; it offers no insight into the origin of this dimension, and it is more cumbersome for computational purposes than the model map (1). In the second class of proposals the ergodicity of the Gauss shifts is generalised to a renormalisation operation on the space of critical circle maps and the action of the renormalisation group is conjectured to be hyperbolic [13]. It is not clear how any numbers are to be extracted out of this scheme. The third proposal uses the Farey-tree scaling functions [9]‡ which, however, are marred by non-universal harmonic tails [9, 14]. The Farey tree renormalisation formulation [10, 11] belongs to the first category above, as it resolves this problem by renormalising onto the golden mean unstable manifold. Furthermore, all of the known formulations exhibit 'phase transitions' [11, 15] which make establishing convergence a delicate problem.

The paper is organised as follows. In section 2 we review the renormalisation group prerequisites to our calculation. In section 3 we state the formula for D_H, and describe its evaluation in section 4. We conclude with several remarks on the thermodynamic averages and the associated phase transitions. The calculation of the universal scaling numbers is described in the appendix.

† For the numerical evidence, see [1, 2]. The proof that the set of irrational windings is of zero Lebesgue measure is given in [3].

‡ The scaling function formalism of [9] is superseded by the cycle expansions discussed in the present paper.

2. Scaling along the critical line

The set of the irrational winding parameter values is formed by excluding from the critical parameter line the mode-locked intervals $\Delta_{P/Q}$ for all rationals P/Q. A finite cover of this set at the 'nth level of resolution' is obtained by selecting a subset $\mathscr{S}_n = \{i\}$ of rational winding numbers P_i/Q_i and deleting the corresponding mode-locked parameter values. This leaves behind a set of complement intervals of widths

$$l_i = \Omega^{\min}_{P_r/Q_r} - \Omega^{\max}_{P_l/Q_l} \tag{2}$$

which provide a finite cover for the irrational winding set. Here $\Omega^{\min}_{P_r/Q_r}$ ($\Omega^{\max}_{P_l/Q_l}$) are, respectively, the lower (upper) edges of the mode-locking intervals Δ_{P_r/Q_r} (Δ_{P_l/Q_l}) bounding l_i, and i is a symbolic dynamics label; for our purposes a natural labelling choice are the entries of the continued fraction representation $P/Q = [a_1, a_2, \ldots, a_n]$ of one of the boundary mode-lockings, $i = a_1a_2 \ldots a_n$. (The other commonly used labelling, the binary Farey labelling [9–11, 14], is equivalent: $a_i - 1$ is the number of consecutive zeros in a binary Farey label.)

One possible hierarchy of finite covers $\mathscr{S}_n$ is given by the continued fraction partitioning. In this partitioning of rationals the $\mathscr{S}_1 = \{a_1\}$ level is obtained by deleting from the critical line all mode-lockings whose continued fraction expansion is of length 1; their complement are the covering intervals $l_1, l_2, \ldots, l_{a_1}, \ldots$ which contain all windings, rational and irrational, whose continued fraction expansion starts with $[a_1, \ldots]$ and is of length at least 2. The $\mathscr{S}_2 = \{a_1a_2\}$ level is obtained by deleting from each l_{a_1} interval all mode-lockings with winding number of the form $P/Q = [a_1, a_2 + 1]$, and so forth†. The object of interest, the set of the irrational winding parameter values, is in this partitioning given by $\mathscr{S}_\infty = \{a_1a_2a_3 \ldots\}$, $a_k \in \mathbb{Z}^+$, i.e. the set of rotation numbers with infinite continued fraction expansions. The continued fraction labelling is convenient because of its close connection to the renormalisation transformations; the Gauss map

$$f(x) = \begin{cases} \dfrac{1}{x} - \left[\dfrac{1}{x}\right] & x \neq 0 \\ 0 & x = 0 \end{cases} \tag{3}$$

([ ] denotes the integer part) acts as a shift on the continued fraction representation of numbers on the unit interval

$$x = [a_1, a_2, a_3, \ldots] \to f(x) = [a_2, a_3, \ldots] \tag{4}$$

and maps 'daughter' intervals $d = a_1a_2a_3 \ldots$ into the 'mother' interval $m = a_2a_3 \ldots$. Associated with this shift is a renormalisation transformation R^*, reviewed in the appendix. We shall concentrate here only on the parameter-scaling aspect of R^*, the scaling function [9–11, 15]

$$\sigma_d = l_d / l_m \tag{5}$$

or, more generally, a transfer operator T multiplicative along a trajectory generated by the Gauss shifts (4). The transfer operator appropriate to the evaluation of the Hausdorff dimension that we shall use here is

$$T_{dm} = \omega_d \, |\sigma_d|^{-\tau}. \tag{6}$$

† See, for example, figure 13 of [23] for a sketch of continued fraction partitioning cover of the Ω parameter axis.

d and m indices run over all irrational numbers $\mathcal{S}_\infty = \{a_1 a_2 a_3 \ldots\}$, so T_{dm} is a linear operator that acts on an infinite-dimensional vector space. Depending on how the l_i edges are defined in (2), σ_d might be non-positive; in the present application we shall need only the absolute value of σ_d. The exponent τ will soon be related to the Hausdorff dimension, and for the time being we set $\omega_i = 1$. The main reason for introducing transfer operators is the observation that, unlike the interval lengths l_i which shrink exponentially with the level of resolution, the transfer matrix entries T_{dm} tend to finite limits [9–11]. The simplest example is the sequence of scaling ratios (5) with $\sigma^{(k)}_{111\ldots1} = l_{111\ldots1}/l_{11\ldots1}$, corresponding to the continued fraction approximants to the golden mean winding number $W_1 = [1, 1, 1, \ldots] = (\sqrt{5} - 1)/2$. It is the fundamental result of the renormalization theory [5, 6] for critical circle maps that these ratios converge to the universal limit $\sigma_1 = \lim_{k\to\infty} \sigma^{(k)}_{111\ldots1} = 1/\delta_1$; for critical circle maps with a cubic inflection point, $\delta_1 = -2.833\,61\ldots$. More generally, if the winding number is a quadratic irrational W_p whose continued fraction expansion has the form of an infinitely repeated block $p = a_1 a_2 \ldots a_{n_p}$

$$W_p = [a_1, a_2, \ldots, a_{n_p}, a_1, a_2, \ldots, a_{n_p}, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots \cfrac{1}{a_{n_p} + W_p}}} \tag{7}$$

then the product of scaling functions along n_p Gauss shifts (4)

$$\sigma^{(k)}_{a_1 a_2 \ldots a_{n_p} a_1 \ldots} \sigma^{(k-1)}_{a_2 \ldots a_{n_p} a_1 a_2 \ldots} \cdots \sigma^{(k-n_p+1)}_{a_{n_p} a_1 \ldots} = l_{a_1 a_2 \ldots a_{n_p} a_1 \ldots} / l_{a_1 \ldots}$$

converges to a universal limit

$$\sigma_p = \lim_{k\to\infty} \sigma^{(k)}_{a_1 a_2 \ldots a_{n_p} a_1 \ldots} \sigma^{(k-1)}_{a_2 \ldots a_{n_p} a_1 a_2 \ldots} \cdots \sigma^{(k-n_p+1)}_{a_{n_p} a_1 \ldots} = 1/\delta_p. \tag{8}$$

If the block repeats, the associated δ factorizes (for example $\delta_{2323} = \delta_{23}^2$), so we always take $p = a_1 a_2 \ldots a_{n_p}$ to be primitive (a non-repeating continued fraction block). As δ_p is a product (8) of σ_i along a cycle, it is cyclically symmetric (for example, $\delta_{137} = \delta_{371} = \delta_{713}$), so only one δ_p per cycle needs to be computed.

We shall not need the full transfer operator T here; our calculation will rely only on the following aspects of the 'hyperbolicity conjecture' of [9–11, 13, 14].

(i) *Limits* for periodic products (8) *exist* and are universal. This should follow from the renormalization theory developed in [5, 6], though a general proof is still lacking.

(ii) δ_p grow *exponentially* with n_p, the length of the continued fraction block p.

(iii) δ_p for $p = a_1 a_2 \ldots n$ with a large continued fraction entry n grows as a *power* of n. According to [1, 9, 16]

$$\lim_{n\to\infty} \delta_p \propto n^3. \tag{9}$$

This follows from methods akin to those used in describing intermittency [17] and could perhaps be turned into a systematic asymptotic expansion. However, in the present calculation we shall not use explicit values of the asymptotic exponents and prefactors, only the assumption that the growth of δ_p with n is not slower than a power of n.

The δ_p for quadratic irrationals with short repeated blocks p and small continued fraction entries are easily estimated. Consider the rth rational approximations to a quadratic irrational winding number W_p whose continued fraction expansion consists of r repeats of block p. Let Ω_r be the parameter for which the map (1) has a superstable cycle of rotation number $P_r/Q_r = [p, p, \ldots, p]$. The δ_p is found from [4]

$$\Omega_r - \Omega_{r+1} \propto \delta_p^{-r}. \tag{10}$$

When the repeated block is not large, the rate of increase of denominators Q_r is not large, and (10) is a viable scheme for estimating the δ values. However, for long repeating blocks, the rapid increase of the Q_r makes the Newton method hard to implement and leads to a loss of numerical accuracy. Instead, we extract δ values from the unstable manifold of the corresponding renormalization group fixed point. The details of this calculation are given in the appendix; the resulting list of values of the δ_p used in the evaluation of D_H in this paper is presented in table 1.

3. The Hausdorff dimension in terms of cycles

Given the above information, the thermodynamic formalism [18, 19] can now be used to relate the Hausdorff dimension of irrational windings to the periodic circle map renormalizations. Consider the sum

$$\Gamma_n(\tau) = \sum_{i \in \mathcal{S}_n} |l_i|^{-\tau}. \tag{11}$$

In the thermodynamic formalism the Hausdorff dimension D_H is given [20] by the $n \to \infty$ limit of the values $\tau = -D_n$ for which $\Gamma_n(-D_n) = O(1)$. Strictly speaking, D_n extracted from (11) converge to D_H only if the cover $\{l_i, i \in \mathcal{S}_n\}$ is *optimal*; if the covers are chosen too large, D_∞ provides only an upper bound to D_H. The covering intervals l_i defined by (2) are expected to be optimal, as they cover the irrational winding parameter values with no slack—they exactly fit the separation between the edges of pairs of neighbouring mode-locking intervals. The available numerical evidence supports this claim, but we have not proven that D_H is indeed attained. In any case, even establishing that our formulation yields a rigorous upper bound on D_H would already be an interesting result from a purely mathematical point of view.

Quantities associated with a given interval, such as $|l_{a_1a_2a_3\ldots}|^{-\tau}$, are recovered from transfer operators in terms of matrix products

$$|l_{a_1a_2a_3\ldots}|^{-\tau} = T_{a_1a_2\ldots,a_2a_3\ldots} T_{a_2a_3\ldots,a_3a_4\ldots} \cdots T_{a_{n+1}a_n\ldots,a_na_{n-1}\ldots} |l_{a_na_{n-1}\ldots}|^{-\tau}. \tag{12}$$

By the usual thermodynamic arguments [18], the sum (11) can be related to products of transfer operators $\Gamma_n \propto T^n$ and is dominated by its leading eigenvalue

$$\Gamma_n(\tau) \propto \lambda^n(\tau).$$

The leading eigenvalue λ of T is given by the leading zero of $\det(1 - zT)$. This determinant can be expressed in terms of the traces $\operatorname{tr} T^n$ and written as the Euler product over periodic orbits in the standard way [18, 22]

$$\det(1 - zT) = 1/\zeta(z) = \prod_p (1 - t_p). \tag{13}$$

Table 1. A list of the values of the δ for short cycles, used in the numerical estimates of D_H in this paper. δ_p are numerically stable under the renormalization 12-term truncation transformations of the appendix to the digits quoted. δ_1–δ_8 from [32].

$\hat{t}_p$	p	δ_p
$\hat{t}_1$	1	$-2.833\,610\,6560 \pm 10^{-10}$
	2	$-6.799\,225\,161 \pm 2\times 10^{-9}$
	3	$-13.760\,282\,37 \pm 3\times 10^{-8}$
	4	$-24.620\,347\,98 \pm 7\times 10^{-8}$
	5	$-40.386\,913 \pm 2\times 10^{-6}$
	6	$-62.140\,406 \pm 2\times 10^{-6}$
	7	$-90.995\,97 \pm 10^{-5}$
	8	$-128.080\,11 \pm 5\times 10^{-5}$
	9	$-174.5198 \pm 2\times 10^{-4}$
	10	$-231.439 \pm 2\times 10^{-3}$
$\hat{t}_{12}$	1 2	17.669
	1 3	31.621
	1 4	50.809
	1 5	76.012
	1 6	108.06
	1 7	147.91
	1 8	196.44
$\hat{t}_{112}$	1 1 2	−52.044
	1 1 3	−98.324
	1 1 4	−165.89
	1 1 5	−259.18
$\hat{t}_{23}$	2 3	91.290
	2 4	157.08
	2 5	246.75
	2 6	365.40
$\hat{t}_{122}$	1 2 2	−122.76
	1 2 3	−234.41
	1 2 4	−398.49
$\hat{t}_{1112}$	1 1 1 2	145.42
	1 1 1 3	269.08
	1 1 1 4	444.99
$\hat{t}_{132}$	1 3 2	−234.41
	1 3 3	−449.86
$\hat{t}_{1122}$	1 1 2 2	356.28
	1 1 2 3	689.77
$\hat{t}_{11112}$	1 1 1 1 2	−414.52
	1 1 1 1 3	−774.08
t_{34}	3 4	335.53
t_{142}	1 4 2	−396.88
t_{223}	2 2 3	−624.07
t_{1213}	1 2 1 3	562.90
t_{1132}	1 1 3 2	689.31
t_{1222}	1 2 2 2	831.52
t_{11212}	1 1 2 1 2	−924.02
t_{11122}	1 1 1 2 2	−1000.3
t_{111112}	1 1 1 1 1 2	1171.7

Here the index p runs through all distinct prime cycles (8). For the Hausdorff dimension calculation (6), (11), the weight associated with the p cycle is $t_p = z^{n_p}|\delta_p|^\tau$. In order to satisfy the Hausdorff dimension condition $\Gamma_n \approx 1$, we need to find the smallest value of τ for which $\lambda(\tau) = 1$. Substituting $z = \lambda = 1$, $\tau = -D_{\mathrm{H}}$ into (13), we obtain

$$0 = \prod_p (1 - 1/|\delta_p|^{D_{\mathrm{H}}}). \tag{14}$$

This formula is one of our two main results (the other is table 1); it relates the Hausdorff dimension of irrational windings to the universal parameter scaling ratios δ_p.

4. Cycle expansions

We now turn to the evaluation of D_{H} from this relation, using the *cycle expansion* technique of [21–23]. Expanding (13) as a power series in z, we obtain

$$1/\zeta = 1 - \hat{t}_1 - (\hat{t}_{12} - t_1\hat{t}_2) - (\hat{t}_{112} - t_1\hat{t}_{12}) - (\hat{t}_{23} - t_2\hat{t}_3) - (\hat{t}_{122} - t_{12}\hat{t}_2) - (\hat{t}_{1112} - t_1\hat{t}_{112}) - \ldots \tag{15}$$

where $\hat{t}_p$ stands for t_p together with its infinite tail sequences of cycles of form†

$$\hat{t}_1 = \sum_{a=1}^{\infty} t_a \qquad \hat{t}_{12} = \sum_{a=2}^{\infty} t_{1a} \qquad \hat{t}_{a_1\ldots a_{k-1}a_k} = \sum_{a=a_k}^{\infty} t_{a_1\ldots a_{k-1}a}. \tag{16}$$

The sums in (16) extend over all $a_1, a_2, \ldots$ and have to be truncated in numerical evaluations. The underlying idea behind cycle expansions is very simple; when the expansion is arranged well, the dominant terms in the expansion are accounted for first, and the subdominant terms are combined into small *curvature* corrections. Table 1 illustrates the size of contributions to (15). The weights we shall use will be of form $|\delta_p|^{-0.87\ldots}$. At that value of the exponent, the infinite fixed point sequence $\delta_1, \delta_2, \delta_3, \ldots$ dominates; other subsequences, such as $\delta_{12}, \delta_{13}, \delta_{14}, \ldots,$ $\delta_{112}, \delta_{113}, \delta_{114}, \ldots$ are exponentially suppressed, roughly by powers of δ_1. The fact that the terms in these series fall off only by power laws, not exponentially, makes the estimates as delicate as evaluations of the Riemann zeta function. We evaluate $\hat{t}_p$ by splitting them into a head $\sum_{a=1}^{N} t_a$, to be evaluated directly, and the tail $R_N = \sum_{a=N+1}^{\infty} t_a$, to be estimated asymptotically in a. According to (9), $\delta_a \approx a^3$ for large a, so $\hat{t}_1$ can be estimated from a finite number of δ_a by matching them up with an a^3 tail. However, in practice we do not do such matchings, as in the cases we have studied we obtain the best numerical convergence by using the Levin logarithmic convergence acceleration method [15, 24, 25] to estimate the tails.

In the fixed-points approximation the cycle expansion (15) is given by

$$\zeta^{(1)}(z, \tau) = 1 - z \sum_{a=1}^{\infty} |\delta_a|^\tau. \tag{17}$$

Using the first $N = 10$ fixed points of table 1, together with the Levin method

† See, for example, table 3 of [23].

estimate [15, 24, 25] of R_N in

$$0 = 1 - \sum_{a=1}^{N} |\delta_a|^{-D_H^{(1)}} + R_N$$

we obtain

$$D_H^{(1)} = 0.8416 \pm 0.0003 \tag{18}$$

(throughout this paper the numerical results quoted are numerically stable to the digits specified). This number is already within 5% of $D_H = 0.870\ldots$, the best estimate available [1, 15]. In contrast, without the tail estimate the $N = 10$ truncation yields $D_H^{(1)} = 0.76\ldots$, i.e. N term truncations of sums like (17) lead to catastrophically slow convergence (familiar from the theory of Riemann ζ functions) of order N^{-3D_H}. Good accuracy in the determination of the fixed-points estimate $D_H^{(1)}$ is a prerequisite for our next step, evaluation of the curvature corrections.

As (15) is a typical curvature expansion of [22] (longer cycles contribute in the form of deviations from their approximations by shorter cycles), we expect fast convergence, provided that every infinite series $\sum t_{\ldots n}$ in (16) is summed up in the same way as $\hat{t}_1$ before being included into the cycle expansion (15). The curvature expansions are expected to converge exponentially, as each counterterm sequence is dominated by its head (low-a terms in the $t_{\ldots a}$ series). We estimate the leading curvature corrections to the fixed-points approximation (17) by considering successive truncations of (15):

$$\begin{aligned}\zeta^{(2)}(z, \tau) &= 1 - \hat{t}_1 - (\hat{t}_{12} - t_1\hat{t}_2)\\ \zeta^{(3)}(z, \tau) &= 1 - \hat{t}_1 - (\hat{t}_{12} - t_1\hat{t}_2) - (\hat{t}_{112} - t_1\hat{t}_{12}), \ldots\end{aligned} \tag{19}$$

These truncations, supplemented by the Levin method estimates yield, respectively

$$D_H^{(2)} = 0.876 \pm 0.003 \qquad D_H^{(3)} = 0.867 \pm 0.003 \tag{20}$$

in agreement with previous estimates [1, 15]. The convergence of $D_H^{(k)}$ with k is in qualitative agreement with our expectations of exponential convergence. However, it should be emphasized that the numerical algorithms we use have no reliable error estimates, and better methods for estimating sums like (17) need to be developed.

Beyond the Hausdorff dimension discussed above, we have explored a variety of 'thermodynamic' averages over the irrational winding set. In the thermodynamic formalism [18, 19] a function $\tau(q)$ is defined by the requirement that the $n \to \infty$ limit of generalized sums (11)

$$\Gamma_n(\tau, q) = \sum_{i \in \mathcal{S}_n} \frac{p_i^q}{|l_i|^\tau} \tag{21}$$

is finite. The $q = q(\tau)$ function is evaluated by substituting the available cycle weights t_p into the cycle expansion of the Euler product (13) and determining its zeros. Both the 'level of resolution' n and the weights p_i in (21) are arbitrary, as a hierarchical presentation of the irrational winding set depends on the choice of organization [15] of rationals on the unit interval. As this Cantor set is generated by scanning the parameter space, not by dynamical stretching and kneading, there is no 'natural' measure, and a variety of measures have been investigated [1, 10, 15, 19, 25]. In the present continued fraction thermodynamics example $p_i = e^{-nq}$, where n is the length of the corresponding continued fraction block. We have explored other choices of p_i in (21), such as the Farey tree partitioning introduced in [9, 27–29] whose associated thermodynamics is discussed in detail in [10, 11, 15]. The Farey tree and the continued fraction partitioning of the unit interval differ only in the choice of measure (or the multiplicative weight ω_p in the transfer operator (6), or the definition of the topological cycle length.) For the Farey tree partitioning the nth level consists of all quadratic irrationals ξ whose repeated block continued fraction entries add up to n. The corresponding prime cycle weight in the cycle expansion (15) is

$$t_p = z^n \, |\delta_p|^\tau \qquad n = \sum_{j=1}^{n_p} a_j \tag{22}$$

The problem with the 'Farey tree thermodynamics' is that a phase transition occurs precisely at the Hausdorff dimension [8, 15], and our estimates are unreliable. We have estimated the Hausdorff dimension level by level, and used a polynomial fit to extrapolate to $D_{\mathrm{H}} = 0.870 \pm 0.005$.

Furthermore we have computed a variety of $q(\tau)$ and $f(\alpha)$ functions for the Farey tree, continued fractions partitioning and other choices of weights p_i, checked the locations of phase transitions, and studied the ways in which they affect the convergence. All versions of the thermodynamic formalism that we have examined here exhibit phase transitions. For example, for the continued fraction partitioning choice of weights t_p, the cycle expansions (15) behave as hyperbolic averages only for sufficiently negative values of τ; hyperbolicity fails at the 'phase transition' [15, 25] value $\tau = -\frac{1}{3}$, due to the power-law divergence (9) of the harmonic tails $\delta_{\ldots n} \approx n^3$. In the above investigations we were greatly helped by the availability of a number theory model [15, 26]: the $k = 0$ limit of (1) is just the Gauss map (3), for which the universal scaling ratios δ_p reduce to quadratic irrationals. This is very useful in testing the quality of our estimates [25]; the associated thermodynamics is discussed in detail in [23].

Unlike D_{H}, the thermodynamic functions depend explicitly on the choice of p_i in (21), and as we know of no physical guiding principle for such choice, we forgo here further detailed discussion of such functions.

The point of the above exercises is not to obtain the best estimate of the Hausdorff dimension; that can be done more directly from the map (1), and in any case even the most careful experimental measurements [30] cannot yield D_{H} to more than one or two significant figures. Our motivation was to check that the relation (14) between the Shenker *et al* and the Jensen *et al* universality is indeed a convergent relation.

In summary, we have stablished here that the universality of the critical irrational winding Hausdorff dimension follows from the universality of quadratic irrational scalings. The formulae we have used here are formally identical to those used for description of dynamical strange sets [22], the deep difference being that here the cycles are not dynamical trajectories in the coordinate space, but renormalization group flow in the parameter space. The 'cycle eigenvalues' are in the present context the universal quadratic irrational scaling numbers. The crucial insight is the observation that the Hausdorff dimension can be expressed in terms of the renormalization group cycles. The implementation of this relation requires the unstable manifold renormalisation methods (see the appendix), the zeta function formalism, and control of the logarithmic convergence of D_{H} estimates from finite numbers of scaling ratios δ_p.

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Note added in proof. δ_1 through δ_8 have recently been computed to a high precision by A Eriksson [32]. We are grateful for permission to include these values in table 1.

Appendix. The unstable manifold renormalization

According to [5, 6] the renormalization group appropriate to cubic critical circle maps at winding number $W_n = [n, n, n, \ldots]$ acts on pairs of functions $F = (\zeta, \eta)$, where ζ is defined on the interval $[\eta(0), 0]$, η on $[0, \zeta(0)]$, satisfying the following conditions:

$$\begin{array}{lll}
\text{(i)} & \zeta(0) = \eta(0) + 1 & \\
\text{(ii)} & 0 < \zeta(0) < 1 & \\
\text{(iii)} & \zeta'(0) = \eta'(0) = 0 & \\
\text{(iv)} & \zeta''(0) = \eta''(0) = 0 & \\
\text{(v)} & \zeta'''(0) \neq 0 & \eta'''(0) \neq 0 \\
\text{(vi)} & \zeta^n(\eta(0)) > 0 & \zeta^{n-1}(\eta(0)) < 0 \\
\text{(vii)} & \eta(\zeta(0)) = \zeta(\eta(0)) & \\
\text{(viii)} & \zeta'(\eta(0))\eta'''(0) = \eta'(\zeta(0))\zeta'''(0). &
\end{array} \tag{A1}$$

(To make connection with the dynamics of a single circle map $h(x)$, define $\zeta(x) = h(x)$, $\eta(x) = h(x) - 1$. Then if h is smooth and has a cubic inflection point at the origin, ζ and η will satisfy (A1).) Acting on these functions, the renormalization transformation is

$$R_n \begin{pmatrix} \zeta \\ \eta \end{pmatrix} = \begin{pmatrix} \alpha\zeta^{n-1} \circ \eta \circ \alpha^{-1} \\ \alpha\zeta^{n-1} \circ \eta \circ \zeta \circ \alpha^{-1} \end{pmatrix} \tag{A2}$$

where α satisfies

$$\alpha = \frac{1}{\zeta^{n-1}(\eta(0)) - \zeta^{n-1}(\eta(\zeta(0)))} < -1. \tag{A3}$$

It can be shown that conditions (A1) are preserved under R_n.

Acting on a map with winding number $[a_1, a_2, a_3, \ldots]$, R_{a_1} yields a map with winding number $[a_2, a_3, \ldots]$, so a fixed point of R_n has a quadratic irrational winding number $W_n = [n, n, n, \ldots]$. R_n has exactly one [5, 6] relevant direction with eigenvalue δ_n. Similarly, the renormalization transformation $R_{a_{n_p}} \ldots R_{a_2} R_{a_1} \equiv R_{a_{n_p} \ldots a_1}$ has a fixed point of winding number $W_p = [a_1, a_2, \ldots, a_{n_p}, a_1, a_2, \ldots]$, with a unique relevant eigenvalue δ_p.

The object of our calculation is to compute the (one-dimensional) unstable manifold of $R_p = R_{a_{n_p} \ldots a_1}$, from which will follow δ_p. The strategy is simple in concept, and becomes complicated only in the details. Essentially, we represent the functions ζ and η as power series in both the argument x and the parameter (Ω in (1), denoted ω here). However, since R_p has a relevant direction, repeated application will expand the range of rotation numbers, and reduce the range (and hence precision) of parameters for a given range of rotation numbers. This reduction in the range of parameters can be avoided by the application of R_p^*, which reparametrizes the family of functions after the application of R_p, R_p^* is defined below, and under its application, the family of functions converge to the unstable manifold of R_p (see [31] for a more complete explanation of R_p^*).

Explicitly, we represent ζ and η by

$$\zeta(\omega, x) = \sum_{i,j=0}^{\infty} a_{ij} x^{3i} \omega^j \qquad \eta(\omega, x) = \sum_{i,j=0}^{\infty} b_{ij} x^{3i} \omega^j, \tag{A4}$$

(note we only need terms in x^3 since we are interested in maps with a cubic inflection point [6]). For implementation on the computer we truncate these series to N terms. This creates the problem that application of R^* no longer preserves all of the conditions (A1); we will have to make corrections to ensure that these boundary conditions are maintained. We rescale the parameter axis at each iteration so that the winding numbers satisfy

$$W(F(-\omega_0, x)) = [p] \qquad W(F(\omega_0/\delta, x)) = [p, p] \tag{A5}$$

where $p = a_1 a_2 \dots a_{n_p}$ is the repeated block in question. δ is its value from the previous iteration. We now apply R_p^* to give the new F, $\tilde{F}$, by the following steps.

(i) Find ω^* such that $W(F(\omega^*, x)) = [p, p, p]$.

(ii) Rescale ω by a linear transformation $h(\omega)$ such that

$$h(-\omega_0) = \omega_0/\delta \qquad h(\omega^*) = -\omega_0. \tag{A6}$$

(iii) Apply R_p to $F(\omega^*, x)$, using equation (A2). This gives $\tilde{F}$.

(i)–(iii) define R_p^* and contain the expansion of the family of functions along the unstable manifold. The particular rescaling of the parameter axis, (A5), was chosen so that ω^* will be close to 0 as the procedure is iterated. Once $\tilde{F}$ is obtained, the new coefficients $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ can be found by first taking N values of x (chosen to be distributed evenly in x^3), for a given ω_k, and solving

$$\zeta(\omega_k, x_l) = \sum_{i=0}^{N} A_i(\omega_k) x^{3i} \qquad \eta(\omega_k, x_l) = \sum_{i=0}^{N} B_i(\omega_k) x^{3i} \tag{A7}$$

for $A(\omega_k)$ and $B(\omega_k)$; then from N values of ω_k, solve for $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ via

$$A_i(\omega_k) = \sum_{j=0}^{N} \tilde{a}_{ij} \omega_k^j \qquad B_i(\omega_k) = \sum_{j=0}^{N} \tilde{b}_{ij} \omega_k^j. \tag{A8}$$

It is here that we must make corrections so that equalities (vi) and (vii) of (A1) are obeyed. We do this by altering the coefficients of $x^{3(N-1)}$ and x^{3N} by the appropriate amounts. Thus we write

$$\tilde{\eta}_c = \tilde{\eta} + c_{N-1} x^{3(N-1)} + c_N x^{3N} \tag{A9}$$

where $\tilde{\eta}_c$ is the corrected $\tilde{\eta}$. (We require two unknowns, c_{N-1} and c_N, since we are imposing two conditions, (vi) and (vii) of (A1). It is easily found that condition (vi) requires

$$c_{N-1} A_0^{3(N-1)} + c_N A_0^{3N} = \tilde{\zeta}(B_0) - \tilde{\eta}(A_0) \tag{A10}$$

and (vii) requires

$$3(N-1)c_{N-1}A_0^{3N-4}+3Nc_NA_0^{3N-1}=B_1\tilde{\xi}'(B_0)/A_1-\tilde{\eta}'(A_0). \qquad \text{(A11)}$$

These linear equations are solved for c_{N-1} and c_N, which are then used to update $\tilde{\eta}$. When this correction has been made, we have completed the iteration: we now have a new set of coefficients for ζ and η, which are then iterated in the same way. This procedure converges to a universal unstable manifold for each p, independent of the initial ζ and η (provided they are within the basin of attraction). At each iteration, δ is determined from ω^*; as ζ and η converge to the unstable manifold, δ converges to its universal value. In this way, we determine δ_p for each p.

N is chosen large enough so that the coefficients a_{ij} and b_{ij} are small for i, j close to N, but small enough so that the computation does not take too long. Typically, we used $N=12$, though we also tried other values. For p corresponding to a large denominator in continued fraction $[p]$, the method runs into precision problems when trying to find ω^*, since finding the cycle with winding number $[p, p, p]$ involves fairly large numbers of iterations (of order 10^3). We were able to go to the tenth level, using 80 bit reals on a 68020 based 'super-micro' computer. This is, however, far deeper than one could go by taking differences to determine the δ values.

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