REVEALING THE GEOMETRY OF TURBULENT PIPE FLOW ATTRACTOR BY SYMMETRY REDUCTION

Predrag Cvitanović*[a)], Ashley P. Willis** & Marc Avila***
*School of Physics, Georgia Institute of Technology, Atlanta, GA 30332, USA
**School of Mathematics and Statistics, University of Sheffield, S3 7RH, UK
***Max Planck Institute for Dynamics and Self-Organisation,
Bunsenstrasse 10, D-37073 Göttingen, Germany

Summary Symmetry reduction by the 'method of slices' is applied to a pipe flow in order to quotient the continuous symmetries of turbulent flow states under stream-wise translations and azimuthal rotations. Within the symmetry-reduced state space, traveling wave solutions reduce to equilibria, and relative periodic orbits reduce to periodic orbits. Projections of these solutions and their unstable manifolds from their infinite-dimensional symmetry-reduced state space onto suitably chosen 3-or 3-dimensional subspaces reveal their interrelations and the role they play in organizing turbulence in wall-bounded shear flows. Visualizations of the flow within the slice and its linearization at equilibria enable us to trace out the unstable manifolds, determine close recurrences, identify connections between different traveling wave solutions, and find, for the first time for pipe flows, relative periodic orbits that are embedded within the chaotic attractor and capture turbulent dynamics at transitional Reynolds numbers.

The numerical discovery of unstable travelling waves in pipe flow, together with glimpses of them in experiments [4], has spurred interest in obtaining a description of turbulent flow in terms of a handful of key exact solutions of Navier-Stokes equations. In this approach, the dynamics of turbulent flows at moderate Reynolds number is visualized using equilibrium solutions to define dynamically invariant, intrinsic, and representation independent coordinate frames [3]. Within the dynamical framework that has emerged in the last decade, turbulence is viewed as a walk through a forest of such solutions. The resulting visualizations show the role exact solutions play in shaping turbulence: the observed coherent structures are the physical images of the flow's least unstable invariant solutions, with turbulent dynamics arising from a sequence of transitions between these states. The long-term goals of this research program are to develop this vision into quantitative, predictive description of moderate-*Re* turbulence, and to use this description to control flows and explain their statistics.

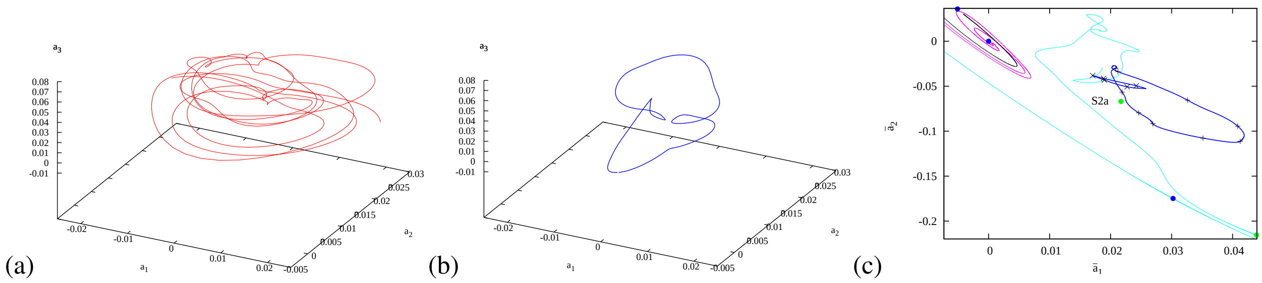

Figure 1. Symmetry reduction replaces each full state space trajectory $a(t)$ by a simpler reduced state space trajectory $\bar{a}(t)$, with continuous group induced drifts quotiented out. Here this is illustrated by a relative periodic orbit (a) traced in the full state space for just two $T = 36.72$ periods, (b) restricted to the symmetry-reduced state space, both projected onto the same 3-dimensional Frenet-Serret frame. In the full state space a relative periodic orbit traces out quasi-periodically a highly contorted 2-torus; in the reduced state space it closes a periodic orbit in one period. (c) Relative periodic orbits embedded within turbulence, a 2-dimensional symmetry-reduced state space projection. All known equilibria are plotted as well, together with the unstable manifold of an 'edge' travelling wave. Evaluated for a short $L \simeq 2.5D$-periodic pipe computational domain $\Omega = [1, \pi, 2\pi/1.25]$ at $Re = 2400$ for which the turbulence is empirically sustained for very long times.

Let $g(\phi, \ell)$ be the shift operator such that $g(\phi, 0)$ denotes an azimuthal rotation by ϕ about the pipe axis, and $g(0, \ell)$ denotes the stream-wise translation by ℓ, $g(\phi, \ell)\,[u, v, w](r, \theta, z) = [u, v, w](r, \theta - \phi, z - \ell)$. The Navier-Stokes equations for pipe flow are equivariant under these transformations. Pipe flow has a non-zero mean axial velocity and hence its unstable invariant solutions are generically stream-wise traveling solutions. While a large number of unstable travelling waves have been identified in pipe flow [1, 6, 5], their neighborhoods are visited for only 10-20% of the time [7], and so it is expected that relative periodic orbits capture most of the natural measure of the turbulent flow. A relative periodic orbit is an orbit of a fluid state which exactly recurs after a fixed relative period T, but shifted by a group action g. One of the main difficulties in identifying relative periodic orbits embedded in turbulence is that each of them travels downstream with its own mean phase velocity. Therefore there is no single co-moving frame that can simultaneously reduce *all* relative periodic orbits to periodic orbits and all traveling waves to equilibria. This problem is here resolved by the method of

[a)] Corresponding author. E-mail: predrag.cvitanovic@physics.gatech.edu.

slices [2], in which the group orbit of any full-flow structure is represented by a single point, the group orbit's intersection with a fixed co-dimension hyperplane or *'slice'*. In the method of slices $\dot{a} = v(a)$, the velocity in the full state space, is decomposed into the sum of $\bar{v}$, the velocity component in the slice, and $\dot{\phi}\,\mathbf{t}$, the velocity component along the group tangent space. The equations for the reduced state space flow, $\dot{\bar{a}} = \bar{v}(\bar{a})$, confined to the slice are

$$\bar{v}(\bar{a}) = v(\bar{a}) - \dot{\phi}(\bar{a})\,\mathbf{t}(\bar{a})\,, \qquad \dot{\phi}(\bar{a}) = \langle v(\bar{a})|\mathbf{t}'\rangle/\langle \mathbf{t}(\bar{a})|\mathbf{t}'\rangle\,. \tag{1}$$

An example of the power of the method is the comparison of a full state space orbit of figure 1 (a) with the corresponding reduced state space orbit of figure 1 (b).

The main advances reported here are the symmetry reduced state space visualization of moderate-*Re* turbulent pipe flow, and the determination of new relative periodic orbits and their unstable manifolds. We demonstrate that this new tool enables us to commence a systematic exploration of the hierarchy of dynamically important invariant solutions of pipe flow. Symmetry reduction is here combined with $3D$ spatial visualization of instantaneous velocity fields to elucidate the physical processes underlying the formation of unstable coherent structures. Running concurrently, the ∞-dimensional state-space representation [3], enables us to track the unstable manifolds of invariant solutions, the heteroclinic connections between them, and provides us with new insights into the nonlinear state space geometry and dynamics of moderate *Re* wall-bounded flows. Starting in neighborhoods of the known travelling waves as initial conditions and then searching for close recurrences in the reduced state space yields educated guesses for locations of relative periodic orbits. Applying Newton–Krylov methods to these initial guesses leads to the discovery reported here, the first examples of relative periodic orbits embedded into pipe turbulence, see figures 1 and 2.

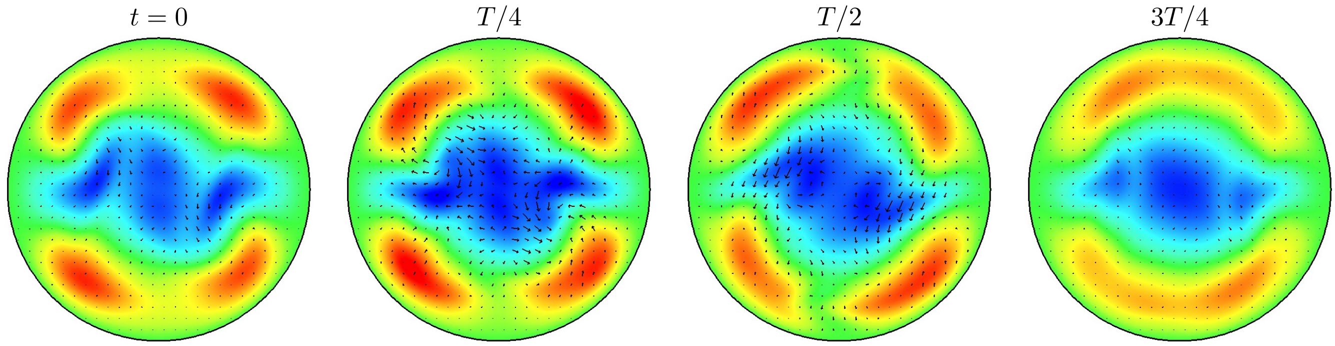

Figure 2. Four snapshots of the $T = 36.72$ relative periodic orbit, reduced by the method of slices into a periodic orbit solution. A colormap of stream-wise velocity is shown, with red (blue) indicating positive $w = 0.6\,U$ (negative $w = -0.7\,U$) velocity with respect to laminar flow.

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