Continuous symmetry reduced trace formulas

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(Dated: October 29, 2007)

Trace formulas relate short time dynamics (unstable periodic orbits) to long time invariant state space densities (natural measure). Higher dimensional dynamics requires inclusion of higher-dimensional compact invariant sets, such as partially hyperbolic invariant tori, into trace formulas. A trace formula for a partially hyperbolic $(N+1)$-dimensional compact manifold invariant under a global continuous symmetry is derived here. In this extension of "periodic orbit" theory there are no or very few periodic orbits - the relative periodic orbits that the trace formula has support on are almost never eventually periodic.

PACS numbers: 05.45.Mt, 46.40.Cd, 46.40.Ff, 62.30.+d

The classical trace formula for smooth continuous time flows [1, 3] relates the spectrum of the evolution operator

$$\mathcal{L}^t(x',x) \,=\, \delta\big(x'-f^t(x)\big)\, e^{\beta\cdot A^t(x)} \tag{1}$$

to the unstable periodic orbits p of the flow $f^t(x)$,

$$\sum_{\alpha=0}^{\infty}\frac{1}{s-s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty}\frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1}-\mathbf{J}_p^r\right)\right|}\,.$$

This formula (and the associated spectral determinants and cycle expansions [4]) is valid for fully hyperbolic flows.

Here we derive the corresponding formula for dynamics invariant under a compact group of symmetry transformations, a close relative of the semiclassical Gutzwiller type trace formula derived by Creagh [5] in 1993. Recent progress by Viswanath [7] in computing exact relative periodic solutions of the full Navier-Stokes plane Couette flow necessitates revisiting this problem in the context of deterministic chaotic flows.

The new trace formula follows from the full reducibility [8] of representations of a compact group G acting linearly on a vector space V, with irreducible representations labeled by sets of integers $m = (m_1,\cdots,m_N)$, and the vector space V decomposed into invariant subspaces V_m. For a N-dimensional compact Lie group G the fundamental result is the Weyl full reducibility theorem, with projection operator onto the V_m irreducible subspace given by

$$P_m = d_m\int_G dg\ \chi_m(g)D_m(g^{-1})\,. \tag{2}$$

The group elements $g = g(\theta_1,\ldots,\theta_N) = e^{i\theta\cdot T}$ are parametrized by N real numbers $\{\theta_1,\ldots,\theta_N\}$ of finite range, hence designation "compact."

The character χ is the trace $\chi_m(g) = \mathrm{tr}\, D_m(g) = \sum_{i=1}^{d_m} D_m(g)_{ii}$, where $D_m(g)$ is a $[d_m\times d_m]$-dimensional matrix representation of action of the group element g on the irreducible subspace V_m. The group integral is weighted by the normalized Haar measure, $\int_G dg = 1$, and d_m is the multiplicity of degenerate eigenvalues in representation m.

If action of every element g of a compact group G commutes with the flow $\dot{x} = v(x)$,

$$D(g)v(x) = v(D(g)x)\,,\ D(g)f^t(x) = f^t(D(g)x)\,,$$

G is a global symmetry of the dynamics. The finite time evolution operator (1) can be written [13] as $\mathcal{L}^t = e^{t\mathcal{A}}$ in terms of the time evolution generator

$$\mathcal{A}\rho(x) = \lim_{\delta\tau\to 0^+}\frac{1}{\delta\tau}\left(\mathcal{L}^{\delta\tau}-I\right)\rho(x) \,=\, -\partial_i(v_i(x)\rho(x))\,. \tag{3}$$

The operator $e^{t\mathcal{A}}$ commutes with all symmetry transformations $e^{i\theta\cdot T}$. For a given state space point x together they sweep out a $(N{+}1)$-dimensional manifold of equivalent orbits.

In other words, the time evolution itself is a (noncompact) 1-parameter Lie group. Thus all continuous symmetries can be considered as being on on the same footing.

A symmetry group element acts on $\mathcal{L}(x,y)$, the kernel of $\mathcal{L}^t$ in the state space representation (1), as

$$g^{-1}\mathcal{L}(y,x) = \mathcal{L}(D(g^{-1})y,x) = \mathcal{L}(y,D(g)x)\,. \tag{4}$$

The irreducible eigenspaces of G are also eigenspaces of the dynamical evolution operator $\mathcal{L}^t$, with the decomposition of the evolution operator to irreducible subspaces, $\mathcal{L} = \sum_m \mathcal{L}_m$, following immediately by application of the projection operator (2):

$$\mathcal{L}^t_m(y,x) = d_m\int_G dg\ \chi_m(g)\mathcal{L}^t(D_m(g^{-1})y,x)\,. \tag{5}$$

To evaluate the contribution of a prime cycle p of period T_p, restrict the integration to an infinitesimally thin manifold $\mathcal{M}_p$ enveloping the cycle and all of its rotations by G, pick a point on the cycle, and choose a local coordinate system with a longitudinal coordinate

$dx_\parallel$ along the direction of the flow, N coordinates dx_G along the invariant manifold swept by p under the action of the symmetry group G, and $(d-N-1)$ transverse coordinates $x_\perp$. The trace $\mathrm{tr}\,_p\mathcal{L}^t_m$ is given by

$$d_m \int_G dg\ \chi_m(g) \int_{\mathcal{M}_p} dx_\perp dx_\parallel dx_G\, \delta\big(x - D_m(g) f^t(x)\big)\,. \tag{6}$$

The integral along the longitudinal coordinate was computed in refs. [1, 3]: Eliminating the time dependence by Laplace transform one obtains

$$\int_0^\infty e^{-st} \oint_p dx_\parallel\, \delta\big(x_\parallel - f^t(x_\parallel)\big) \;=\; T_p \sum_{r=1}^\infty e^{-sT_p r}\ . \tag{7}$$

The m subspace group integral is simple:

$$\int_G dg\ \chi_m(g) \int_{\mathcal{M}_p} dx_G\, \delta\big(x_G - D_m(g) f^{rT_p}(x_G)\big) = \chi_m(g_p^r). \tag{8}$$

For the remaining transverse coordinates the Jacobian matrix is defined in a $(N+1)$-dimensional surface of section $\mathcal{P}$ of constant $(x_\parallel, x_G)$. Linearization of the periodic flow transverse to the orbit yields

$$\int_{\mathcal{P}} dx_\perp \delta\big(x_\perp - D_m(g_p^r) f^{rT_p}(x_\perp)\big) = \frac{1}{\left|\det\left(\mathbf{1} - \tilde{\mathbf{J}}^r_{m,p}\right)\right|}\ , \tag{9}$$

where $\tilde{\mathbf{J}}_{m,p} = D_m(g_p)\mathbf{J}_p$ is the p-cycle $[(d-1-N)\times(d-1-N)]$ symmetry reduced Jacobian matrix, computed on the reduced surface of section and rotated by g_p. We assume hyperbolicity, that is, that the magnitudes of all transverse eigenvalues are bounded away from unity.

The *classical symmetry reduced trace formula for flows* follows by substituting (7) - (9) into (6):

$$\sum_{\beta=0}^\infty \frac{1}{s - s_{m,\beta}} = d_m \sum_p T_p \sum_{r=1}^\infty \chi_m(g_p^r) \frac{e^{r(\beta A_p - sT_p)}}{\left|\det\left(\mathbf{1} - \tilde{\mathbf{J}}^r_{m,p}\right)\right|}\ . \tag{10}$$

The sum is over all prime *relative periodic orbits* p and their repeats, orbits in state space which satisfy

$$x(t) = D(g_p) x(t + T_p) \tag{11}$$

for a fixed *relative period* T_p and a fixed *shift* g_p.

For example, for the full Navier-Stokes plane Couette flow defined in a box periodic in stream-wise and span-wise directions, a relative periodic solution is a solution that recurs at time T_p with exactly the same disposition of velocity fields over the entire box, but shifted by a 2-dimensional translation g_p.

The $m = (0, 0, \cdots, 0)$ subspace is the one of most relevance to chaotic dynamics, as its leading eigenfunction, with least nodes and the slowest decay rates, corresponds to the natural measure observed in the long time dynamics.

Discussion. One of the goals of nonlinear dynamics is to describe the long time evolution of ensembles of trajectories, when individual trajectories are exponentially unstable. The main tool in this effort have been trace formulas because they make explicit the duality between individual short time trajectories, and long time invariant densities (natural measures, eigenfunctions of evolution operators). So far, the main successes have been in applications to low dimensional flows and iterated mappings, where the compact invariant sets of short-time dynamics are equilibria, periodic points and periodic orbits. Dynamics in higher dimensions requires extension of trace formulas to higher-dimensional compact invariant sets, such as partially hyperbolic invariant tori.

Here we have used a particularly simple direct product structure of a global symmetry that commutes with the flow to reduce the dynamics to a symmetry reduced $(d-N-1)$-dimensional state space $\mathcal{M}/G$.

Amusingly, in this extension of “periodic orbit” theory from unstable 1-dimensional closed orbits to unstable $(N+1)$-dimensional compact manifolds invariant under continuous symmetries, there are no or very few periodic orbits. Relative periodic orbits are almost never eventually periodic, that is, they almost never lie on periodic trajectories in the full state space [6], unless forced to do so by a discrete symmetry, so looking for periodic orbits in systems with continuous symmetries is a fool’s errand.

Restriction to compact Lie groups in derivation of the trace formula (10) was a matter of convenience, as the general case is more transparent than particular implementations (such as $SO(2)$ and $SO(3)$ rotations, with their explicit Haar measures and characters). This can be relaxed as the need arises - much powerful group theory developed since Cartan-Weyl era is at our disposal. For example, the time evolution is in general non-compact (a generic trajectory is an orbit of infinite length). Nevertheless, the trace formulas have support on compact invariant sets in $\mathcal{M}$, such as periodic orbits and $(N+1)$-dimensional manifolds generated from them by action of the global symmetry groups. Just as existence of a periodic orbit is a consequence of given dynamics, not any global symmetry, higher-dimensional flows beckon us on with nontrivial higher-dimensional compact invariant sets (for example, partially hyperbolic invariant tori) for whom the trace formulas are still to be written.

Acknowledgments. Author thanks Glen Robinson Jr. for support, and the July 2, 2006 Delta Airlines flight 96, where this work was done, for hospitality. This work is not supported by the National Science Foundation.

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