Cycle expansion for power spectrum

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ABSTRACT

A cycle expansion method is applied to calculation of power spectrum of chaotic one-dimensional maps. It is shown that the broad–band part of the spectrum can be represented as a diffusion constant of some auxiliary process, and this constant is then represented in terms of periodic orbits. Accuracy of the method is also considered.

1. INTRODUCTION

Chaotic oscillations have broad–band power spectrum, and this is possibly the easiest way to recognize them in experiments. However, when chaos is studied theoretically (numerically), other characteristics (like, e.g., Lyapunov exponents) are more convenient, firstly, because they are, contrary to the power spectrum, invariants of the motion, secondly, because they are easier to calculate. The aim of this paper is to extend recent methods of calculation invariant properties of chaos to the case of power spectrum. We show how the power spectrum of chaotic discrete one–dimensional mapping may be represented through the properties of periodic orbits.

It is known that the periodic orbits are everywhere dense in chaotic attractors and may be considered as a *sceleton* of chaos. Recently an efficient method of cycle expansion was proposed[1], which allows to represent different characteristics of chaotic motion through the properties of periodic orbits[2]. Below we apply this method to study of power spectrum. First, we present the cycle expansion method for calculation of the diffusion constant[3]. Then we show how the power spectrum may be represented as a diffusion constant. Finally, we discuss cases of simple and complex chaotic dynamics.

2. RECYCLING DIFFUSION

Let us consider a diffusion process, driven by a one–dimensional map

$$x_{t+1} = f(x_t) \tag{1a}$$
$$w_{t+1} = w_t + \phi(x_t), \quad w_0 = 0. \tag{1b}$$

Here the first equation is assumed to generate a chaotic process, which acts in the second equation as an external "random" force. Diffusion properties of the variable w are described with the logarithm of the generating function:

$$Q(\beta) = \lim_{t\to\infty} \frac{1}{t} \log < e^{\beta w_t} > \tag{2}$$

Using this function we can calculate drift velocity and diffusion constant:

$$\frac{dQ(0)}{d\beta} = \lim_{t\to\infty} \frac{< w_t >}{t}, \quad \frac{d^2Q(0)}{d\beta^2} = \lim_{t\to\infty} \frac{< w_t^2 > - < w_t >^2}{t}. \tag{3}$$

In order to find $Q(\beta)$ let us define an operator $\mathcal{L}^t$:

$$\mathcal{L}^t(y, x) = e^{\beta w_t} \delta(y - x_t)$$

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which allows to represent the generating function as

$$< e^{\beta w_t} >= \int dx\, dy\, e^{\beta w_t} \delta(y - x_t)\rho(x), \tag{4}$$

where $\rho(x)$ is initial probability distribution density. Comparing (2) and (4), we conclude that $e^{Q(\beta)}$ is the leading eigenvalue of $\mathcal{L}^t$. This means that $z = e^{-Q(\beta)}$ is the smallest root of the equation

$$\det(1 - z\mathcal{L}) = 0. \tag{5}$$

Representing determinant through the trace formula gives

$$\det(1 - z\mathcal{L}) = \exp[\mathrm{tr} \log(1 - z\mathcal{L})] = \exp[-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\mathcal{L}^n] \tag{6}$$

where

$$\mathrm{tr}\mathcal{L}^n = \sum_{i \in \mathcal{F}_n} \frac{e^{\beta w_i}}{|1 - \Lambda_i|}.$$

Here $\mathcal{F}_n$ is a set of all fixed points of the mapping $f^n(x)$, w_i - corresponding value of w and Λ_i is multiplicator of the fixed point. As a result we obtain the equation for determining $Q(\beta)$ in the form

$$S(Q, \beta) = \exp[-\sum_{n=1}^{\infty} \frac{e^{-nQ}}{n} \sum_{i \in \mathcal{F}_n} \frac{e^{\beta w_i}}{|1 - \Lambda_i|}] = 0. \tag{7}$$

$S(Q, \beta)$ is the Selberg zeta function. First note that for $\beta = 0$ the operator $\mathcal{L}$ is the Frobenius–Perron operator with the leading eigenvalue equal to 1. This means that $S(0,0)$=0. Differentiating (7) we get

$$\begin{aligned} \frac{dQ}{d\beta} &= -\frac{S_\beta}{S_Q} \\ \frac{d^2Q}{d\beta^2} &= -\frac{S_{\beta\beta} + 2Q_\beta S_{Q\beta} + Q_\beta^2 S_{QQ}}{S_Q}. \end{aligned} \tag{8}$$

The Selberg zeta function may be represented as a product of Ruelle zeta functions. Using the formula

$$\frac{1}{|1 - \Lambda_i|} = \frac{1}{|\Lambda_i|} \sum_{k=0}^{\infty} \Lambda_i^{-k}$$

we can rewrite (7) as

$$S(Q, \beta) = \prod_{k=0}^{\infty} \zeta_k^{-1}(Q, \beta)$$

where

$$\zeta_k^{-1}(Q, \beta) = \exp(-\sum_{n=1}^{\infty} \frac{e^{-nQ}}{n} \sum_{i \in \mathcal{F}_n} \frac{e^{\beta w_i}}{|\Lambda_i|} \Lambda_i^{-k}).$$

It is known that zero – order Ruelle zeta function ζ_0 and Selberg zeta function have the same root, so both can be used to determine diffusion constant.

Ruelle zeta function (below we shall use notation $R(Q, \beta) = \zeta_0^{-1}$) can be simply expressed through characteristics of prime cycles of the map f. Suppose that a fixed point $i \in \mathcal{F}_n$ belongs to the orbit of a prime cycle p with length n_p so that $n = rn_p$. Then we use the fact that the Lyapunov number is multiplicative along a trajectory and the diffusing variable w is additive:

$$|\Lambda_i| = |\Lambda_p|^r \quad e^{\beta w_i} = (e^{\beta w_p})^r.$$

Also taking into account that there are exactly n_p different fixed points of f^n that give the same contribution, we get

$$R = \exp(-\sum_p \sum_{r=1}^{\infty} (\frac{e^{\beta w_p - Q n_p}}{|\Lambda_p|})^r \frac{1}{r}) = \prod_{p \in \mathcal{P}} [1 - t_p(Q,\beta)]$$

where

$$t_p = \frac{e^{\beta w_p - Q n_p}}{|\Lambda_p|}$$

and $\mathcal{P}$ is a set of all prime cycles of the mapping f. The zeta function may be represented as a power series

$$\prod_{p \in \mathcal{P}} [1 - t_p(Q,\beta)] = 1 + \sum_{p_1,\dots,p_k} (-1)^k t_{p_1} t_{p_2} \dots t_{p_k}$$

with a sum over all distinct nonrepeating combinations of prime cycles. Then the derivatives (8) are expressed as

$$\frac{dQ}{d\beta} = \frac{\sum (-1)^k (w_{p_1} + \dots + w_{p_k})(|\Lambda_{p_1} \dots \Lambda_{p_k}|)^{-1}}{\sum (-1)^k (n_{p_1} + \dots + n_{p_k})(|\Lambda_{p_1} \dots \Lambda_{p_k}|)^{-1}}$$

and a similar, but more complicated expression for $d^2Q/d\beta^2$.

3. POWER SPECTRUM AS A DIFFUSION CONSTANT

Consider a mapping

$$x_{t+1} = f(x_t) \tag{9}$$

which produces a chaotic sequence $x_1, x_2, \dots, x_N$. Our goal is to calculate the power spectrum of some observable $\phi(x)$. Let us define Fourier transform as

$$s(\omega, N) = \sum_{k=1}^{N} e^{i2\pi\omega k} \phi(x_k)$$

and average it:

$$< |s(\omega, N)|^2 >= N \sum_{m=-N}^{N} (1 - \frac{|m|}{N} C(m)) e^{i2\pi\omega m}$$

where

$$C(m) =< \phi(x_k)\phi(x_{k+m}) >$$

is a correlation function. Generally, power spectrum consists of broad band noise $S(\omega)$ and of discrete spectrum $D(\omega)$:

$$< |s(\omega, N)|^2 >\sim N S(\omega) + N^2 D(\omega).$$

Let us complement the mapping (9) with

$$\xi_{t+1} = e^{i2\pi\omega} \xi_t + \phi(x_t), \quad \xi_0 = 0. \tag{10}$$

Then

$$\xi_N = e^{i2\pi\omega N} s(-\omega, N)$$

and

$$< |\xi_N|^2 >\sim N S(\omega) + N^2 D(\omega). \tag{11}$$

Comparing (11) with (2),(3), we see that $S(\omega)$ is nothing else as diffusion constant for quantity ξ, and $D(\omega)$ is the drift term.

4. ZETA FUNCTION FOR POWER SPECTRUM

The main difficulty in application of cycle expansion technique of Sect.2 to calculation of power spectrum is that generally ξ, unlike w in (1b) is not additive along a trajectory. As one can see from (10), ξ is always multiplied by $\exp(i2\pi\omega)$ and this does not allow to express zeta function through prime cycles. However, for *rational frequencies* we can reduce the system (9),(10) almost exactly to the system (1).

Consider a rational frequency $\omega = l/q$. We can rewrite (9),(10) as

$$x_{t+q} = f^q(x_t) \tag{12a}$$

$$\xi_{t+q} = \xi_t + \tilde{\phi}(x_t, \omega) \tag{12b}$$

where

$$\tilde{\phi}(x_t, \omega) = \phi(x_t)e^{i2\pi\omega q} + \phi(x_{t+1})e^{i2\pi\omega(q-1)} + \cdots + \phi(x_{t+q-1})$$

is Fourier transform of the sequence $\phi(x_t), ..., \phi(x_{t+q-1})$. Now (12) coincides with (1) and all the machinery of cycle expansion may be used to compute power spectrum density for frequency ω as a diffusion constant (albeit ξ is complex variable, for application of formulae of section 1 it can be splited into real and imaginary parts). Note also that in order to obtain proper scaling of the spectrum the diffusion constant for the eq.(12) must be divided by q. We shall do this by multiplying Q in the zeta function by q.

4.1 Complete binary tree

Let us start with the map, whose symbolic dynamics is described by full (0,1) binary tree (i.e. all sequences of "0" and "1" are possible). As an example we shall use in this subsection a skewed tent map

$$f(x) = \begin{cases} ax & \text{if } 0 \leq x \leq a^{-1}, \\ \frac{a}{a-1}(1-x) & \text{if } a^{-1} \leq x \leq 1. \end{cases} \tag{13}$$

For this map the correlation function was calculated by Grossmann and Thomae[4]:

$$C(m) = < (x_t - < x >)(x_{t+m} - < x >) > = \frac{1}{12}\left(\frac{2-a}{a}\right)^m$$

The power spectrum is

$$S(\omega) = 6^{-1}(a-1)(a(a-2)(1+\cos 2\pi\omega)+2)^{-1} \tag{14}$$

and we shall compare this exact solution with the values obtained from the cycle expansion.

Consider first zero frequency $\omega = 0$. In this case Ruelle zeta function

$$R = (1-t_0)(1-t_1)(1-t_{01})(1-t_{001})... = 1 - t_0 - t_1 - [t_{01} - t_0 t_1]...$$

consists of main terms and curvature corrections, and calculation of diffusion constant is straightforward. Figure shows that when using Ruelle zeta function convergence is exponential, while Selberg zeta function gives faster superexponential convergence.

For non zero frequencies ω we must consider zeta functions of higher iterates of the mapping f. We shall show now that these zeta functions may be expressed in terms of prime cycles of the map f. Consider first the case $q = 2$. The mapping f^2 is described with 4 symbols (a, b, c, d) which correspond to fixed points of f^2 labeled in the old (0,1) representation as $a = 00$, $b = 11$, $c = 01$, $d = 10$. Now the zeta function is

$$R^{(2)} = (1-t_a)(1-t_b)(1-t_c)(1-t_d)(1-t_{ab})...$$

Consider the term

$$t_a = \frac{\exp(\beta\xi_a - 2Qn_a)}{|\Lambda_a|} = \frac{\exp(\beta\xi_{00} - 2Qn_{00})}{|\Lambda_{00}|}$$

Note that $|\Lambda_{00}| = |\Lambda_0|^2$, $n_{00} = 1$ and $\xi_{00} = \phi(x_0) - \phi(x_0) = 0$, so

$$t_a(Q,\beta) = t_0^2(Q,\beta)$$

Also, $t_b(Q,\beta) = t_1^2(Q,\beta)$. Now mention that $\xi_c = -\xi_d$ and $|\Lambda_c| = |\Lambda_d| = |\Lambda_{01}|$, so $t_c(Q,\beta) = t_d(Q,-\beta) = t_{01}(Q,\beta)$. This means that the product $(1-t_c)(1-t_d)$ does not contribute to the first derivative S_β (indeed, periodic motion does not contribute to drift velocity) so as far as zeta function and its second derivative with respect to β is considered, we can write

$$(1-t_a)(1-t_b)(1-t_c)(1-t_d) = (1-t_0^2(Q,0))(1-t_1^2(Q,0))(1-t_{01}(Q,\beta))^2$$

Analogously, one can easily find that each primary cycle of the mapping f with odd period gives term $(1-t_p^2(Q,0))$ while cycles of even period appear in pairs and give term $(1-t_p(Q,\beta))^2$. So the zeta function for the mapping f^2 that we need for calculation of power spectrum at frequency $\omega = 1/2$ has the form

$$R^{(2)} = \prod_{p\in\mathcal{P}_e}(1-t_p(Q,\beta)^2 \prod_{p\in\mathcal{P}_o}(1-t_p^2(Q,0) \qquad (15)$$

where $\mathcal{P}_e$ and $\mathcal{P}_o$ are sets of prime cycles with even and odd periods, respectively. From this formula we see that only cycles with even periods contribute to the power spectrum at frequency $\omega = 1/2$. Representation of $R^{(2)}$ as a power series gives shadowing similar (but not exactly the same) as for R, the leading term comes from period 2 cycle "01".

Consider now an arbitrary rational frequency $\omega = l/q$. We show how it's zeta function may be represented through prime cycles of the mapping f. The prime cycles of f^q can be divided into two sets

(1) cycles which are multiples of prime cycles of the map f

(2) cycles which are prime cycles of the map f

Consider a prime cycle p' of the map f^q of the type 1, its length being $n_{p'}$. If it is a multiple of a cycle p of the map f with length n_p, then $jn_p = qn_{p'}$ and j repetitions of the cycle p give exactly the cycle p'. It is easy to see that for this cycle $\xi_{p'} = 0$:

$$\xi_{p'} = \sum_{k=0}^{qn_{p'}-1} e^{i2\pi\frac{lk}{q}} x_k = \sum_{k=0}^{n_p-1} e^{i2\pi\frac{lk}{q}} \sum_{\nu=0}^{j-1} e^{i2\pi\frac{\nu n_p}{q}} = \sum_{k=0}^{n_p-1} e^{i2\pi\frac{lk}{q}} \sum_{\nu=0}^{j-1} e^{i2\pi\frac{\nu n_{p'}}{j}} = 0$$

and $|\Lambda_{p'}| = |\Lambda_p|^j$, so $(1-t_{p'}(Q,\beta)) = (1-t_p^j(Q,0))$.

Cycle p' of the map f^q of type 2 appears simultaneously with q other cycles belonging to the same trajectory of the mapping f (for example, for f^3 fixed points labeled by "001", "010" and "100" appear as three different prime fixed points). For these cycles the quantities ξ_n have the same absolute value, but phases are shifted by $2\pi l/q$:

$$\xi_n = e^{-i2\pi\frac{l(n-1)}{q}}\xi_1, \quad n = 1, ..., q$$

So we have $\sum_n \xi_n = 0$ and $|\xi_1|^2 = ... = |\xi_q|^2$, and as far as only zeta function and its second derivative over β are considered, we can write

$$(1-t_{p'}(Q,\beta)) = (1-t_p(Q,\beta))^q$$

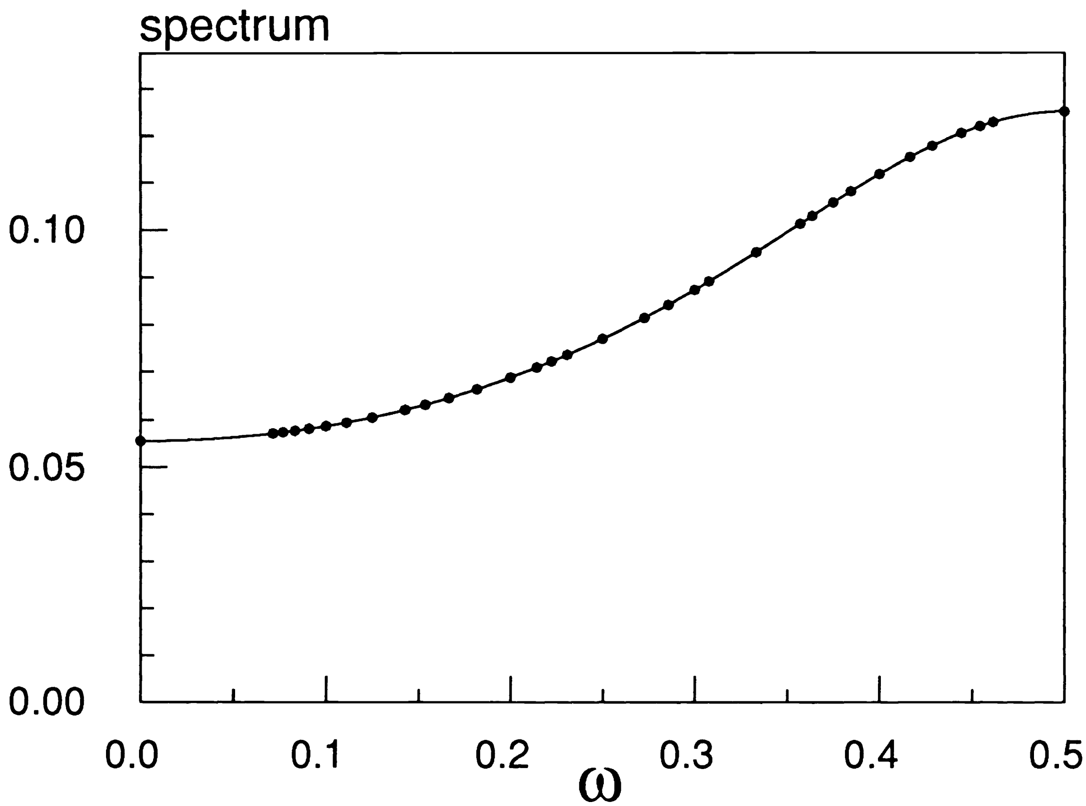

Fig. 1. Power spectrum for skewed tent map (13) with $a = 2.5$ from the Selberg zeta function for all rational frequencies $\omega = l/q$, $q \leq 14$. Solid line - analytical formula (14).

Finally, we can represent zeta function as

$$R^{(q)} = \prod_{p \in \mathcal{P}_q} (1 - t_p(Q, \beta))^q \prod_{p \in \mathcal{P} \setminus \mathcal{P}_q} (1 - t_p^j(Q, 0)), \quad j = \frac{\mathrm{lcm}(n_p, q)}{q} \tag{16}$$

where lcm means least common multiple and $\mathcal{P}_q$ is a set of all prime cycles with periods $q, 2q, 3q....$

As we can see from Eq.(16), that to the power spectrum at frequency $\omega = p/q$ only cycles with periods $q, 2q, ...$ contribute, all cycles with other periods are "filtered out". From the other point of view, cycle with period q gives contribution only to the power spectrum at the frequencies $0, 1/q, 2/q, ..., (q-1)/q$, i.e. at the harmonics of the cycles' "frequency".

We illustrate this section with example of computation of power spectrum for the tent map (13)(Fig.1). In the Fig.2 we illustrate convergence for the Selberg and Ruelle zeta functions.

4.2 More complicated symbolic dynamics

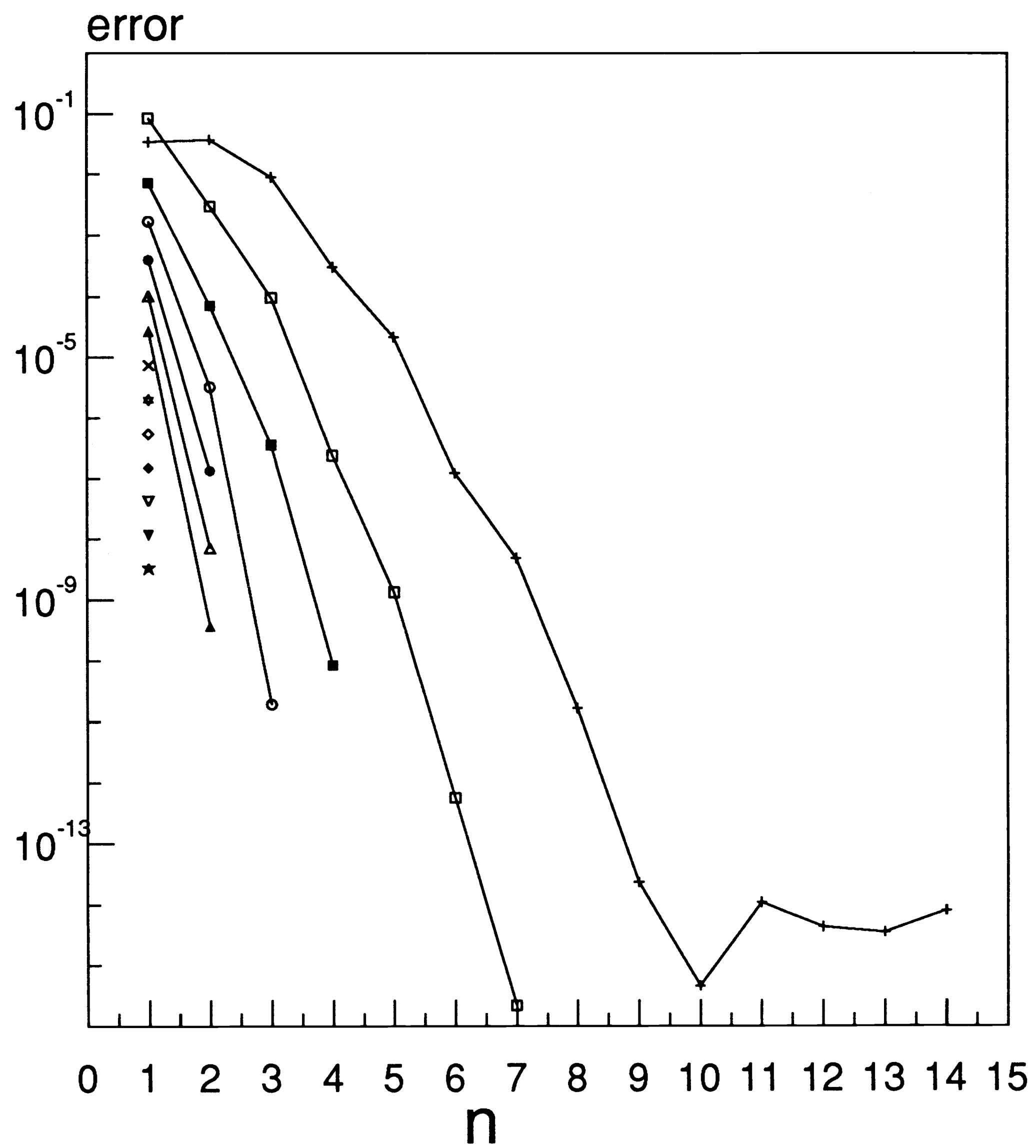

Fig.2. Errors in determining spectrum Fig.1 versus length of maximal used cycle. pluses - $\omega = 0$, squares - $\omega = 1/2$, other symbols from top to bottom correspond to frequencies $1/3,\ 1/4, ..., 1/14$.

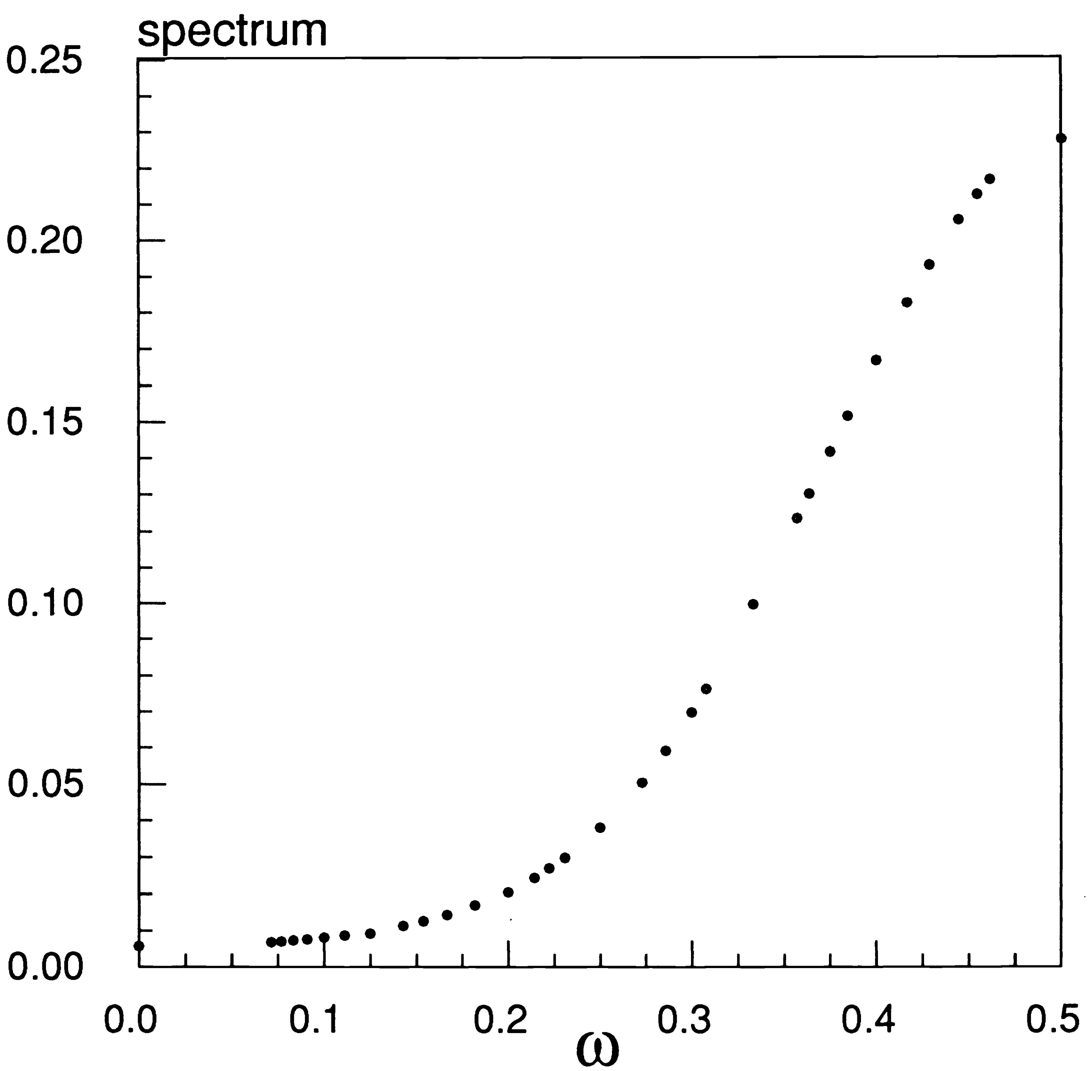

Fig.3. Power spectrum for a pruned symmetric tent map with $a = (1 + \sqrt{5})/2$. No analytic formula for spectrum is available.

It is known that when symbolic dynamics is described by a subshift of a finite type, cycle expansion gives good convergence. Figure 3 shows power spectrum calculation for a symmetric tent map with pruning _00_. In this case the equation (16) works well.

Some difficulties arises when symbolic dynamics leads to qualitative changes in power spectrum. For example, the tent map $x_{t+1} = 1 - a|x_t|$ for $a = \sqrt{2}$ has two nonoverlaping bonds. The system is no more mixing and in the power spectrum a delta – peak at $\omega = 1/2$ appears. In symbolic dynamics only sequences having "1" at all odd or at all even places are allowed. Let us focus our attention on the frequency $\omega = 1/2$ and on the zeta function $R^{(2)}$ (15). Now the set of cycles with odd periods is empty (except for the fixed point "1", and it also disappears for $a < \sqrt{2}$) and

$$R^{(2)} = \prod_{p \in \mathcal{P}_e} (1 - t_p(Q, \beta))^2 \qquad (17)$$

However, straightforward differentiation of $R^{(2)}$ does not give correct result: the drift term vanishes and the diffusion constant diverges. The reason is that the mapping f^2 is not mixing and has two symmetric attractors (indeed the zeta function (17) is a product of two zeta functions for these attractors). the probability distribution function for ξ_t does not tend to Gaussian hump as $t \to \infty$, but instead to two symmetric humps, these humps are drifting away from the origin in opposite directions (that is why the drift term for $R^{(2)}$ vanishes). In order to describe the power spectrum for $\omega = 1/2$ correctly, we must take one hump. This corresponds to considering one of the symmetric attractors of the map f^2. In terms of zeta function this means that we must consider square root of zeta function (17):

$$R^{(2)'} = \prod_{p \in \mathcal{P}_e} (1 - t_p(Q, \beta))$$

From this zeta function correct values of discrete and continuous components of the power spectrum at $\omega = 1/2$ are obtained as the drift term and the diffusion constant.

5. CONCLUSION

We have presented the method for calculating the power spectrum of chaotic motion through the properties of periodic orbits. It is worth to mention that sometime the term "spectrum of chaotic motion" is used in a different sense, meaning the spectrum of eigenvalues of the Frobenius–Perron operator. These eigenvalues have some relation to the power spectrum, because they give the asymptotics of the correlation function. However, because the power spectrum is not an invariant (it depends on the observation function $\phi(x)$), it cannot be calculated exactly from the invariant characteristics of the Frobenius–Perron operator.

6. REFERENCES

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