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THE PLANAR SECTOR OF FIELD THEORIES

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Predrag CVITANOVIĆ

Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

P.G. LAUWERS

NIKHEF, Sectie H, Postbus 41882, NL-1009 DB Amsterdam, The Netherlands

P.N. SCHARBACH

Rutherford Appleton Laboratory, Chilton, Didcot OX11 0QX, UK

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The full field-theoretic apparatus for the description of planar field theories is developed including planar generating functionals and planar Dyson-Schwinger equations. It is shown that planar field theory is characterized by continued fractions rather than exponentials. Planar QCD is studied in detail and the planar Ward identities are derived.

1. Introduction

The planar approximation to QCD, i.e. the $N \to \infty$, g^2N fixed, limit of the SU(N) gauge theory of quarks and gluons, has attracted much interest since its introduction by 't Hooft [1].

To leading N, at the nth order of perturbation theory, only (constant)^n of the number $n!$ of Feynman diagrams survive [2]. At the qualitative level the approximation of retaining only this relatively small subset, that of planar diagrams, meets with considerable phenomenological success [3] and may form a bridge between QCD and the dual string model [1, 4]. In this limit, the Bethe-Salpeter equation can be solved in two space-time dimensions and a meson spectrum of approximately straight Regge trajectories emerges [5]. The summation of the planar diagrams of QCD in two dimensions is facilitated by a property that is unique to the two-dimensional case: an appropriate choice of gauge eliminates the gluon self-couplings. The summation of the large-N limit of four-dimensional QCD, on the other hand, remains a difficult and open problem, for which it is likely that the full apparatus of quantum field theory will be required.

As the first step one has to formulate a planar field theory, i.e. a theory whose perturbation expansion consists only of planar Feynman diagrams. One needs the planar Ward identities to establish gauge invariance and renormalizability and the planar Dyson-Schwinger equations in order to formulate the bound-state equations. Then one has to identify the dominant contributions to binding (leading infrared singularities, for example [6]) and attempt to sum them.

Our present purpose is to develop the formal tools required for carrying out this program; we will give a formulation of planar field theory, the planar QCD program Dyson-Schwinger equations and the planar Ward identities. In a previous work [6] we solved the problem of the division of an arbitrary gauge theory into gauge-invariant sectors. In particular, we showed that the planar sector can be defined for any gauge theory without resorting to the $N \to \infty$ limit. This result motivated us to attempt a direct formulation of *planar field theory*, i.e. a theory whose perturbation expansion consists only of planar Feynman diagrams. Three approaches to this latter problem have been made. In ref. [7] a functional approach based on generating functionals of non-commuting sources was introduced. In ref. [8] a canonical formalism based on conjugate "planar fields" obeying a particular commutation relation is discussed. Finally, ref. [9] describes an alternative functional scheme, in which sources commute and an auxiliary non-relativistic fermion field is introduced to provide the necessary ordering of the external legs of planar Green functions. It is the first of these formulations that we wish to elaborate here.

In sect. 2, we examine the $N \to \infty$ limit of $\mathrm{U}(N)$ Yang-Mills theory. With the aid of several examples we show how the planar sector of the theory is isolated and motivate our prescription for the construction of the generating functionals of the planar theory in terms of non-commuting fields. Sect. 3 is a discussion of the properties of the corresponding sources. In sect. 4 we set up a functional formalism for planar theories and relate the planar connected and one-particle irreducible (1PI) planar Green functions to the full planar Green functions. The solution of the free planar field theory is given. We derive the Dyson-Schwinger equations, which generate the planar perturbation expansion, in sect. 5; $\Phi^3 + \Phi^4$ is considered as an example. In sect. 6 we examine zero-dimensional field theories: the planar Dyson-Schwinger equations are solved and the solution is applied to the problem of counting planar Feynman diagrams. Finally, in sect. 7 we return to planar QCD. We give the Dyson-Schwinger equations for this theory and derive the Ward identities for the planar full Green functions.

2. U(N) planar Yang-Mills theory

It is well known that the set of planar Feynman diagrams form the leading approximation to the SU(N) gauge theory in the large-N limit [1]. In fact, it is an exact result independent of the choice of gauge group that the planar sector of an arbitrary Yang-Mills theory is a gauge-invariant subset of the perturbation series [6]. In this section we use the example of U(N) Yang-Mills theory to develop intuition and motivation for the formulation of planar field theory to be given in the following sections.

Let us consider an $[N \times N]$ hermitian Bose field $(A_\alpha)^a_b$ with an action of the form

$$S[A] = \hat{\gamma}_{\alpha\beta}\mathrm{tr}[A_\beta A_\alpha] + \hat{\gamma}_{\alpha\beta\gamma}\mathrm{tr}[A_\gamma A_\beta A_\alpha] + \hat{\gamma}_{\alpha\beta\gamma\delta}\mathrm{tr}[A_\delta A_\gamma A_\beta A_\alpha] + \cdots . \quad (2.1)$$

Here a, b are colour indices that run from 1 to N, while Greek indices denote all other field dependence: position, Lorentz index, etc. Repeated indices indicate summation over discrete variables and integration over continuous ones. Yang-Mills theories are of the above type. Since the trace is cyclically symmetric, the momentum space vertex factors $\hat{\gamma}$ may be replaced by cyclically symmetric vertices γ, e.g.

$$\gamma_{\alpha\beta\gamma\delta} \equiv (\hat{\gamma}_{\alpha\beta\gamma\delta} + \hat{\gamma}_{\beta\gamma\delta\alpha} + \hat{\gamma}_{\gamma\delta\alpha\beta} + \hat{\gamma}_{\delta\alpha\beta\gamma}) . \quad (2.2)$$

In terms of these cyclically symmetric vertices the action becomes

$$S[A] = \tfrac{1}{2}\gamma_{\alpha\beta}\mathrm{tr}[A_\beta A_\alpha] + \tfrac{1}{3}\gamma_{\alpha\beta\gamma}\mathrm{tr}[A_\gamma A_\beta A_\alpha] + \tfrac{1}{4}\gamma_{\alpha\beta\gamma\delta}\mathrm{tr}[A_\delta A_\gamma A_\beta A_\alpha] + \cdots . \quad (2.3)$$

We shall assert that the action corresponding to the planar sector of the theory is

$$S[A] = \gamma_{\alpha\beta}A_\beta A_\alpha + \gamma_{\alpha\beta\gamma}A_\gamma A_\beta A_\alpha + \gamma_{\alpha\beta\gamma\delta}A_\delta A_\gamma A_\beta A_\alpha + \cdots , \quad (2.4)$$

obtained from the action (2.3) by replacing the traces $\mathrm{tr}[A_\mu \ldots A_\alpha]/m$ by products of non-commuting fields $A_\mu \ldots A_\alpha$. To motivate this statement, let us consider, as an example, the U(N) Yang-Mills action

$$S[A] = -\tfrac{1}{4}\int \mathrm{d}^4x\, \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) , \quad (2.5)$$

with the field strength F expressed in terms of the gauge field A by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. Fourier transforming the fields and explicitly cyclically symmetrizing the momentum-space factors, one can rewrite (2.5) in the form (2.3):

$$\begin{aligned} S[A] = {} & (2g^{\alpha\beta}k_1\cdot k_2 - k_1^\alpha k_2^\beta - k_2^\alpha k_1^\beta)\tfrac{1}{2}\mathrm{tr}[A_\beta(k_2)A_\alpha(k_1)] \\ & + g[g^{\alpha\beta}(k_2-k_1)^\gamma + g^{\beta\gamma}(k_3-k_2)^\alpha + g^{\alpha\gamma}(k_1-k_3)^\beta] \\ & \times \tfrac{1}{3}\mathrm{tr}[A_\gamma(k_3)A_\beta(k_2)A_\alpha(k_1)] \\ & + g^2(2g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma})\tfrac{1}{4}\mathrm{tr}[A_\delta(k_4)A_\gamma(k_3)A_\beta(k_2)A_\alpha(k_1)] . \end{aligned} \quad (2.6)$$

(Momentum integrations and momentum-conserving delta functions have been suppressed.) The Feynman rules are obtained from the action by functional differentiation:

$$\frac{\delta}{\delta(A^{\alpha})^a_b}\cdots\frac{\delta}{\delta(A^{\gamma})^e_f}S[A]\bigg|_{A=0}=\gamma^{a}_{\alpha b},\ldots{}^{e}_{\gamma f}. \tag{2.7}$$

Thus the inverse propagator is derived from the quadratic term in the action:

$$-\Delta^{-1}{}^{a}_{\alpha b,}{}^{c}_{\beta d}=\delta^a_d\delta^c_b\gamma_{\alpha\beta}. \tag{2.8}$$

Similarly, the three-gluon vertex is

$$\frac{\delta^3 S[A]}{\delta(A^{\alpha})^a_b\delta(A^{\beta})^c_d\delta(A^{\gamma})^e_f}\bigg|_{A=0}=\gamma_{\alpha\beta\gamma}\delta^e_d\delta^c_b\delta^a_f+\gamma_{\alpha\gamma\beta}\delta^e_b\delta^a_d\delta^c_f, \tag{2.9}$$

and the four-gluon vertex is given by

$$\begin{aligned}&\frac{\delta^4 S[A]}{\delta(A^{\alpha})^a_b\delta(A^{\beta})^c_d\delta(A^{\gamma})^e_f\delta(A^{\delta})^g_h}\bigg|_{A=0}\\&\quad=\gamma_{\alpha\beta\gamma\delta}\delta^a_h\delta^c_b\delta^e_d\delta^g_f+\gamma_{\alpha\beta\delta\gamma}\delta^a_f\delta^c_b\delta^e_h\delta^g_d+\gamma_{\alpha\gamma\beta\delta}\delta^a_h\delta^c_f\delta^e_b\delta^g_d\\&\qquad+\gamma_{\alpha\delta\beta\gamma}\delta^a_f\delta^c_h\delta^e_d\delta^g_b+\gamma_{\alpha\gamma\delta\beta}\delta^a_d\delta^c_h\delta^e_b\delta^g_f+\gamma_{\alpha\delta\gamma\beta}\delta^a_d\delta^c_f\delta^e_h\delta^g_b.\end{aligned} \tag{2.10}$$

Finally, we add to the action (2.5) the gauge-fixing and ghost contributions

$$\begin{aligned}S[A,C,\bar{C}]=&-\frac{1}{\alpha}k^{\alpha}k^{\beta}\mathrm{tr}[A_{\beta}(k)A_{\alpha}(-k)]-k^2\mathrm{tr}[\bar{C}(k)C(-k)]\\&+gk_3^{\alpha}\mathrm{tr}[\bar{C}(k_3)[A_{\alpha}(k_2),C(k_1)]],\end{aligned} \tag{2.11}$$

where C and $\bar{C}$ are ghost fields and α is the covariant gauge parameter. The ghost-gluon coupling is

$$\frac{\delta^3 S[A,C,\bar{C}]}{\delta C^c_d\delta(A_{\alpha})^a_b\delta\bar{C}^e_f}\bigg|_{A,C,\bar{C}=0}=gk_3^{\alpha}(\delta^e_d\delta^c_b\delta^a_f-\delta^e_b\delta^a_d\delta^c_f). \tag{2.12}$$

The Feynman rules (2.9), (2.10) and (2.12) are represented in fig. 1, where use has been made of the convenient double-line notation [1] for the Kronecker-delta colour factors.

We wish to isolate the leading contributions to a Feynman diagram in the $N\to\infty$ limit. As QCD is believed to be a confining theory only colour singlets are of

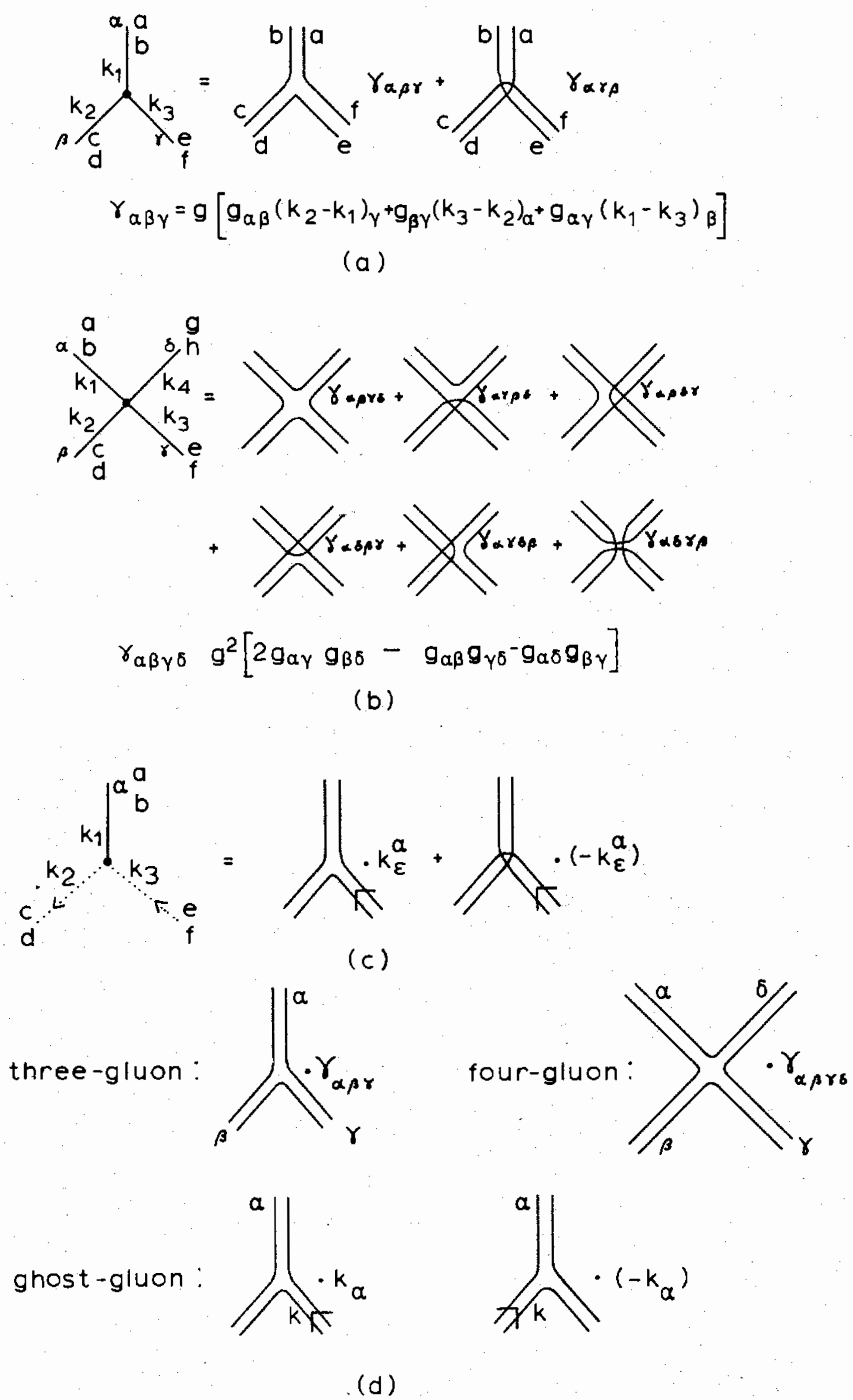
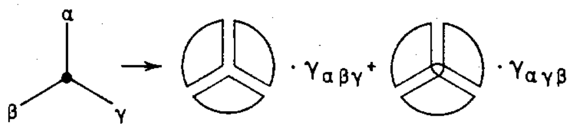


Fig. 1. U(N) vertices in double-line notation. All momenta are directed into the vertex: (a) three-gluon vertex, (b) four-gluon vertex, (c) ghost-gluon vertex; (d) planar vertices for U(N) theory.

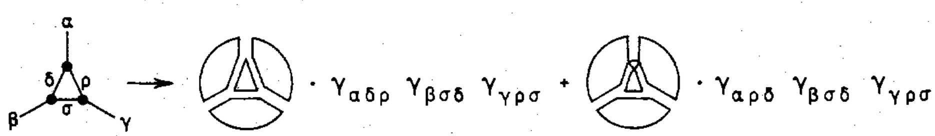
physical relevance. We are therefore interested in diagrams contributing to $\langle \mathrm{tr}(A_\gamma \ldots A_\beta A_\alpha)\rangle$; their colour weights are represented diagrammatically by vacuum bubbles. Consider for example the contribution of the three-gluon vertex:



$$= N^3 \cdot \gamma_{\alpha\beta\gamma} + N \cdot \gamma_{\alpha\gamma\beta}\,. \tag{2.13}$$

For large N, the leading term is given by that part of the vertex (2.9) whose colour flow can be drawn in a plane; we shall refer to this part as the "planar vertex".

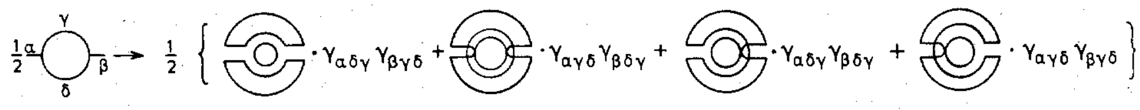
A slightly more complicated example is



+ 6 other nonplanar contributions

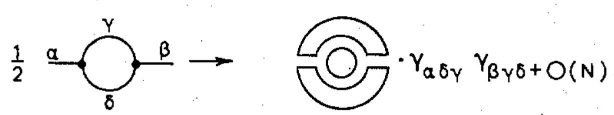
$$= N^4 \cdot \gamma_{\alpha\delta\rho}\gamma_{\beta\sigma\delta}\gamma_{\gamma\rho\sigma} + \mathrm{O}(N^2)\,. \tag{2.14}$$

Again, only the planar vertex contributes to leading order in N. However, one can easily find examples where non-planar parts of the vertices also contribute to the leading order term:



(2.15)

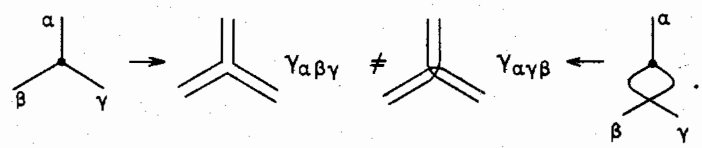
The first two terms, one planar and the other with two non-planar vertices, are of order N^3. The second "non-planar" contribution can, however, be deformed into the planar diagram, i.e. they have identical colour and momentum factors, with the result



(2.16)

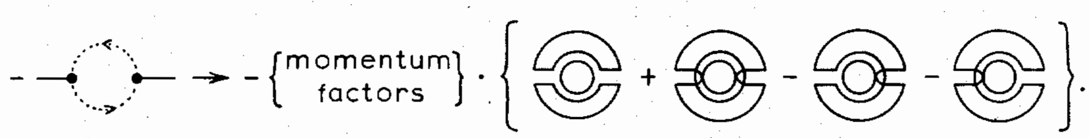
The Bose symmetry factor has been cancelled. We could have immediately obtained the leading result by retaining only the planar part of the 3-gluon vertex and dropping the combinatoric factors.

One can convince oneself by studying further examples that this is a general feature of the theory: one obtains the leading contribution by dropping the combinatoric factors and keeping just the planar part of the vertices. In general there are no combinatoric weights in planar field theories. Such factors are a manifestation of the Bose symmetry of the complete theory. At leading order in N, only the planar part of the vertices is retained. As we have seen earlier, the planar sector is only cyclically symmetric; the Bose symmetry of the theory is lost. Thus planar vertices are rigid in the plane. For example, the 3-gluon vertex of the planar theory is not symmetric under the interchange of legs:



(2.17)

We must also consider diagrams with ghost loops. The simplest example is the one-loop ghost contribution to the gluon two-point function:



(2.18)

The first two terms are identical and are of order N^3, the remaining two are $O(N)$. Thus the leading terms can be obtained directly by introducing two different rigid planar vertices, as is shown in fig. 1d. There we have tabulated the $U(N)$ planar vertices; the rules reproduce the leading-N part of the perturbation expansion [1].

In summary, taking the large-N limit of a $U(N)$ gauge theory reduces the theory to its planar sector, whose characteristic features are:

(i) all diagrams can be drawn in a plane with gluon legs ordered around the edge;

(ii) all vertices are rigid; legs cannot be interchanged;

(iii) there are no combinatoric factors.

Our purpose now is to develop a general formalism for planar perturbation theory embodying these features. In the case of Yang-Mills theory, the planar perturbative expansion should coincide automatically with that found by the large-N limit of the complete theory.

We shall consider a theory with action S to be defined by its perturbation expansion, the set of all Feynman diagrams constructed from the propagators and vertices of S. The planar sector of the theory consists of the subset of diagrams which can be drawn in a plane with no overlapping of lines. The key to the construction of the planar field theory is the observation that the constraint of planarity requires that Green functions possess no symmetry under the interchange of external legs: crossing the legs of a planar diagram destroys its planarity. Thus external sources introduced in the definition of generating functionals for planar

Green functions neither commute (as in the complete Bose theory, whose Green functions are symmetric under the interchange of external legs) nor anticommute (as in the complete Fermi theory, whose Green functions are antisymmetric in external legs). The ordering of non-interchangeable legs in planar Green functions will be enforced by the introduction of non-c-number sources, which we discuss in the next section.

3. Non-commuting sources

We shall set up our formulation of planar field theory in analogy with the usual functional formalism of the complete theory [10]. In complete bosonic theories, Green functions are symmetric under the interchange of external legs. In that case the full Green functions are generated by the expansion of the functional $Z[j]$ in terms of commuting c-number sources j:

$$Z[j]=\sum_{m=0}^{\infty}\frac{1}{m!}Z_{ij\dots m}j_m\cdots j_j j_i. \tag{3.1}$$

Here the indices $i, j, \dots m$ represent all discrete and continuous variables specifying the state of an external particle. Throughout this paper, repeated indices will denote summation over discrete variables and integration over continuous ones. The combinatorial factor $1/m!$ prevents overcounting when Green functions are recovered by ordinary functional differentiation:

$$Z_{ij\dots k}=\frac{\delta}{\delta j_i}\cdots\frac{\delta}{\delta j_k}Z[j]\bigg|_{j=0}. \tag{3.2}$$

In complete theories with Fermi statistics, the Green functions are antisymmetric under the interchange of external legs. Again, the full Green functions can be generated by a generating functional $Z[j]$ and the properties of the sources must reflect the antisymmetric character of the Green functions. For that reason the sources are anticommuting Grassman variables,

$$\{j_i, j_k\}=0. \tag{3.3}$$

Eqs. (3.1) and (3.2) remain valid if differentiation with respect to anticommuting sources is defined by

$$\frac{\delta}{\delta j_i}j_k=\delta_{ik}-j_k\frac{\delta}{\delta j_i},$$

$$\frac{\delta}{\delta j_i}\frac{\delta}{\delta j_k}=-\frac{\delta}{\delta j_k}\frac{\delta}{\delta j_i}. \tag{3.4}$$

We adopt a similar strategy for planar theories. In these theories Green functions are neither symmetric nor antisymmetric under interchange of two external legs, they do not in general have any kind of symmetry under interchange of two external legs because interchanging two legs destroys the planarity of the Green function.

The sources of the planar theory must therefore reflect this non-commuting property. The planar full Green functions are generated by the expansion of the planar functional $Z[j]$ in terms of the planar non-commuting sources j:

$$Z[j] = \sum_{m=0}^{\infty} Z_{ij\ldots m} j_m \ldots j_j j_i . \tag{3.5}$$

Differentiation with respect to these non-commuting sources is defined by

$$\frac{\delta}{\delta j_i}\left(a j_j j_k \ldots j_m\right) = a \delta_{ij} j_k \ldots j_m ,$$

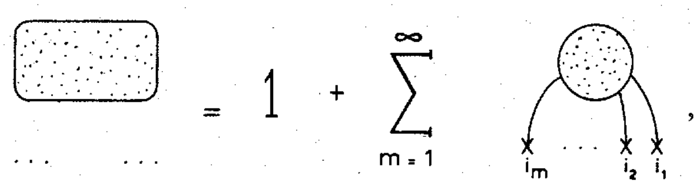
$$\frac{\delta}{\delta j_i} a = 0 , \tag{3.6}$$

where a is any c-number numerical constant. From now on we shall call quantities with this non-commuting property non-c-numbers. The planar full Green functions are obtained by differentiating $Z[j]$:

$$Z_{ij\ldots k} = \frac{\delta}{\delta j_i}\frac{\delta}{\delta j_j}\cdots\frac{\delta}{\delta j_k} Z[j]\bigg|_{j=0} . \tag{3.7}$$

It is obvious that because of the above definition of differentiation with respect to non-commuting sources, there is no combinatorial factor $1/m!$ in eq. (3.5).

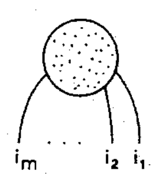
Eq. (3.5) may be represented diagrammatically as



(3.5′)

where a cross at the end of an external line i_k represents contraction of the Green function

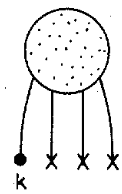
$Z_{i_1 i_2 \ldots i_m}$ =



(3.7′)

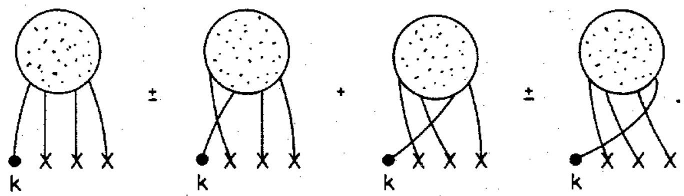
with the source j_{i_k}. (In our ordering convention the legs are labelled clockwise around a Green function.) $\delta/\delta j$ is the operation of pulling out the left-most leg in the expansion (3.5′). The non-c derivative (3.6) has been introduced to ensure that only planar diagrams contribute. For example, in the planar theory

$$\frac{\delta}{\delta j_k}\left(Z_{mnpq}j_q j_p j_n j_m\right) =$$



while in the complete theory differentiation yields non-planar diagrams as well:

$$\frac{\delta}{\delta j_k}\left(Z_{mnpq}j_q j_p j_n j_m\right) =$$



Here $+$ $(-)$ sign is to be taken for the Bose (Fermi) theory. From the definition of the differentiation (3.6), it follows that the operator $j_i\delta/\delta j_i$ projects out the non-c-number part of the functional $f[j]$:

$$j_i\frac{\delta}{\delta j_i}f[j] = f[j] - f[0]. \tag{3.8}$$

The derivative of a product of two functionals of non-c-number variables is

$$\frac{\delta}{\delta j_i}\{f[j]g[j]\} = \frac{\delta f[j]}{\delta j_i}\cdot g[j] + f[0]\cdot\frac{\delta g[j]}{\delta j_i}. \tag{3.9}$$

Note that the ordering of the non-c-number functionals must be preserved. A chain rule can also be formulated. Let f be an implicit functional of j: $f = f[g]$, $g = g[j]$. Then, expanding f and g in terms of their non-c-number argument,

$$f[g] = f_{\alpha\beta\ldots\varepsilon}g_\varepsilon\cdots g_\beta g_\alpha,$$

$$g_\alpha[j] = g_\alpha^{km\ldots p}j_p\cdots j_m j_k,$$

we have

$$\frac{\delta f[g]}{\delta j_i} = \frac{\delta g_\alpha}{\delta j_i}\cdot\frac{\delta f[g]}{\delta g_\alpha}, \tag{3.10}$$

provided that $g_\alpha[j]$ does not contain c-number terms.

After these preliminaries we are now in a position to discuss the planar perturbation theory.

4. Planar generating functionals

We begin by relating the generating functionals for planar connected Green functions,

$$W[J]=\sum_{m=1}^{\infty} W_{i_1 i_2 \dots i_m} J_{i_m} \dots J_{i_2} J_{i_1}, \tag{4.1}$$

to the generating functional of full Green functions $Z[j]$ introduced in sect. 3. Denoting the connected Green function by a hatched blob, the relation between Z and W is represented by fig. 2. That is, a given leg enters a connected diagram whose other legs are separated by disconnected parts. The functional statement of fig. 2 is

$$Z[j]=1+W[jZ[j]], \tag{4.2a}$$

or

$$Z[j]=1+W[Z[j]j]. \tag{4.2b}$$

Because of planarity, each leg of the connected piece is followed by all possible disconnected diagrams, and it is convenient to define the product $J_i \equiv j_i Z[j]$ as the (non-c-number) source for the connected functional $W[J]$. From now on a hatched blob will stand for a connected piece with the J_i as sources. The derivatives with respect to j_i and J_i are related by the chain rule (3.10):

$$\frac{\delta}{\delta j_i}=\frac{\delta J_k}{\delta j_i}\cdot\frac{\delta}{\delta J_k}=Z[j]\frac{\delta}{\delta J_i}=(1+W[J])\frac{\delta}{\delta J_i}. \tag{4.3}$$

In particular, we have

$$\frac{\delta}{\delta j_i}Z[j]=Z[j]\frac{\delta W[J]}{\delta J_i}, \tag{4.4}$$

which may be represented as in fig. 3. For an arbitrary number of derivatives (4.4)

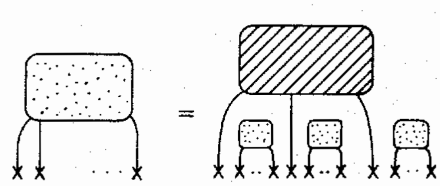


Fig. 2. Relation between full and connected planar Green function, eq. (4.2). Full Green functions are denoted by dotted blobs, connected ones by hatched blobs.

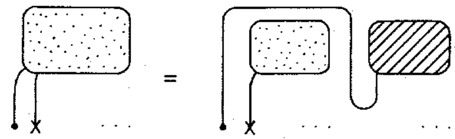


Fig. 3. Extracting a leg from a full planar Green function, eq. (4.3).

generalizes to

$$\frac{\delta}{\delta j_i}\cdots\frac{\delta}{\delta j_m}\frac{\delta}{\delta j_n}Z[j]=Z[j]\left(\frac{\delta W[J]}{\delta J_i}+\frac{\delta}{\delta J_i}\right)\cdots\left(\frac{\delta W[J]}{\delta J_m}+\frac{\delta}{\delta J_m}\right)\frac{\delta W[J]}{\delta J_n}. \quad (4.5)$$

Eq. (4.4) is formally similar to the relation

$$\frac{\delta Z[j]}{\delta j_i}=Z[j]\frac{\delta W[j]}{\delta j_i} \quad (4.6)$$

of the complete theory [10] for which the sources j are c-numbers and $\delta/\delta j$ is the usual functional derivative. The content, however, is quite different. Integration of (4.6) gives an exponential relation $Z=\exp[W]$ between full and connected generating functionals. This relation has a very different form in the planar theory, as we can see from the free field case. The solution of the free planar field theory follows directly from (4.2). In the absence of interactions there is only one connected diagram,

$$W[J]=\Delta_{ik}J_kJ_i, \quad (4.7)$$

where Δ is the Feynman propagator, so that the generating functional for full Green functions is

$$\begin{aligned} Z[j]&=1+W[jZ[j]]\\ &=1+\Delta_{ik}j_kZ[j]j_iZ[j]\\ &=\{1-\Delta_{ik}j_kZ[j]j_i\}^{-1} \end{aligned} \quad (4.8)$$

in terms of the expansion $(1-x)^{-1}=1+x+x^2\ldots$. Iteration generates the free planar theory in the form of a continued fraction:

$$Z[j]=1+\Delta_{ik}j_kj_i+(\Delta_{ij}\Delta_{km}+\Delta_{im}\Delta_{jk})j_mj_kj_jj_i+\cdots$$

$$=\cfrac{1}{1-j_i\cfrac{\Delta_{ki}}{1-j_l\cfrac{\Delta_{ml}}{1-j_p\cfrac{\Delta_{qp}}{\vdots}j_q}j_m}j_k} \quad (4.9)$$

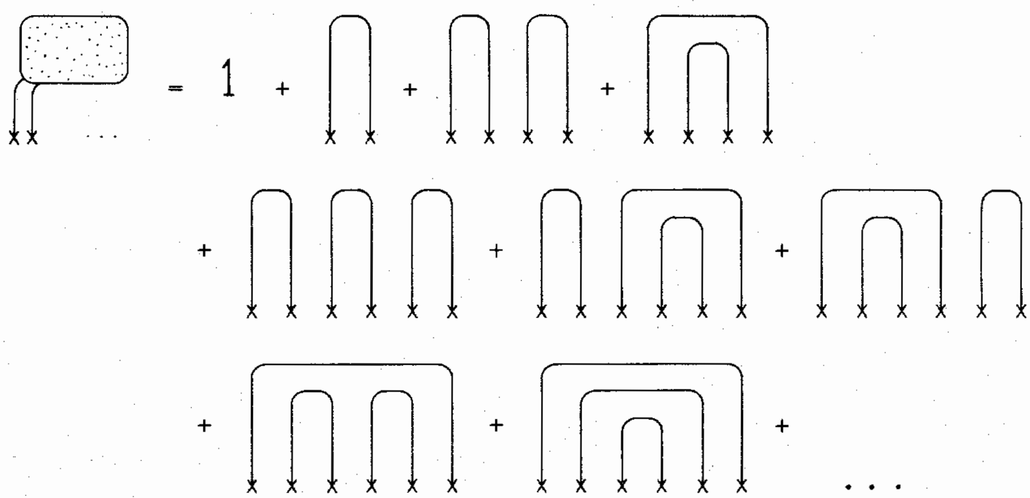


Fig. 4. Full Green function for free planar theory, eq. (4.9).

Diagrammatically this is represented by fig. 4. In contrast, for the complete bosonic free field theory,

$$Z[j]=\exp\left(\tfrac{1}{2}j_i\Delta_{ki}j_k\right).$$

The generating functional $\Gamma[\Phi]$ of 1PI planar Green functions (proper vertices) may now be constructed from $W[J]$. We define a field Φ by

$$\Phi_i\equiv\frac{\delta W[J]}{\delta J_i}. \tag{4.10}$$

A leg entering a connected diagram must either end on a source J_k or a proper tadpole Γ_k, or may enter connected parts of the diagram through a proper self-energy Π or proper vertices Γ, as in fig. 5. Additional (suppressed) sources J are hidden in the fields $\Phi_i[J]$ defined by (4.10), so the fields Φ_i are non-c-number quantities. Fig. 5 may be stated as

$$\Phi_i=\Delta_{ki}(J_k+\Gamma_k+\Pi_{mk}\Phi_m+\Gamma_{nmk}\Phi_m\Phi_n+\cdots). \tag{4.11}$$

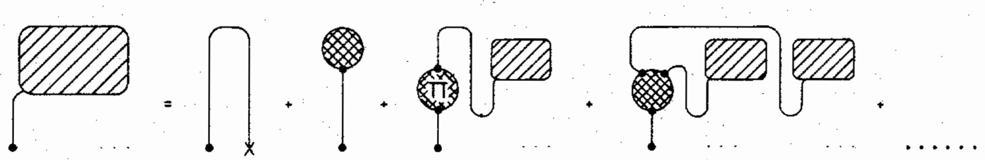


Fig. 5. Expansion of a connected Green function in terms of 1PI Green functions, denoted by cross-hatched blobs, eq. (4.11). All sources J_i are implicitly contained in the connected Green functions (hatched blobs).

Now an "interaction" functional $\Gamma_{\rm I}[\Phi]$ can be defined:

$$\Gamma_{\rm I}[\Phi] \equiv \Gamma_i\Phi_i + \Pi_{ij}\Phi_j\Phi_i + \Gamma_{ijk}\Phi_k\Phi_j\Phi_i + \cdots . \tag{4.12}$$

In terms of $\Gamma_{\rm I}[\Phi]$ eq. (4.11) takes the form

$$\Phi_i = \Delta_{ki}\left(J_k + \frac{\delta\Gamma_{\rm I}[\Phi]}{\delta\Phi_k}\right). \tag{4.13}$$

The iteration of (4.13) yields an expansion of Φ in terms of dressed planar trees. As the right-hand side of this equation contains a term linear in Φ, it is convenient to define the effective action

$$\Gamma[\Phi] \equiv \Gamma_{\rm I}[\Phi] - \Delta_{ki}^{-1}\Phi_i\Phi_k, \tag{4.14}$$

which obeys the equation

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi_i} + J_i = 0. \tag{4.15}$$

As in the complete theory, transforming from $\Gamma_{\rm I}$ to Γ amounts to summing proper self-energy insertions into full propagators $W_{ij} = \Delta_{ik}\Sigma[(\Pi\Delta)^n]_{kj}$ on external legs.

The Legendre transformation between $W[J]$ and $\Gamma[\Phi]$ follows directly from (4.1), (4.10) and (4.15):

$$J_i\Phi_i = J_i\frac{\delta W[J]}{\delta J_i} = W[J],$$

$$\Phi_i J_i = -\Phi_i\frac{\delta\Gamma[\Phi]}{\delta\Phi_i} = -\Gamma[\Phi]. \tag{4.16}$$

Hence the Legendre transformation is given by

$$W[J] = \Gamma[\Phi] + J_i\Phi_i + \Phi_i J_i. \tag{4.17}$$

The only formal difference from the Legendre transformation of the complete theory is that because of the non-c-number character of sources J_i and fields Φ_i, the right-hand side of (4.17) now contains the symmetric sum $J_i\Phi_i + \Phi_i J_i$.

The relations between the connected and proper n-point functions are derived from (4.10), (4.15) and (4.17). In general these relations are made tedious by proliferation of tadpoles. However, if the tadpoles vanish ($\Gamma_i = 0$, $W_i = 0$), the simple chain rule (3.10) can be used. In the remaining part of this section we shall limit the discussion to this case. (QCD belongs to this category.) Differentiating (4.10) gives

$$0=\frac{\delta}{\delta\Phi_k}\left\{\frac{\delta W[J]}{\delta J_i}-\Phi_i\right\}=\frac{\delta J_m}{\delta\Phi_k}\cdot\frac{\delta^2 W[J]}{\delta J_m \delta J_i}-\delta_{ik}, \tag{4.18}$$

while differentiation of (4.15) gives

$$\frac{\delta J_m}{\delta\Phi_k}=-\frac{\delta^2\Gamma[\Phi]}{\delta\Phi_k\delta\Phi_m}. \tag{4.19}$$

Now, defining $\Gamma[\Phi]_{km}\equiv\delta^2\Gamma/\delta\Phi_k\delta\Phi_m$, etc., we obtain from eqs. (4.18) and (4.19)

$$\Gamma[\Phi]_{km}\cdot W[J]_{mi}+\delta_{ik}=0. \tag{4.20}$$

Similarly, differentiation of eqs. (4.10) and (4.15) with respect to the source J gives

$$W[J]_{km}\cdot\Gamma[\Phi]_{mi}+\delta_{ki}=0. \tag{4.21}$$

We conclude from eqs. (4.20) and (4.21) that $-\Gamma[\Phi]_{mi}$ is the inverse of $W[J]_{km}$, the full planar propagator in the presence of external sources.

Indeed, we may return to the "interaction" functional, eq. (4.12), and make the replacement

$$\Gamma[\Phi]_{km}=-\Delta^{-1}_{km}+\Pi[\Phi]_{km},$$

where $\Pi[\Phi]_{km}\equiv\Gamma_{\mathrm{I}}[\Phi]_{km}$.

Using eq. (4.13), we may express $W[J]_{km}$ as a geometric series of proper self-energy insertions:

$$\begin{aligned} W[J]_{ij}&=(\Delta+\Delta\Pi[\Phi]\Delta+\Delta\Pi[\Phi]\Delta\Pi[\Phi]\Delta+\cdots)_{ij}\\ &=\left(\frac{1}{\Delta^{-1}-\Pi[\Phi]}\right)_{ij}. \end{aligned}$$

The rest of this section describes the expansion of connected Green functions in terms of proper vertices and full planar propagators.

As a first step toward this goal we shall derive a useful simple relation. We take the functional derivative with respect to Φ_s of both sides of (4.21) and using (3.9) we obtain

$$W[0]_{km}\Gamma[\Phi]_{smi}+\frac{\delta W[J]_{km}}{\delta\Phi_s}\Gamma[\Phi]_{mi}=0,$$

or, using eq. (4.20),

$$\frac{\delta W[J]_{kr}}{\delta \Phi_s} = W[0]_{km}\Gamma[\Phi]_{smi}W[J]_{ir}, \tag{4.22}$$

where $W[0]_{km}$ is the c-number part of the full propagator $W[J]_{km}$. Now we can express all functional derivatives with respect to the sources J_i in terms of propagators and functional derivatives with respect to the fields Φ_i. Using the chain rule (3.10) we obtain:

$$\frac{\delta}{\delta J_i} = \frac{\delta \Phi_m}{\delta J_i}\cdot\frac{\delta}{\delta \Phi_m} = W[J]_{im}\frac{\delta}{\delta \Phi_m}. \tag{4.23}$$

The content of (4.23) is clear: in the transition from connected to 1PI Green functions, all insertions on external legs are factorized.

All higher derivatives can be expressed in a similar fashion by repeated use of (3.9), (4.22), and (4.23). The simplest example is

$$\begin{aligned}\frac{\delta}{\delta J_k}\frac{\delta}{\delta J_i} &= W[J]_{kl}\frac{\delta}{\delta \Phi_l}W[J]_{im}\frac{\delta}{\delta \Phi_m}\\ &= W[J]_{kl}W[0]_{is}\Gamma[\Phi]_{lst}W[J]_{tm}\frac{\delta}{\delta \Phi_m}\\ &\quad + W[J]_{kl}W[0]_{im}\frac{\delta}{\delta \Phi_l}\frac{\delta}{\delta \Phi_m}.\end{aligned} \tag{4.24}$$

It is now straightforward to obtain the desired expansions. One expands the derivatives of the fields, eq. (4.10), with respect to the sources J as in (4.24) and sets the remaining sources and fields equal to zero. For example, from (4.24) we have

$$\frac{\delta^3 W[J]}{\delta J_k \delta J_i \delta J_l} = W[J]_{kj}W[0]_{im}\Gamma[\Phi]_{jmn}W[J]_{nl}, \tag{4.25}$$

as illustrated in fig. 6a, from which follows the expression for the connected 3-point function in terms of the proper vertex with dressed legs, fig. 6b. Note the appearance of the c-number full propagator $W[0]_{im}$ rather than the non-c-number functional $W[J]_{im}$. This guarantees that the next external leg can be inserted only in planar fashion. Thus a further derivative of (4.25) gives

$$\begin{aligned}\frac{\delta^4 W[J]}{\delta J_s \delta J_k \delta J_i \delta J_l} &= W[J]_{skj}W[0]_{im}\Gamma[\Phi]_{jmn}W[J]_{nl}\\ &\quad + W[0]_{kj}W[0]_{im}W[J]_{sr}\Gamma[\Phi]_{rjmn}W[J]_{nl}\\ &\quad + W[0]_{kj}W[0]_{im}\Gamma[0]_{jmn}W[J]_{snl},\end{aligned} \tag{4.26}$$

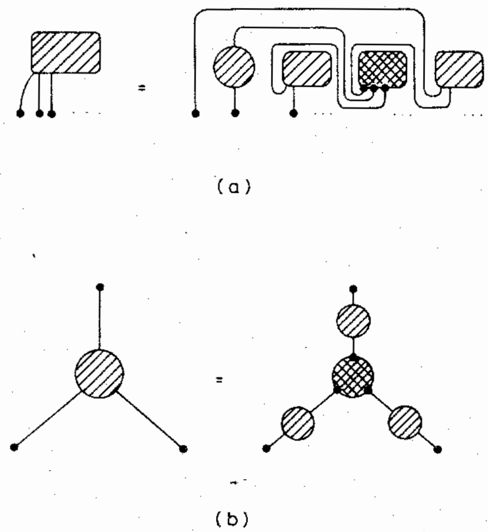


Fig. 6. (a) Extraction of three legs from a generating functional for the connected planar Green functions eq. (4.25). A circular (oblong) blob indicates the c-number Green function (the generating functional). (b) Relation between connected and proper planar three-point functions.

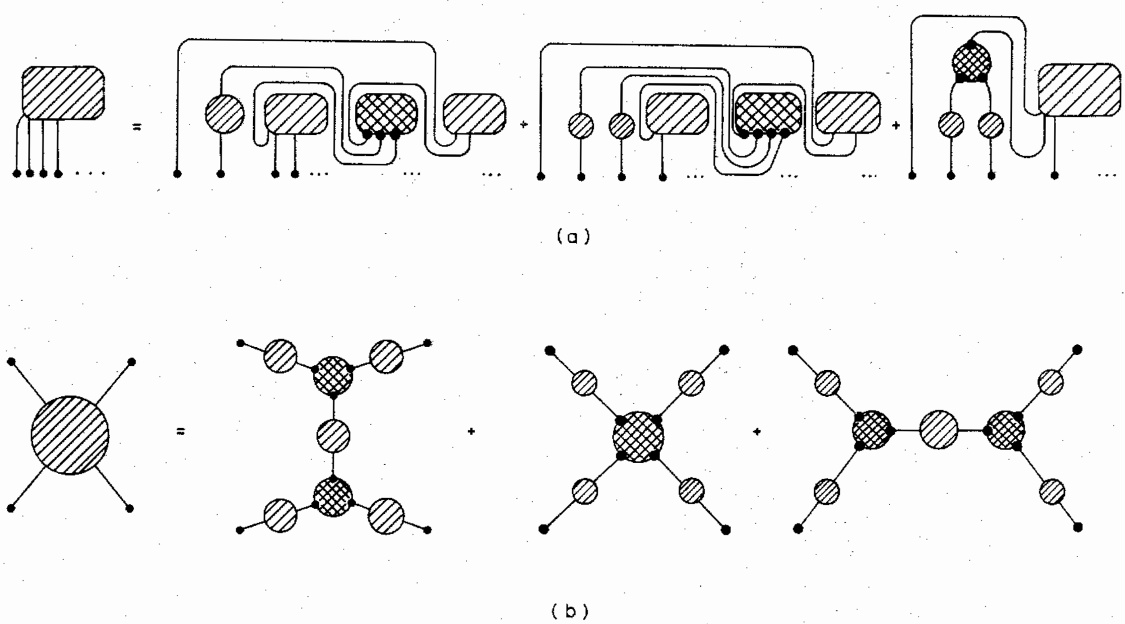


Fig. 7. (a) Extracting four legs from a connected planar generating functional eq. (4.26). A circular blob stands for a Green function. (b) Relation between connected and proper planar four-point functions. In (a) and fig 6a we have drawn the generating functionals in one-to-one correspondence with the algebraic notation (4.26) and (4.25). This is convenient for keeping track of the (ordered) planar derivatives and the fermionic signs. One can also draw the diagrammatic relations in the conventional way, i.e. think of the blobs in figs. 6b and 7b as planar generating functionals, with all sources ordered around the periphery of the diagram. In that notation the planar derivative acts by pulling a leg from all blobs on the edge of a given sector. For example, the three terms of b are obtained by inserting a leg into fig. 6b in all possible planar ways.

as shown in fig. 7a. The corresponding expansion of the 4-point function, fig. 7b, includes, indeed, only planar terms.

5. Planar Dyson-Schwinger equations

It is straightforward to write down the Dyson-Schwinger equations (DSE) which generate the full planar perturbation expansion. As an illustration we take $\Phi^3 + \Phi^4$ theory and examine the effect of extracting a leg from a full Green function. The leg may either terminate on a source or enter a vertex as is shown diagrammatically in fig. 8. Thus

$$\frac{\delta Z[j]}{\delta j_i} = \Delta_{ki}\left\{ Z[j]\, j_k + \gamma_{nlk}\frac{\delta}{\delta j_l}\frac{\delta}{\delta j_n} + \gamma_{mnlk}\frac{\delta}{\delta j_l}\frac{\delta}{\delta j_n}\frac{\delta}{\delta j_m}\right\} Z[j], \tag{5.1}$$

where γ denotes a rigid planar vertex. We remind the reader that the indices i, j,... stand for all field dependence, such as position, Lorentz indices, flavour, etc.

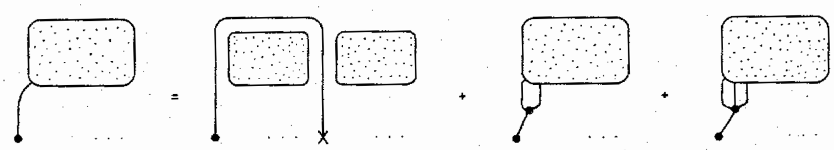
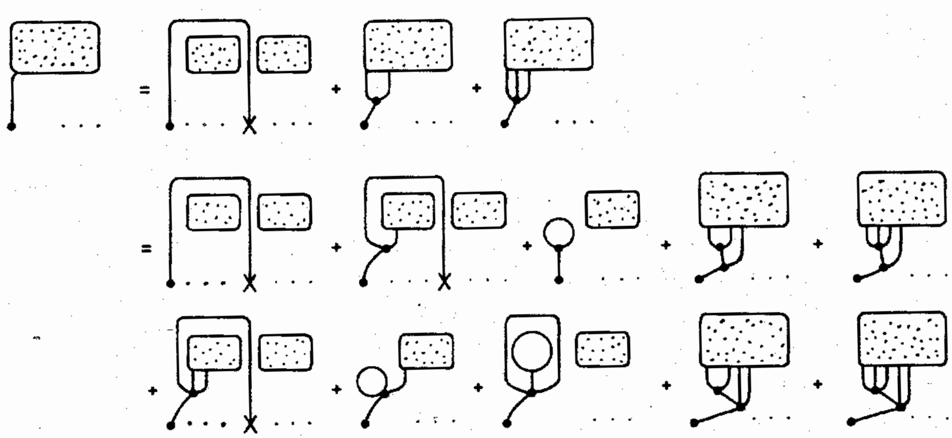


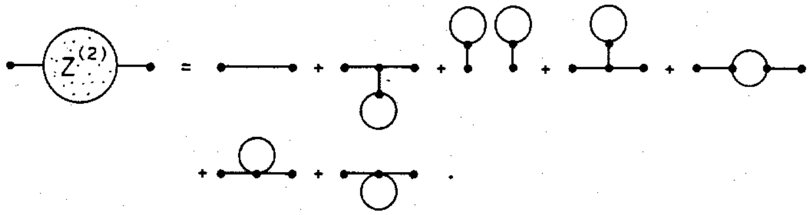
Fig. 8. Dyson-Schwinger equation for the generating functional of the full planar Green functions in the case of the $\Phi^3 + \Phi^4$ theory, eq. (5.1).

Iteration of this equation generates the planar sector of the theory's perturbation expansion. As an example we derive the $O(g^2)$ expansion of the planar two-point function. Repeated use of the DSE, eq. (5.1) gives



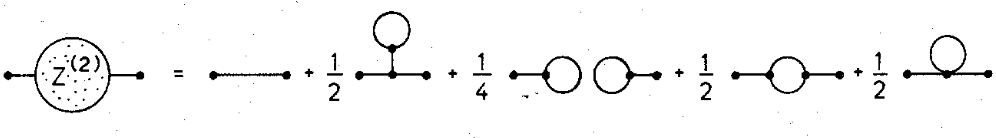
= etc. (5.2)

So to order g^2, the planar two-point function is



(5.3)

It is instructive to compare this result with the equivalent expansion in the complete bosonic theory:



(5.4)

As in our earlier example (2.14), planar Feynman diagrams have no combinatoric weights. Recall that the Feynman diagrams of the planar sector are constructed from the non-deformable planar parts of the vertices of the complete theory; we see once again that, in terms of these, the combinatoric factors of the complete theory cancel in the planar sector.

A planar action $S[\Phi]$ can also be defined:

$$S[\Phi] = -\Delta_{ik}^{-1}\Phi_k\Phi_i + \gamma_{mlk}\Phi_k\Phi_l\Phi_m + \gamma_{nmlk}\Phi_k\Phi_l\Phi_m\Phi_n, \tag{5.5}$$

with γ the planar vertices as in eq. (5.1). The DSE can then be rephrased as the equation of motion for $Z[j]$:

$$\left\{\frac{\delta S[\delta/\delta j]}{\delta \Phi_i} + Z[j]j_i\right\}Z[j] = 0. \tag{5.6}$$

The starting point for the derivation of the DSE for planar connected Green function are eqs. (4.2) and (5.1). Remembering the definition $J_i \equiv j_i Z[j]$ and using (4.5), we can rewrite (5.1) as

$$\begin{aligned}\frac{\delta W[J]}{\delta J_i} = \Delta_{ki}\Bigg\{ J_k &+ \gamma_{nlk}\left[\frac{\delta^2 W[J]}{\delta J_l \delta J_n} + \frac{\delta W[J]}{\delta J_l}\frac{\delta W[J]}{\delta J_n}\right] \\ &+ \gamma_{mnlk}\left[\frac{\delta W[J]}{\delta J_l}\frac{\delta W[J]}{\delta J_n}\frac{\delta W[J]}{\delta J_m} + \frac{\delta W[J]}{\delta J_l}\frac{\delta^2 W[J]}{\delta J_n \delta J_m}\right. \\ &\left. + \frac{\delta^2 W[J]}{\delta J_l \delta J_n}\frac{\delta W[J]}{\delta J_m} + \frac{\delta^3 W[J]}{\delta J_l \delta J_n \delta J_m} + \frac{\delta W[0]}{\delta J_n}\frac{\delta^2 W[J]}{\delta J_l \delta J_m}\right]\Bigg\},\end{aligned} \tag{5.7}$$

which is the DSE for the planar connected Green functions.

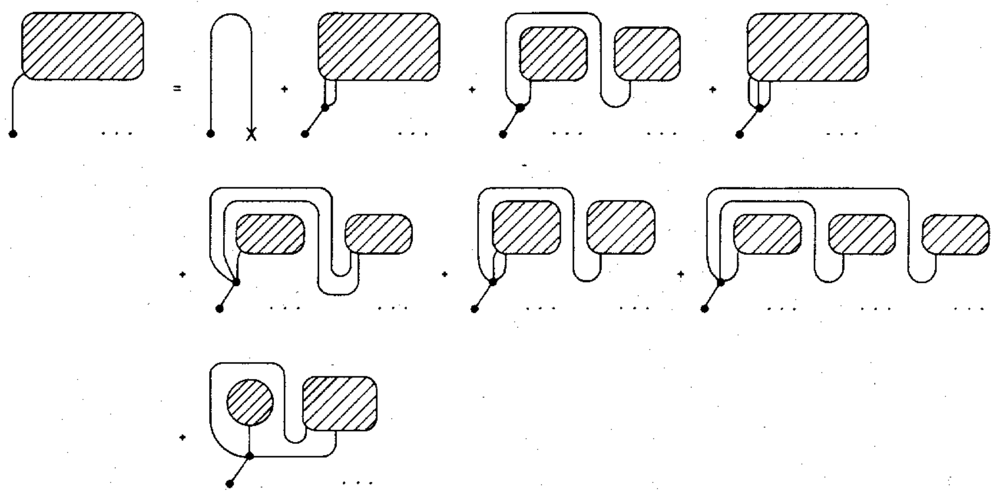


Fig. 9. Dyson-Schwinger equation for the generating functional of the connected planar Green functions in the case of the $\Phi^3 + \Phi^4$ theory, eq. (5.7).

The physical content of this equation becomes obvious in its diagrammatic representation, fig. 9: a line enters a connected Green function to end on an external source or a bare vertex whose remaining legs continue into a connected part or separate connected parts. The generalization to the case of an arbitrary planar action $S[\Phi]$ is

$$J_i + \frac{\delta}{\delta\Phi_i} S\left[\frac{\delta W[J]}{\delta J} + \frac{\delta}{\delta J}\right] = 0. \tag{5.8}$$

It is now a simple task to derive the DSE for planar amputated 1PI Green functions, at least for the theories for which tadpoles vanish ($\Gamma_i = 0$ and $W_i = 0$). We apply the transformations (4.10), (4.15) and (4.23) to eq. (5.8):

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi_i} = \frac{\delta}{\delta\Phi_i} S\left[\Phi_k + W[J]_{kj}\frac{\delta}{\delta\Phi_j}\right]. \tag{5.9}$$

The meaning of this equation may be illustrated by considering the specific case of the $\Phi^3 + \Phi^4$ planar theory. In this case eq. (5.9) takes the form

$$\begin{aligned}\frac{\delta\Gamma_{\mathrm{I}}[\Phi]}{\delta\Phi_i} = {} & \gamma_{mli}(\Phi_l\Phi_m + W[J]_{lm}) \\ & + \gamma_{nmli}(\Phi_l\Phi_m\Phi_n + \Phi_l W[J]_{mn} + W[J]_{lm}\Phi_n \\ & + W[J]_{lk}W[0]_{mt}\Gamma[\Phi]_{ktu}W[J]_{un}),\end{aligned} \tag{5.10}$$

for the interaction functional $\Gamma_{\mathrm{I}}[\Phi]$. The effect of the $W[J]$ terms is to create dressed loops as illustrated in fig. 10.

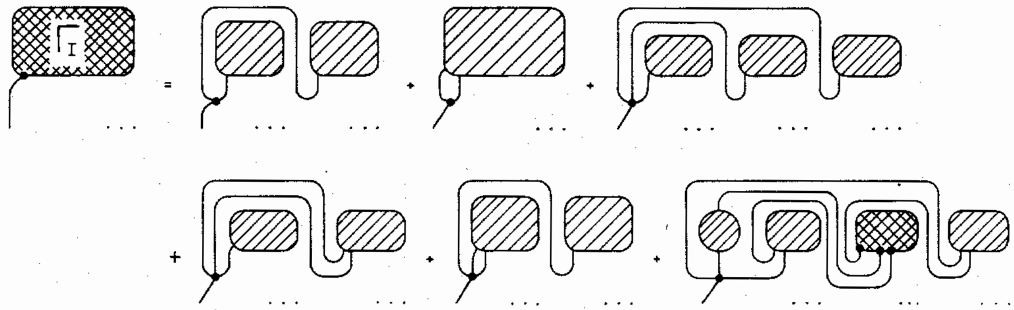


Fig. 10. Dyson-Schwinger equation for the generating functional of the proper planar Green functions, $\Phi^3 + \Phi^4$ theory, eq. (5.10).

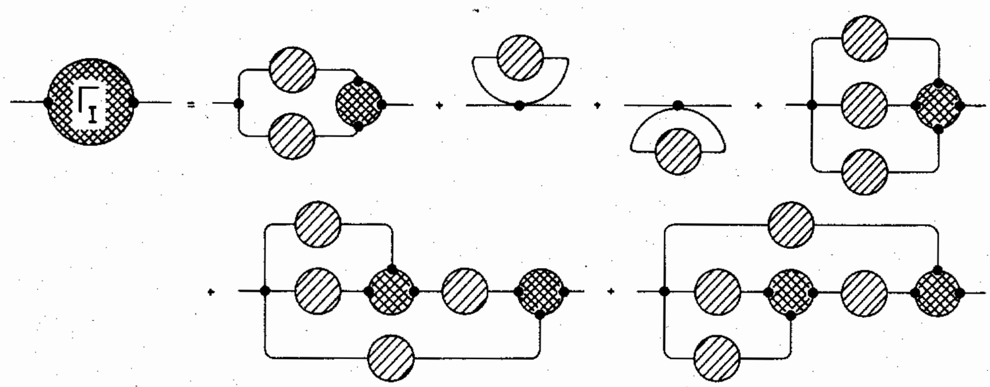


Fig. 11. Dyson-Schwinger equations for the proper planar two-point Green function (self-energy), $\Phi^3 + \Phi^4$ theory. Tadpole contributions have been suppressed.

For example, differentiating (5.10) with respect to Φ and then setting all fields and sources equal to zero, we obtain the DSE for the proper self-energy shown in fig. 11. Successive differentiations produce the DSE's for the higher proper vertices of the planar theory.

6. Zero-dimensional theories

It is amusing and instructive to consider field theories in the zero-dimension limit because the planar Dyson-Schwinger equations can be solved explicitly in this case. In this limit space-time collapses to a single point, and there is only a single source j, propagators take the value unity, and each vertex is simply a factor g, the coupling constant. The generating functionals become ordinary c-number valued functions,

$$Z[j] = \sum_m Z^{(m)} j^m = \sum_{m,k} g^k Z^{(m,k)} j^m , \tag{6.1}$$

where $Z^{(m,k)}$ is the number of diagrams with m legs and k vertices. Furthermore, the planar functional derivative becomes trivial: for a function $f[j]$ admitting a power-series expansion $f[j]=\sum_{m=0}^{\infty} f^{(m)}j^m$, we have in zero dimensions

$$\frac{\delta f[j]}{\delta j}=\frac{f[j]-f[0]}{j}. \tag{6.2}$$

Thus the DSE's become algebraic equations whose solution counts the number of planar diagrams generated by the theory under consideration. The relevance of the asymptotic behaviour of the number of diagrams to the possible convergence of the planar perturbative expansion has been discussed in ref. [2].

We illustrate the above remarks with two examples, the most trivial of which is the counting problem for the free field theory. In this first example, the DSE (5.1) becomes

$$\frac{\delta Z}{\delta j}-Z^2 j=0, \tag{6.3}$$

or, using (6.2),

$$j^2Z^2-Z+1=0 \tag{6.4}$$

[of course, in this simple case, we could have arrived at (6.4) directly from (4.2) and (4.7)]. Noting that $Z[j]$ is a power series in j, we immediately get an explicit summation of the continued fraction (4.9).

$$Z=\frac{1-\sqrt{1-4j^2}}{2j^2}=\sum_{n=0}^{\infty}\frac{(2n)!}{n!(n+1)!}j^{2n}. \tag{6.5}$$

The coefficient of j^{2n} gives the number of diagrams with $2n$ legs (see fig. 4).

As a less trivial example, let us consider Φ^3 theory. In zero dimensions the DSE (5.1) becomes

$$\frac{\delta Z}{\delta j}=jZ^2+g\frac{\delta^2 Z}{\delta j^2}, \tag{6.6}$$

with

$$\frac{\delta^2 Z}{\delta j^2}=\frac{Z-Z^{(1)}j-1}{j^2}. \tag{6.7}$$

The solution of this quadratic equation is

$$Z=\frac{1}{2j^2}\left\{\left(1-\frac{g}{j}\right)-\sqrt{\left(1-\frac{g}{j}\right)^2-4j^2\left(1-\frac{g}{j}-gZ^{(1)}\right)}\right\}. \tag{6.8}$$

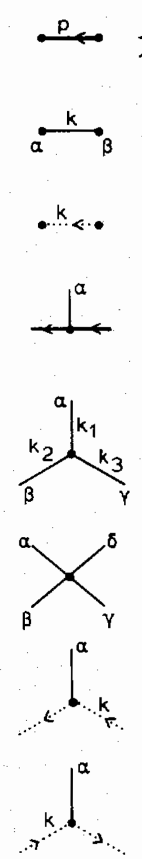
By requiring that Z be a power series of the form (6.1), one finds after some algebra [11] explicit expressions for $Z^{(m,k)}$.

In a similar way, the number of planar connected and 1PI diagrams with a given number of legs and vertices may be found by solving the zero-dimensional versions of the relevant DSE's. The planar formalism provides a simple and more direct method of diagram counting than that of taking the large-N limit of a scalar $[N \times N]$ matrix field theory [11].

7. Planar QCD

In sect. 2 we have used 't Hooft's [1] large-N limit of U(N) QCD to motivate the formal development of planar field theory carried out in sects. 3 to 6. In this section we return to QCD and describe the planar QCD perturbation expansion.

The contribution of a Feynman diagram to an amplitude consists of three parts: a combinatoric factor, a colour weight and a momentum-space factor. As we have



$$\frac{i}{\not p - m + i\varepsilon}$$

$$\frac{(-i)}{k^2+i\varepsilon}\left[g^{\alpha\beta} - \tfrac{1}{2}\left(k^\alpha f(k)^\beta + f(k)^\alpha k^\beta\right)\right]$$

$$\frac{-i}{k^2+i\varepsilon}$$

$$-ig_p\gamma^\alpha$$

$$-ig_p\left[g^{\alpha\beta}(k_1-k_2)^\gamma + g^{\beta\gamma}(k_2-k_3)^\alpha + g^{\gamma\alpha}(k_3-k_1)^\beta\right]$$

$$+ig_p^2\left[2g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right]$$

$$ig_p h^\alpha(k)$$

$$-ig_p h^\alpha(k)$$

$$h^\alpha \equiv k^\alpha + k_\beta \frac{(k^\beta f^\alpha - f^\beta k^\alpha)}{2}$$

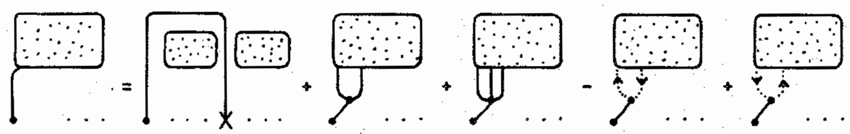
Fig. 12. Feynman rules for planar QCD. If $f^\alpha(k)$ is set equal to $(1-\alpha)k^\alpha/k^2$, one obtains the rules for the covariant gauges; if $f^\alpha(k)$ is set to $2n^\alpha/(n\cdot k) - n^2k^\alpha/(n\cdot k)^2$, the rules for the axial gauge are obtained. In the latter case the ghosts decouple.

argued above, due to the lack of Bose symmetry, all planar combinatoric factors are equal to unity. In ref. [6] we have exploited the colour weight structure of QCD to show that the planar sector of QCD is gauge invariant and defined for *any* non-abelian gauge group. The colour weights of planar diagrams are all proportional to $C_2(\mathrm{R})^k$, where $2k$ is the order in perturbation theory, and $C_2(\mathrm{R})$ is the quadratic Casimir operator for the defining representation. Hence all planar colour weights can be absorbed into the coupling constant by redefinition

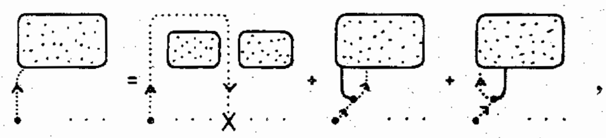
$$g_{\mathrm{p}}^2 = g^2 C_2(\mathrm{R}). \tag{7.1}$$

(In the large-N limit $g_{\mathrm{p}}^2 \to g^2 N$, the usual coupling constant for $1/N$ expansions). The only non-trivial part of the planar QCD Feynman rules is their momentum-space structure. This has been discussed in sect. 2 and is summarized in fig. 12. The essential point is that the planar vertices are rigid: they are not symmetric under leg interchanges.

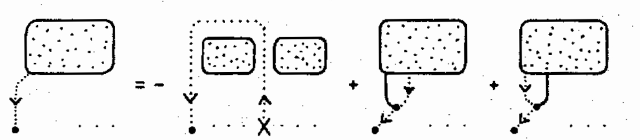
The planar perturbation expansion is generated by iterating the Dyson-Schwinger equations of sect. 5. For planar QCD the Dyson-Schwinger equations for the generating functional of the full Green functions are



(7.2)



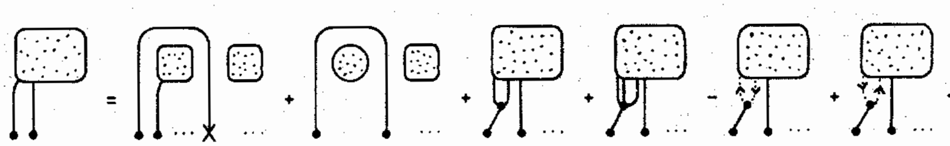
(7.3a)



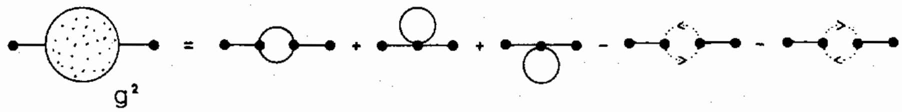
(7.3b)

We are omitting quarks in this section for reasons of simplicity, their inclusion is straightforward. In order to obtain the full perturbation expansion for a particular Green function, e.g. the full gluon propagator, one should use the DSE's (7.2) and (7.3) to follow the left-most leg entering a blob further into the blob. Note the minus signs in (7.2) and (7.3b); they will generate the fermionic factor -1 for each ghost loop.

As an example, we expand the gluon self-energy to the order g_{p}^2. The gluon Dyson-Schwinger equation (7.2) gives



Now, using (7.2) and (7.3) and then setting all remaining sources equal to zero, we obtain to order g_p^2


(7.4)

(We drop the tadpoles, as in QCD they vanish identically.) Even though the combinatorics look different, the Feynman integrals for the planar gluon self-energy and the full QCD gluon self-energy are the same to this order. The reason has been explained in sect. 2; there is a compensation between the combinatoric factors and the colour weights.

Now that we have the planar QCD Feynman rules and the planar Dyson-Schwinger equations, we are ready to derive the planar Ward identities. They are needed for the proof of gauge invariance and renormalizability. Of course, we know [6] that the planar QCD is gauge invariant because the complete theory is gauge invariant – but for any attempt at the direct summation of the planar theory, the planar Ward identities are an essential prerequisite.

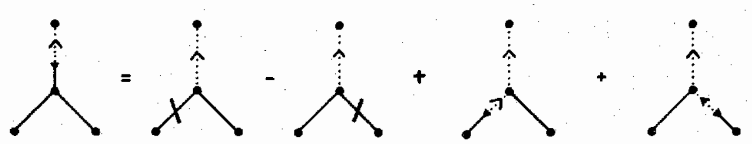
In QED the proof of Ward identities follows from the Feynman identity for the electron-photon vertex:

$$\not{k} = (\not{k} + \not{p} - m) - (\not{p} - m). \tag{7.5}$$

The corresponding QCD identity for the three-gluon vertex is

$$k_{3\gamma}\left\{g^{\alpha\beta}(k_1 - k_2)^\gamma + g^{\beta\gamma}(k_2 - k_3)^\alpha + g^{\gamma\alpha}(k_3 - k_1)^\beta\right\}$$
$$= \left(g^{\alpha\beta}k_2^2 - k_2^\alpha k_2^\beta\right) - \left(g^{\alpha\beta}k_1^2 - k_1^\alpha k_1^\beta\right). \tag{7.6}$$

Sandwiching the above identity between two gluon propagators (fig. 12) we obtain the 't Hooft [12] identity for the three-gluon vertex:


(7.7)

The auxiliary vertices and propagators are defined in fig. 13. The corresponding 't Hooft identities for the planar four-gluon and gluon-ghost vertices are easily checked:


(7.8)

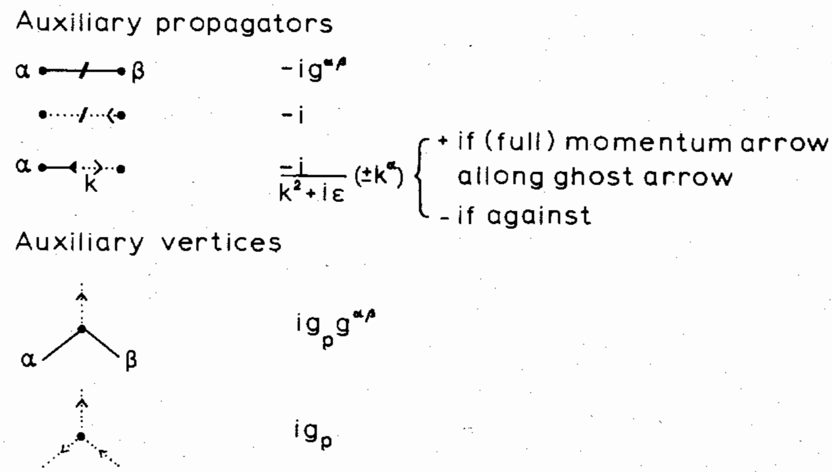
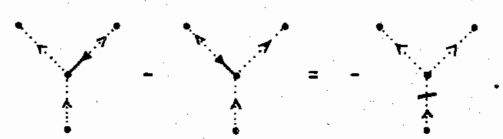


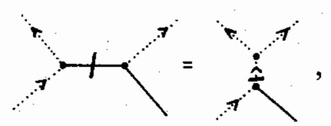
Fig. 13. Auxiliary propagators and vertices used in the derivation of the planar Ward identity.



(7.9)

We have used momentum conservation together with $k_\alpha h^\alpha(k) = k^2$ (cf. fig. 12) in deriving the last relation. The above are the Ward identities for the bare planar vertices. To derive the Ward identities for the full planar Green functions we shall also need the following trivial identities (true by the definitions of figs. 12, 13):

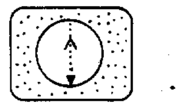
(7.10)



(7.11)

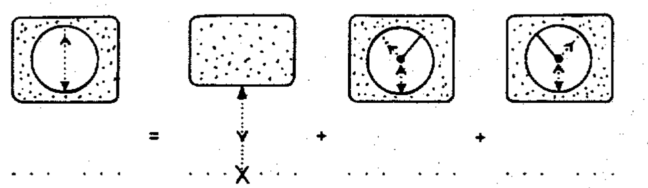
(7.12)

The 't Hooft identity for the three-gluon vertex (7.7) has two types of terms. The auxiliary propagator terms cancel against similar terms generated by (7.8) and (7.9). The last two terms in (7.7) can be interpreted as propagation of a "longitudinal gluon" k^μ through the diagram; in each application of the 't Hooft identity k^μ "eats" a gluon line, leaving a ghost line in its wake. We start our proof of Ward identities by considering an arbitrary full Green function (external legs suppressed) with a longitudinal gluon somewhere inside the diagram,



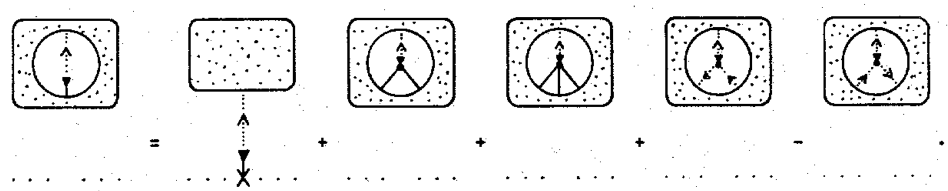
(7.13)

Under an infinitesimal variation of a gauge parameter, gauge variation insertions will appear on every gluon line. The above diagram represents the sum of all such terms. This unusual way of drawing Green functions is forced upon us by the planarity; the dotted blob represents the remainder of a planar Green function. Using the ghost Dyson-Schwinger equation (7.3a) we obtain



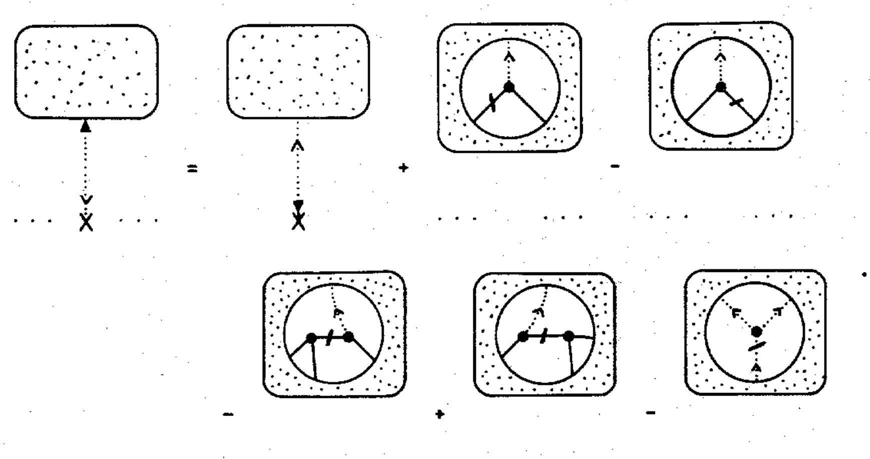
(7.14)

We may instead use the gluon Dyson-Schwinger equation (7.2) to obtain



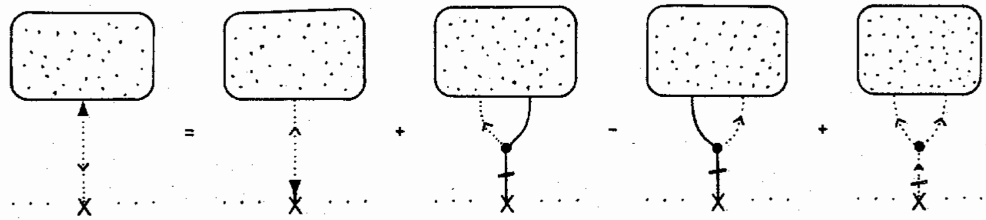
(7.15)

Now we use the 't Hooft identities (7.7)–(7.9). The last two terms in (7.14) cancel against terms generated by (7.7) and we obtain



(7.16)

It is clear that the fourth and fifth term of the right-hand side of this equation will cancel against identical terms generated in the Dyson-Schwinger expansion of the second and third term. Expanding the sixth term with (7.3b) and using identities (7.10)–(7.12) we finally obtain the *planar Ward identity*



(7.17)

Modulo combinatoric factors, the planar Ward identities have the same form and the same interpretation as the complete QCD Ward identities: insertion of a longitudinal gluon (left-hand side) results in a sum of Green functions with vanishing mass-shell factors (right-hand side).

Eq. (7.17) is the planar QCD equivalent of the Becchi-Rouet-Stora identity [13] in the complete theory. Due to the non-commutativity of the sources, it is not possible to write this identity in its usual form (with functional derivatives acting on $Z[j]$ from the left). The Ward identities for the connected and 1PI Green functions can be obtained by the methods of sect. 4.

To summarise, in this paper we have developed the full field-theoretic apparatus for the description of planar QCD: the planar generating functionals, the planar Dyson-Schwinger equations and the planar Ward identities. Planar field theory is characterized by continued fractions (rather than exponentials), which gives us hope that it may be convergent and summable. The stage is now set for a serious study of the high order UV behaviour (planar β function) and IR behaviour (bound states) of planar QCD.

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