Nuclear Physics B188 (1981) 373–396

NEGATIVE DIMENSIONS AND E_7 SYMMETRY

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Received 6 August 1980
(Revised 21 April 1981)

The invariant length and volume which characterize the Lorentz group are extended to a quadratic and a quartic supersymmetric invariant. The symmetry group of the Grassmann sector can be SO(2), SU(2), SU(2) × SU(2) × SU(2), Sp(6), SU(6), SO(12) or E_7, which are also possible global symmetries of extended supergravities. Diophantine conditions which yield this classification follow from the corresponding conditions in d bosonic dimensions by the replacement $d \to -d$.

1. Introduction

Parisi and Sourlas [1] have suggested that a Grassman vector space of dimension d can be interpreted as an ordinary vector space of dimension $-d$. It is perhaps not widely appreciated that semisimple Lie groups abound with examples in which a $d \to -d$ substitution can be interpreted in this way [2]. An early example were Penrose's [3] binors, which are representations of SU(2) = Sp(2) constructed as SO(−2). This is a special case of a general relation between SO(d) and Sp($-d$); if symmetrizations and antisymmetrizations are interchanged, representations of SO(d) become Sp($-d$) representations. Here I shall illustrate such relations by working out in detail an example which is suggestive in the light of Cremmer and Julia's [4] surprising discovery of a global E_7 symmetry in supergravity.

I shall extend the Minkowski space into Grassmann dimensions by requiring that the invariants of SO(4) (or SO(3, 1); compactness plays no role in this analysis) become supersymmetric invariants. SO(4) is the invariance group of $g_{\mu\nu}$ and $\varepsilon_{\mu\nu\sigma\rho}$, hence I am looking for the invariance group of the supersymmetric invariants

$$(x, y) = g_{\mu\nu}x^{\mu}y^{\nu}\,,$$
$$(x, y, z, w) = e_{\mu\nu\sigma\rho}x^{\mu}y^{\nu}z^{\sigma}w^{\rho}\,,$$

where* $\mu, \nu, \ldots = 4, 3, 2, 1, -1, -2, \ldots, -d$. For the quadratic invariant the group is the orthosymplectic [5] OSp($4, d$). This group is orthogonal in the bosonic dimensions and symplectic in the Grassmann dimensions, because if $g_{\mu\nu}$ is symmetric

* My secret motive for thinking of the Grassmann dimension as $-d$ is that I think of the dimension as a trace, $d = \delta^{\mu}_{\mu}$, and in a Grassmann (or fermionic) world eaeh trace carries a minus sign.

in the ν, $\mu > 0$ indices, it must be antisymmetric in the ν, $\mu < 0$ indices. In this way the supersymmetry ties in with the $SO(d) \sim Sp(-d)$ equivalence mentioned above.

Following the line of reasoning just employed to extend the symmetric Minkowski invariant $g_{\mu\nu}$ into Grassmann dimensions, I assume that if the quartic invariant tensor $e_{\mu\nu\sigma\rho}$ is antisymmetric in ordinary dimensions, it is symmetric in the Grassmann dimensions. My task is to determine all groups which admit an antisymmetric quadratic invariant together with a symmetric quartic invariant. The method will be to introduce the invariants one by one, and study the way in which they split up reducible representations. The first invariant might be realizable in many spaces. When the next invariant is added, the group of invariance transformations of the first invariant splits into two subgroups; those transformations which preserve both invariants, and those which do not. Such decompositions yield Diophantine conditions on representation dimensions. These conditions are so constraining that they limit the possibilities to a few that can be easily identified.

In the present example the resulting classification can be summarized by*

Invariants	*Lie algebra (representation dimension)*
symmetric $g_{\mu\nu}$	
antisym. $e_{\mu\nu\sigma\rho}$:	$A_1+A_1(4)$, $G_2(7)$, $B_3(8)$, $D_5(10)$;
antisym. $g_{\mu\nu}$	$SO(2)$, $A_1(4)$, $A_1+A_1+A_1(8)$,
symmetric $e_{\mu\nu\sigma\rho}$;	$C_3(14)$, $A_5(20)$, $D_6(32)$, $E_7(56)$.

From the supergravity point of view it is important to note that the Grassmann space relatives of our SO(4) world include E_7, SO(12) and SU(6) in the same representations as those discovered by Cremmer and Julia. Furthermore, it appears that *all* seven possible groups can be realized as global symmetries of the seven extended supergravities if one vector multiplet is added to $N=1$, 2, 3 and 4 extended supergravities.

The above method has been used in ref. [2] to extend a fascinating group-theoretic construct known as the Freudenthal magic square [6] to a "magic triangle". Here I reproduce a row of this extension using the matrix notation of Okubo [7], rather than the diagrammatic notation [8] used in ref. [2]. Originally the $d \to -d$ relations and the magic triangle arose as byproducts of an investigation of group-theoretic structure of gauge theories undertaken in ref. [8]. At the time they appeared to be mere mathematical curiosities, but since then their possible connection with Grassmann dimensions and supergravities has made them more intriguing. As they emerge from what appears to be a new construction of exceptional Lie groups, the complete presentation [2] is very lengthy. The present paper is not an explanation of the general construction; its purpose is to make specialists aware of the possible connections between supergravity and supersymmetry on one hand and the magic triangle and the negative dimensions on the other. Unfortunately, as the complete

* Semisimple algebras are identified here by their Cartan designations. The corresponding classical groups are given in sect. 6 and the abstract.

presentation [2] is not yet available, this article has to include quite a bit of algebra essential to the understanding of the construction.

Sect. 2 sets up the general procedure. In sects. 3 to 6 I determine the groups which allow a symmetric quadratic together with an antisymmetric quartic invariant. The end result of the analysis is two non-trivial Diophantine conditions together with the explicit projection operators for reduced representations. In sects. 7 to 10 the analysis is repeated for an antisymmetric quadratic together with a symmetric quartic invariant. I find the same Diophantine conditions with dimension d replaced by $-d$, and the same projection operators with symmetrizations and antisymmetrizations interchanged. Possible relations to the extended supergravities are discussed in sect. 11. The summary is given in sect. 12. The calculation of Dynkin indices is described in appendix A. The relation between SO(d) and Sp($-d$) is discussed in appendix B.

2. Reduction procedure

Let $x \in \mathrm{V}$ be a d-dimensional vector x^μ, $\mu = 1, 2, \ldots, d$, and G^ν_μ be a $[d \times d]$ matrix representation of a linear transformation g: $x'^\mu = G^\mu_\nu x^\nu$. A d-dimensional vector x_μ, an element of the dual* vector space $\bar{x} \in \bar{\mathrm{V}}$, transforms under g as $x'_\mu = x_\nu (G^{-1})^\nu_\mu$. By definition, $x_\mu y^\mu$ is an invariant of g, and δ^μ_ν is an invariant tensor. Consider next a tensor $x^{\mu_1\mu_2}_{\mu_3\cdots\mu_m} \in \mathrm{V}_1 \otimes \mathrm{V}_2 \otimes \bar{\mathrm{V}}_3 \otimes \cdots \otimes \bar{\mathrm{V}}_m$. It transforms linearly under g and it can be considered a vector x^μ in a $d_1 d_2 \ldots d_m$ dimensional vector space. We can chose to index this vector with a single index μ ($\mu = 1, 2, \ldots, d_1 d_2 \cdots d_m$] or with the array of indices ${}^{\mu_1\mu_2}_{\mu_3\cdots\mu_m}$. The $[d_1 d_2 \cdots d_m \times d_1 d_2 \cdots d_m]$ dimensional matrices can be indexed in the same way, with the multiplication defined by

$$(AB)^\nu_\mu = A^{\nu_1\nu_2}_{\nu_3\cdots\nu_m},{}^{\sigma_3\cdots\sigma_m}_{\sigma_1\sigma_2} B^{\sigma_1\sigma_2}_{\sigma_3\cdots\sigma_m},{}^{\mu_3\cdots\mu_m}_{\mu_1\mu_2} . \tag{2.1}$$

The distinct roots of the minimal characteristic polynomial of a $[d \times d]$ matrix A,

$$\prod_{i=1}^{M} (A - \alpha_i) = 0 , \tag{2.2}$$

resolve the d-dimensional vector space V into M orthogonormal subspaces V_1, $\mathrm{V}_2, \ldots, V_M$ by means of projection operators [7]

$$P_i = \prod_{j \neq i} \frac{(A - \alpha_j 1)}{(\alpha_i - \alpha_j)} . \tag{2.3}$$

The dimension of the subspace V_i is

$$d_i = \mathrm{tr}\, P_i . \tag{2.4}$$

In particular, if A is invariant under transformations g, the resolution into

* For more rigorous definitions, see ref. [9].

orthonormal subspaces is also invariant. Each new invariant introduces new projection operators which might or might not resolve an invariant space V_i into further subspaces. I shall not have to worry about the further reducibility of such subspaces, as the Diophantine conditions I shall derive do not require irreducibility.

A particularly interesting vector space is $V \otimes \bar{V}$ which contains the generators of infinitesimal transformations $D^\mu_\nu \simeq G^\mu_\nu - \delta^\mu_\nu$. They form a subspace $V_A \in V \otimes \bar{V}$ called the adjoint representation. According to (2.3), an invariant matrix $A^{\nu}_{\mu,}{}^{\gamma}_{\delta}$ resolves $V \otimes \bar{V}$ into invariant subspaces* V_i. To determine which of these subspaces belong to the adjoint representation, consider the invariant

$$A(\bar{x}, y, \bar{u}, v) = A^{\nu}_{\mu,}{}^{\gamma}_{\delta} x_\nu y^\mu u_\gamma v^\delta .$$

The invariance group of A consists of all g such that

$$A(\bar{x}G^{-1}, Gy, \bar{u}G^{-1}, Gv) = A(\bar{x}, y, \bar{u}, v) .$$

Taking infinitesimal $G \simeq 1 + D$ leads to an invairance condition on the infinitesimal generators

$$-D^\nu_{\nu'} A^{\nu'}_{\mu,}{}^{\gamma}_{\delta} + D^{\mu'}_{\mu} A^{\nu}_{\mu',}{}^{\gamma}_{\delta} - D^{\gamma}_{\gamma'} A^{\nu}_{\mu,}{}^{\gamma'}_{\delta} + D^{\delta'}_{\delta} A^{\nu}_{\mu,}{}^{\gamma}_{\delta'} = 0 .$$

If $V_i \subseteq V_A$, then $(P_i)^{\nu\gamma}_{\mu\delta} D^\delta_\gamma \in V_A$ for any D. Hence V_i belongs to the adjoint representation only if it satisfies the *invariance condition*:

$$-(P_i)^{\alpha\nu}_{\beta\nu'} A^{\nu'}_{\mu,}{}^{\gamma}_{\delta} + (P_i)^{\alpha\mu'}_{\beta\mu} A^{\nu}_{\mu',}{}^{\gamma}_{\delta} - (P_i)^{\alpha\gamma}_{\beta\gamma'} A^{\nu}_{\mu,}{}^{\gamma'}_{\delta} + (P_i)^{\alpha\delta'}_{\beta\delta} A^{\nu}_{\mu,}{}^{\gamma}_{\delta'} = 0 . \tag{2.5}$$

Clearly there is a corresponding covariance condition for any invariant tensor: $A^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots\nu_m}$.

As an example of the above reduction procedure consider how the invariant tensor δ^μ_ν reduces $V \otimes \bar{V}$ into representations of SL(d). There are exactly two independent invariant $[d^2 \times d^2]$ matrices:

$$\begin{aligned} &\text{identity:} \quad 1^{\mu}_{\nu,}{}^{\delta}_{\rho} = \delta^\mu_\rho \delta^\delta_\nu , \\ &\text{trace:} \quad T^{\mu}_{\nu,}{}^{\delta}_{\rho} = \delta^\mu_\nu \delta^\delta_\rho . \end{aligned} \tag{2.6}$$

The trace operator has a trivial characteristic equation

$$T^2 = dT ,$$

with roots $\alpha_1 = d$, $\alpha_2 = 0$. The corresponding projection operators (2.3) are

$$P_1 = \frac{1}{d} T , \qquad P_2 = 1 - \frac{1}{d} T \tag{2.7}$$

with dimensions $d_1 = \operatorname{tr} P_1 = 1$, $d_2 = \operatorname{tr} P_2 = d^2 - 1$. In this way the invariant matrix T

* For example, x^μ_ν can be separated into trace x^μ_μ and the traceless part, as in (2.8).

has resolved the space of $[d \times d]$ matrices into

$$\text{singlet:} \qquad (P_1)^{\mu\delta}_{\nu\rho} = \frac{1}{d}\,\delta^{\mu}_{\nu}\delta^{\delta}_{\rho}\,,$$
$$\text{adjoint rep:} \quad (P_2)^{\mu\delta}_{\nu\rho} = \delta^{\mu}_{\rho}\delta^{\delta}_{\nu} - \frac{1}{d}\delta^{\mu}_{\nu}\delta^{\delta}_{\rho}\,. \tag{2.8}$$

V_2 is the space of traceless matrices, i.e. the invariance group of δ^{μ}_{ν} is SL(d). [The invariance conditions (2.5) are in this case satisfied trivially by both representations.]

Before leaving SL(d) one should also quote the classical result of the theory of invariants [9]; beyond δ^{μ}_{ρ}, SL(d) transformations also preserve the Levi–Civita tensor in d dimensions,

$$\varepsilon_{\mu_1\mu_2\cdots\mu_d},\ \varepsilon^{\mu_1\mu_2\cdots\mu_d}\,. \tag{2.9}$$

It is easily verified [2] that P_2 satisfies the appropriate invariance condition of type (2.5). As I shall study subgroups of SL(d), they all will also have Levi–Civita tensors as invariants.

3. The symmetric quadratic invariant

Extend the set of invariant tensors to three:

$$\delta^{\mu}_{\nu}\,, \qquad g^{\mu\nu} = g^{\nu\mu}\,, \qquad g_{\nu\mu} = g_{\mu\nu}\,. \tag{3.1}$$

The matrix $A^{\mu}_{\nu} = g^{\mu\sigma}g_{\sigma\nu}$ must be proportional to unity, otherwise its characteristic equation would decompose the d-dimensional representation. One can choose a normalization such that

$$g^{\mu\sigma}g_{\sigma\nu} = \delta^{\mu}_{\nu}\,. \tag{3.2}$$

The only new $[d^2 \times d^2]$ invariant matrix is the flip matrix F:

$$F^{\mu\delta}_{\nu\rho} = g^{\mu\delta}g_{\nu\rho}\,. \tag{3.3}$$

The characteristic equation

$$F^2 - 1 = 0 \tag{3.4}$$

yields two projection operators (2.3)

$$P_3 = \tfrac{1}{2}(1+F)\,, \qquad P_4 = \tfrac{1}{2}(1-F)\,. \tag{3.5}$$

To find out how the subspaces V_1, V_2 given by (2.7) decompose we need the multiplication rule

$$TF = T\,. \tag{3.6}$$

This gives

$$\begin{aligned} P_3P_1 &= P_1\,, \qquad P_3P_2 = P_3 - P_1 = P_5\,, \\ P_4P_1 &= P\,, \qquad P_4P_2 = P_4\,. \end{aligned} \tag{3.7}$$

In this way the invariant matrices T and F have decomposed the d^2 dimensional vector space $V\otimes\bar{V}$ into three subspaces $V\otimes\bar{V} = V_1\oplus V_4\oplus V_5$ given by projectors

$$\begin{aligned} &\text{singlet:} && (P_1)^{\mu\delta}_{\nu\rho} = \frac{1}{d}\delta^{\mu}_{\nu}\delta^{\delta}_{\rho}\,, \\ &\text{symmetric:} && (P_5)^{\mu\delta}_{\nu\rho} = \tfrac{1}{2}(\delta^{\mu}_{\rho}\delta^{\delta}_{\nu} + g^{\mu\delta}g_{\nu\rho}) - \frac{1}{d}\delta^{\mu}_{\nu}\delta^{\delta}_{\rho}\,, \\ &\text{antisymmetric:} && (P_4)^{\mu\delta}_{\nu\rho} = \tfrac{1}{2}(\delta^{\mu}_{\rho}\delta^{\delta}_{\nu} - g^{\mu\delta}g_{\nu\rho})\,. \end{aligned} \tag{3.8}$$

The dimensions of the subspaces are

$$\begin{aligned} d_1 &= (P_1)^{\mu\nu}_{\nu\mu} = 1\,, \\ d_4 &= \mathrm{Tr}\, P_4 = \frac{d(d-1)}{2}\,, \\ d_5 &= \mathrm{Tr}\, P_5 = \frac{d(d+1)}{2} - 1\,. \end{aligned} \tag{3.9}$$

Clearly this decomposition is just the standard SO(d) decomposition (trace, antisymmetric, traceless symmetric)

$$d\otimes d = 1\oplus\frac{d(d-1)}{2}\oplus\left(\frac{d(d+1)}{2}-1\right).$$

Moreover, the projection operators (3.8) are explicit Clebsch–Gordon coefficients for the decomposition. $(P_4)^{\mu\delta}_{\nu\rho}$ are SO(d) rotation generators, as can be verified by checking the invariance condition

$$(P_4)^{\mu\delta}_{\nu\rho}g^{\rho\varepsilon} + (P_4)^{\mu\varepsilon}_{\nu\rho}g^{\delta\rho} = 0\,. \tag{3.10}$$

The remaining generators of SL(d) from (2.8), $P_2 - P_4 = P_5$, do not leave $g^{\rho\delta}$ invariant. It is also worth noting that nowhere have we made assumptions about the eigenvalues of $g^{\rho\delta}$, so this construction applies equally well to compact and non-compact groups.

4. The antisymmetric quartic invariant

Add to the set of invariants (3.1) a fully antisymmetric tensor

$$e_{\mu\nu\rho\delta} = -e_{\nu\mu\rho\delta} = -e_{\mu\rho\nu\delta} = -e_{\mu\nu\delta\rho}\,. \tag{4.1}$$

The simplest $[d^2 \times d^2]$ matrix constructed from the new invariant is

$$E^{\mu\delta}_{\nu\rho} = g^{\mu\varepsilon} g^{\delta\sigma} e_{\varepsilon\sigma\nu\rho} . \tag{4.2}$$

The multiplication table (3.6) is now extended by

$$TE = 0 , \qquad FE = -E . \tag{4.3}$$

The E invariant is absent from the symmetric subspaces:

$$P_1 E = 0 , \qquad P_5 E = \tfrac{1}{2}(1+F)E = 0 . \tag{4.4}$$

This means that the E invariant can decompose only the V_4 subspace. As I wish to introduce one invariant at a time, I demand that no further independent $[d^2 \times d^2]$ invariant matrices can be constructed from E. In particular, E^2 is not independent:

$$E^2 + bE + cP_4 = 0 . \tag{4.5}$$

This condition incidentally also insures that the $[d \times d]$ matrix $(E^2)^{\mu\gamma}_{\nu\mu}$ is proportional to unity:

$$(E^2)^{\mu\gamma}_{\nu\mu} = -c \frac{d_4}{d} \delta^\gamma_\nu . \tag{4.6}$$

Were this not true, distinct eigenvalues of E^2 matrix could decompose the d-dimensional representation.

If the coefficients in (4.5) can be fixed, V_4 will separate into the new adjoint representation subspace V_6 and the remainder V_7 by means of projection operators:

$$\begin{aligned} &\text{adjoint:} && P_6 = \frac{E - \alpha_7 1}{\alpha_6 - \alpha_7} P_4 , \\ &\text{antisymmetric:} && P_7 = \frac{E - \alpha_6 1}{\alpha_7 - \alpha_6} P_4 , \end{aligned} \tag{4.7}$$

$$\alpha_6 + \alpha_7 = -b , \qquad \alpha_6 \alpha_7 = c .$$

The coefficient c is fixed by the scale of E:

$$\operatorname{Tr} E^2 + c d_4 = 0 . \tag{4.8}$$

To fix the remaining coefficient b, introduce an index permutation on $[d^2 \times d^2]$ matrices:

$$\sigma(A)^{\mu\delta}_{\nu\rho} = A^{\delta\mu}_{\nu\rho} , \qquad \sigma^2 = 1 . \tag{4.9}$$

The invariant matrices map as

$$\sigma(1) = T , \qquad \sigma(F) = F , \qquad \sigma(E) = -E . \tag{4.10}$$

It follows that

$$P_4 \sigma(P_4) = \tfrac{1}{2}(1-F)\tfrac{1}{2}(T-F) = \tfrac{1}{2}P_4 . \tag{4.11}$$

The characteristic equation (4.5) maps under $\sigma(\ )$ and P_4 projection into

$$P_4(\sigma(E^2)-bE+\tfrac{1}{2}c)=0\,. \tag{4.12}$$

In particular, in the adjoint representation subspace V_6 (note $P_6E=\alpha_6P_6$)

$$P_6\left(\sigma(E^2)+\alpha_6^2-\frac{3}{2}\frac{\operatorname{Tr}E^2}{d_4}\right)=0\,. \tag{4.13}$$

To compute $P_6\sigma(E^2)$, one contracts the invariance condition (2.5) for E with another E matrix and uses the antisymmetry of E as well as (4.6). (Parenthetically, such calculations are easy in the diagrammatic notation [2, 8].) The result is

$$P_6\sigma(E^2)=\frac{1}{3}\frac{\operatorname{Tr}E^2}{d}P_6\,. \tag{4.14}$$

Now α_6, α_7 and P_6, P_7 follow from (4.13) and (4.8):

$$\alpha_6=\sqrt{\frac{\operatorname{Tr}E^2}{d_4}\frac{10-d}{6}}\,,\qquad \alpha_7=-\sqrt{\frac{\operatorname{Tr}E^2}{d_4}\frac{6}{10-d}}\,, \tag{4.15}$$

$$\begin{aligned}&\text{adjoint:}\quad P_6=\sqrt{\frac{6(10-d)}{(16-d)^2}\frac{d_4}{\operatorname{Tr}E^2}}\,E+\frac{6}{16-d}P_4\,,\\ &\text{antisym:}\quad P_7=-\sqrt{\frac{6(10-d)}{(16-d)^2}\frac{d_4}{\operatorname{Tr}E^2}}\,E+\frac{10-d}{16-d}P_4\,,\end{aligned} \tag{4.16}$$

with the dimensions

$$\begin{aligned}d_6&=\operatorname{Tr}P_6=\frac{3d(d-1)}{16-d}\,,\\ d_7&=\operatorname{Tr}P_7=\frac{d(d-1)(10-d)}{2(16-d)}\,.\end{aligned} \tag{4.17}$$

This completes decomposition $V\otimes\bar{V}=V_1\oplus V_5\oplus V_6\oplus V_7$. The new subspaces V_6, V_7 have integer dimension only for $d=4$, 6, 7, 8, 10. However, the reduction of $V\otimes\bar{V}\otimes\bar{V}$ undertaken in the next section will eliminate the $d=6$ possibility.

5. Further Diophantine conditions

The reduction of the $V\otimes\bar{V}$ space induced by the invariants δ^{μ}_{ν}, $g_{\mu\nu}$ and $e_{\mu\nu\sigma\rho}$ has led to a very powerful Diophantine condition (4.17). I shall now show that further Diophantine conditions follow from the reduction of higher product spaces $V^p\otimes\bar{V}^q$. As an example, I turn to the reduction of $V_6\otimes\bar{V}\subset V\otimes\bar{V}^2$. The tensor $x^{\mu}_{\nu\rho}$ is an element of the tensor space $V_6\otimes\bar{V}$ if

$$(P_6)^{\mu\nu'}_{\nu\mu'}x^{\mu'}_{\nu'\rho}=x^{\mu}_{\nu\rho}\,, \tag{5.1}$$

and the $[dd_6 \times dd_6]$ matrices are indexed as in (2.1). The two simplest invariant matrices one can write down are

$$\begin{aligned} &\text{identity:} && 1^{\mu}_{\nu\rho},{}^{\alpha\gamma}_{\beta} = (P_6)^{\mu\alpha}_{\nu\beta}\delta^{\gamma}_{\rho}\,, \\ &\text{defining rep:} && R^{\mu}_{\nu\rho},{}^{\alpha\gamma}_{\beta} = (P_6)^{\mu\sigma}_{\nu\rho}\delta^{\sigma'}_{\sigma}(P_6)^{\gamma}_{\sigma'},{}^{\alpha}_{\beta} \end{aligned} \tag{5.2}$$

The factor $\delta^{\sigma'}_{\sigma}$ in R is written out to indicate that R is a mapping $V_6 \otimes \bar{V} \to \bar{V} \to V_6 \otimes \bar{V}$. Indeed, the characteristic equation

$$R^2 = \frac{d_6}{d} R \tag{5.3}$$

yields projection operators

$$P_8 = \frac{d}{d_6} R\,, \qquad P_9 = 1 - \frac{d}{d_6} R\,. \tag{5.4}$$

Hence $V_6 \otimes \bar{V} = \bar{V}_8 \oplus V_9$ with dimensions

$$\begin{aligned} d_8 &= (P_8)^{\mu}_{\nu\rho},{}^{\nu\rho}_{\mu} = d\,, \\ d_9 &= \operatorname{tr} P_9 = d(d_6 - 1)\,. \end{aligned} \tag{5.5}$$

The next obvious invariant matrix we can construct is an index permutation of R:

$$Q^{\mu}_{\nu\rho},{}^{\alpha\gamma}_{\beta} = (P_6)^{\mu\gamma}_{\nu\sigma}(P_6)^{\sigma\alpha}_{\rho\beta}\,. \tag{5.6}$$

In order to find the associated projector operators one has to compute

$$(Q^2)^{\mu}_{\nu\rho},{}^{\alpha\gamma}_{\beta} = (P_6)^{\mu\rho'}_{\nu\sigma}(P_6)^{\sigma\gamma}_{\rho\sigma'}(P_6)^{\sigma'\alpha}_{\rho'\beta}\,.$$

This is achieved by substituting $(P_6)^{\sigma\gamma}_{\rho\sigma'}$ by (4.7) and using the invariance condition (2.5). The result is

$$Q^2 = \frac{1}{2(\alpha_6 - \alpha_7)}\{(\alpha_6 + \alpha_7)Q - \alpha_6 P_8 - \alpha_7 1\}\,. \tag{5.7}$$

The d-dimensional space V_8 is irreducible. V_9 is decomposed by the roots α_{10}, α_{11} of the characteristic equation:

$$\begin{aligned} &P_9\left(Q^2 - \frac{1}{2}\frac{\alpha_6 + \alpha_7}{\alpha_6 - \alpha_7} Q + \frac{1}{2}\frac{\alpha_7}{\alpha_6 - \alpha_7}\right) = 0\,, \\ &\alpha_{10} = \frac{\alpha_7}{\alpha_6 - \alpha_7}\,, \qquad \alpha_{11} = \tfrac{1}{2}\,. \end{aligned} \tag{5.8}$$

The associated projection operators are [substituting (4.15)]

$$\begin{aligned} P_{10} &= \frac{2(16-d)}{28-d}\left(-Q + \tfrac{1}{2}\right)P_9\,, \\ P_{11} &= \frac{2(16-d)}{28-d}\left(Q + \frac{6}{16-d}\right)P_9\,. \end{aligned} \tag{5.9}$$

This completes the decomposition $V \otimes V_6 = V_8 \oplus V_{11}$. To compute the dimensions of V_{10}, V_{11} subspaces we need tr $P_9 Q$. Evaluation yields

$$\mathrm{tr}\, P_9 Q = -\frac{2d(2+d)}{16-d}. \tag{5.10}$$

Finally, I obtain

$$\begin{aligned} d_{10} &= \mathrm{tr}\, P_{10} = \frac{3d(d+2)(d-4)}{28-d}, \\ d_{11} &= \mathrm{tr}\, P_{11} = \frac{32d(d-1)(d+2)}{(16-d)(28-d)}. \end{aligned} \tag{5.11}$$

The important aspect of these relations is that the denominators, and hence the Diophantine conditions, are different from those in (4.17). It is easy to check that of the solutions to (4.17) $d=4, 7, 8, 10$ are also solutions of the present Diophantine conditions. All the solutions are summarized in table 1.

6. Lie algebra identification

As I have shown, symmetric $g_{\mu\nu}$ together with antisymmetric $f_{\mu\nu\sigma\rho}$ invariants cannot be realized in dimensions other than $d=4, 7, 8$. But can they be realized at all? To verify that, one can turn to the tables of Lie algebras [10] and identify these three solutions.

TABLE 1

Representation dimensions for the SO(4) family of invariance groups

Representation	Dimension	A_1+A_1	G_2	B_3	D_5
V = defining	d	4	7	8	10
V_6 = adjoint	$\frac{3d(d-1)}{16-d}$	3	14	21	45
V_7 = antisym.	$\frac{d(d-1)(10-d)}{2(16-d)}$	3	7	7	0
V_5 = symmetric	$\frac{(d+2)(d-1)}{2}$	9	27	35	54
V_{10}	$\frac{3d(d+2)(d-4)}{28-d}$	0	27	48	120
V_{11}	$\frac{32d(d-1)(d+2)}{(16-d)(28-d)}$	8	64	112	320

6.1. SO(4) OR A_1+A_1 ALGEBRA

The first solution, $d=4$ is not a surprise; it was SO(4), Minkowski or euclidean version, that motivated the whole project. The quartic invariant is the Levi–Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$. Even so, the projectors constructed are interesting. Taking

$$E^{\mu\delta}_{\nu\rho}=g^{\mu\varepsilon}g^{\delta\sigma}\varepsilon_{\varepsilon\sigma\nu\rho}\,, \tag{6.1}$$

one can immediately calculate (4.5):

$$E^2=4P_4\,. \tag{6.2}$$

The pojectors (4.16) become

$$P_6=\tfrac{1}{2}P_4+\tfrac{1}{4}E\,,\qquad P_7=\tfrac{1}{2}P_4-\tfrac{1}{4}E\,, \tag{6.3}$$

and the dimensions are $d_6=d_7=3$. Also both P_6 and P_7 satisfy the invariance condition, so the adjoint representation splits into two invariant subspaces. In this way one shows explicitly that the Lie algebra of SO(4) is semisimple A_1+A_1. Furthermore, the projection operators are precisely the η, $\bar\eta$ symbols used by 't Hooft [11] to map self-dual and self-antidual SO(4) antisymmetric tensors onto SU(2) gauge group:

$$\begin{aligned}(P_6)^{\mu\delta}_{\nu\rho}&=\tfrac{1}{4}(\delta^{\mu}_{\rho}\delta^{\delta}_{\nu}-g^{\mu\delta}g_{\nu\rho}+\varepsilon^{\mu\delta}{}_{\nu\rho})\\&=-\tfrac{1}{4}\eta_{a\nu}^{\ \ \mu}\eta_{a\rho}^{\ \ \delta}\,,\\(P_7)^{\mu\delta}_{\nu\rho}&=\tfrac{1}{4}(\delta^{\mu}_{\rho}\delta^{\delta}_{\nu}-g^{\mu\delta}g_{\nu\rho}-\varepsilon^{\mu\delta}{}_{\nu\rho})\\&=-\tfrac{1}{4}\bar\eta_{a\nu}^{\ \ \mu}\bar\eta_{a\rho}^{\ \ \delta}\,.\end{aligned} \tag{6.4}$$

The only difference is that instead of using an index pair ${}^{\mu}_{\nu}$, 't Hooft indexes the adjoint spaces by $a=1, 2, 3$. All identities listed in the appendix of ref. [11] now follow from the relations of sect. 4.

6.2. DEFINING REPRESENTATION OF G_2

The 7-dimensional representation of G_2 is a subgroup of SO(7), so it has invariants $g_{\mu\nu}$ and $\varepsilon_{\mu\nu\delta\sigma\rho\alpha\beta}$. In addition, it has an antisymmetric cubic invariant [12, 8] $f_{\mu\nu\rho}$. (This invariant can be interpreted as the multiplication table for octonions.) The quartic invariant we have inadvertently discovered is

$$e_{\mu\nu\rho\sigma}=\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma}f^{\alpha\beta\gamma}\,.$$

Furthermore, for G_2 we have a powerful algorithm [8, 13] by which any chain of contractions of more than two $f_{\alpha\beta\gamma}$ can be reduced. Projector operators of sects. 4 and 5 yield Clebsch–Gordan series

$$7\otimes 7=1\oplus 27\oplus 14\oplus 7$$

$$7\otimes 14=7\oplus 27\oplus 64\,.$$

TABLE 2
Dynkin indices for the SO(4) family of invariance groups

Representation	Dynkin index	A_1+A_1	G_2	B_3	D_5
V = defining	$\frac{16-d}{4(d+2)}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{8}$
V_6 = adjoint	1	1	1	1	1
V_7 = antisym.	$\frac{(10-d)(d-4)}{4(d+2)}$	0	$\frac{1}{4}$	$\frac{1}{5}$	0
V_5 = symmetric	$\frac{(16-d)}{4}$	3	$\frac{9}{4}$	2	$\frac{3}{2}$
V_{10}	$\frac{7(16-d)(d-4)}{4(28-d)}$	—	$\frac{9}{4}$	$\frac{14}{5}$	$\frac{7}{2}$
V_{11}	$\frac{8(2d+7)}{(28-d)}$	5	8	$\frac{46}{5}$	12

6.3. SO(7) 8-DIMENSIONAL REPRESENTATION

I have not attempted to identify the quartic invariant in this case. However, all the representation dimensions (table 1) as well as their Dynkin indices (table 2) match B_3 representations listed in the tables of Patera and Sankoff [10].

6.4. SO(10) 10-DIMENSIONAL REPRESENTATION

This is a trivial solution; $P_6 = P_4$ and $P_1 = 0$, so that there is no decomposition. The quartic invariant is

$$e_{\mu\nu\sigma\rho} = \varepsilon_{\mu\nu\sigma\rho\alpha\beta\gamma\delta\omega\zeta} C_{\alpha\beta,\gamma\delta,\omega\zeta} \equiv 0 \,,$$

where $C_{\alpha\beta,\gamma\delta,\omega\zeta}$ are the structure constants.

7. Antisymmetric quadratic invariant

Instead of (3.1), consider invariants

$$\delta^\mu_\nu,\, g^{\mu\nu} = -g^{\nu\mu}\,, \qquad g_{\mu\nu} = -g_{\nu\mu}\,. \tag{7.1}$$

As usual, $A^\mu_\nu = g^{\mu\sigma} g_{\sigma\nu}$ must be proportional to unity, but this time I chose the normalization

$$g^{\mu\sigma} g_{\sigma\nu} = -\delta^\mu_\nu\,. \tag{7.2}$$

With this choice eqs. (3.3)–(3.8) apply, but now

$$\mathrm{Tr}\, F = -d\,, \tag{7.3}$$

so d in the dimension formulas (3.9) gets replaced by $-d$:

$$d_4=\frac{d(d+1)}{2}, \qquad d_5=\frac{d(d-1)}{2}-1\,. \tag{7.4}$$

In this way we reach the standard Sp(d) decomposition

$$d\otimes d=1\oplus\frac{d(d+1)}{2}\oplus\left(\frac{d(d-1)}{2}-1\right).$$

In addition d must be even, because (7.2) cannot be satisfied in the odd dimensions.

8. Symmetric quartic invariant

We now add to the set of invariants (7.1) a symmetric tensor

$$e_{\mu\nu\rho\delta}=e_{\nu\mu\rho\delta}=e_{\mu\rho\nu\delta}=e_{\mu\nu\delta\rho}\,. \tag{8.1}$$

Again most of the algebra is the same as in sect. 4. Eqs. (4.2) to (4.8) are the same. We redefine the index permutation (4.9) as

$$\sigma(A)^{\mu\delta}_{\nu\rho}=-A^{\delta\mu}_{\nu\rho}\,, \qquad \sigma^2=1\,. \tag{8.2}$$

Continuing as in sect. 4 we have

$$\sigma(1)=-T, \qquad \sigma(F)=F, \qquad \sigma(E)=-E\,. \tag{8.3}$$

(4.12), (4.13) still apply, but the present redefinition of σ flips sign in (4.14)

$$P_6(E^2)=-\frac{1}{3}\frac{\mathrm{Tr}\,E^2}{d}P_6\,. \tag{8.4}$$

This amounts to replacing $d\to -d$ in all remaining expressions

$$\begin{aligned}
&\text{adjoint:} \quad P_6=\sqrt{\frac{6(10+d)d_4}{(16+d)^2\,\mathrm{Tr}\,E^2}}\,E+\frac{6}{16+d}P_4\,,\\
&\text{symmetric:} \quad P_7=-\sqrt{\frac{6(10+d)d_4}{(16+d)^2\,\mathrm{Tr}\,E^2}}\,E+\frac{10+d}{16+d}P_4\,,
\end{aligned} \tag{8.5}$$

$$\begin{aligned}
d_6&=\frac{3d(d+1)}{16+d}=3d-45+\frac{360}{8+\frac{1}{2}d}\,,\\
d_7&=d_4-d_6\,.
\end{aligned} \tag{8.6}$$

There are 17 solutions to this Diophantine condition, but only 10 will survive the next one.

9. Further Diophantine conditions

Rewriting sect. 5 for an antisymmetric $g_{\mu\nu}$, symmetric $e_{\mu\nu\sigma\rho}$ is absolutely trivial, as these tensors never make an explicit appearance. The only subtlety is that for the reductions of Kronecker products of odd numbers of defining representations (in this case $V \otimes \bar{V}^2$), additional overall factors of -1 appear, such as in the eq. (B.1) of the appendix B. For example, it is clear that the dimension of the quark subspace d_8 in (5.5) does not become negative; $d \to -d$ substitution propagates only through α_6, α_7 and d_6 expressions. The dimensional formulas (5.11) become

$$d_{10} = \frac{3d(d-2)(d+4)}{d+28},$$
$$d_{11} = \frac{32d(d-2)(d+1)}{(d+16)(d+28)}. \tag{9.1}$$

Out of the 17 solutions to (8.2), 10 also satisfy this Diophantine condition; $d = 2, 4, 8, 14, 20, 32, 44, 56, 164, 224$. $d = 44, 164$ and 224 can be eliminated [2] by considering reductions along the columns of the Freudenthal magic square and proving that the resulting subgroups cannot be realized; consequently the groups that contain them cannot be realized either. These considerations are beyond the scope of the present paper. Only the 7 identified solutions listed in table 3 are expected to have antisymmetric $g_{\mu\nu}$ and symmetric $e_{\mu\nu\rho\delta}$ invariants in the defining representation.

10. Lie algebra identification

It turns out that one does not have to work very hard to identify the series of solutions of the preceding section. SO(2) is trivial, and there is extensive literature on the remaining solutions. Mathematicians study them because they form the third row of the (extended) Freudenthal magic square [6]*, and physicists study them because $E_7(56) \to SU(3)_c \times SU(6)$ is one of the favourite unified models [14]. The representation dimensions and the Dynkin indices listed in tables 3 and 4 agree with the above literature, as well as the Lie algebra tables [10]. Here I shall explain only why E_7 is one of the solutions.

The construction of E_7 closest to the spirit of the present paper has been carried out by Brown [15, 16]. He considers a d-dimensional complex vector space V with properties

(i) V possesses a non-degenerate skew-symmetric bilinear form $\{x, y\} = g_{\mu\nu}x^\mu y^\nu$.

(ii) V possesses a symmetric four-linear form $q(x, y, z, w) = e_{\mu\nu\sigma\rho}x^\mu y^\nu z^\sigma w^\rho$.

(iii) If the ternary product $T(x, y, z)$ is defined on V by $\{T(x, y, z), w\} = q(x, y, z, w)$, then $3\{T(x, x, y), T(y, y, y)\} = \{x, y\}q(x, y, y, y)$.

* Further references can be found in [8].

TABLE 3

Representation dimensions for the E_7 family of invariance groups

Representation	SO(2)	A_1	$A_1+A_1+A_1$	C_3	A_5	D_6		E_7		
V = defining	2	4	8	14	20	32	44	56	164	224
V_6 = adjoint	1	3	9	21	35	66	99	133	451	630
V_7 = symmetric	2	7	27	84	175	462	891	1463	13 079	24 570
V_5 = antisym.	0	5	27	90	189	495	945	1639	13 365	24 975
V_{10}	0	6	48	216	540	1728	3696	6480	69 741	134 976
V_{11}	0	2	16	64	70+70	352	616	912	4059	5920

TABLE 4

Dynkin indices for the E_7 family of invariance groups

Representation	SO(2)	A_1	$A_1+A_1+A_1$	C_3	A_5	D_6		E_7		
V = defining		$\frac{5}{2}$	1	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{5}{14}$	$\frac{1}{3}$	$\frac{5}{18}$	$\frac{10}{37}$
V_6 = adjoint		1	1	1	1	1	1	1	1	1
V_7 = symmetric		14	9	9	10	$\frac{63}{5}$	$\frac{108}{7}$	$\frac{55}{3}$	$\frac{406}{9}$	$\frac{2233}{37}$
V_5 = antisym.		5	6	$\frac{15}{2}$	9	12	15	18	45	60
V_{10}		$\frac{35}{4}$	14	$\frac{45}{2}$	$\frac{63}{2}$	$\frac{252}{5}$	70	90	$\frac{2205}{8}$	380
V_{11}		$\frac{1}{4}$	2	4	$\frac{11}{4}+\frac{11}{4}$	$\frac{38}{5}$	9	10	$\frac{107}{8}$	14

The third property is nothing but our invariance condition (2.5) for $e_{\mu\nu\rho\delta}$, as can be verified by substituting P_6 from (8.5). Hence our quadratic, quartic invariants fulfil all three properties assumed by Brown. He then proceeds to prove that the 56-dimensional representation of E_7 has the above properties, and saves us from that labour.

11. The extended supergravities and the magic triangle

The purpose of all this algebra has been to show that the extension of Minkowski space into a superspace can be a non-trivial enterprise. We now have an exhaustive classification, but are there any realizations of it? Surprisingly enough, every single entry in the classification appears to be realized as a global symmetry of an extended supergravity.

Cremmer and Julia [4] have discovered that in $N = 8$ (or $N = 7$) supergravity's 28 vectors together with their 28 duals form a $\underline{56}$ multiplet of a global E_7 symmetry. This is a global symmetry analogous to SO(2) duality rotations of the doublet $(F_{\mu\nu}, F^*_{\mu\nu})$ in $j^\mu = 0$ sourceless electrodynamics. The appearance of E_7 was quite unexpected; it is the first time we see an exceptional Lie group emerge as a symmetry without having been inserted into a model by hand. While the classification I have obtained here does not explain why this happens, it suggests that there is a deep connection between extended supergravities and the exceptional Lie algebras. To establish this connection, observe that Cremmer and Julia's $N = 7, 6, 5$ global symmetry groups E_7, SO(12), SU(6) are included in the present classification. Furthermore, vectors plus their duals form multiplets of dimension 56, 32, 20, so they belong to the defining representations in my classification. For $N \leq 4$ extended supergravities the numbers of vectors do not match the dimensions of the defining representations. However, if one adds one vector multiplet*, the numbers match up, and $N = 1, 2, \ldots, 7$ extended supergravities exhaust the present classification. This is summarized in table 5. As I have not explicitly constructed $N = 1, 2, 3, 4$ supergravities with the extra vector multiplet, at present this list of global symmetries is only a conjecture. However, there are examples of such supergravities in the literature [17]. Scherk has argued that they are physically preferable [18] to supergravities without extra multiplets. They might also be natural from the dimensional reduction point of view; for example, extended supergravity in five spacetime dimensions [19] reduces to $N = 2$ extended supergravity with an extra vector multiplet.

As mentioned in the preceding section, the present classification is a row of the magic triangle [2]. This is an extension of Freudenthal's magic square, an octonionic construction of exceptional Lie algebras of great interest to mathematicians [6]. The remaining rows are obtained [2] by applying the methods of the present paper to various kinds of quadratic and cubic invariants, while the columns are subgroup

* I am indebted to Poul Howe for this observation.

TABLE 5

Conjectured global symmetries of extended supergravities

						Global symmetry			Local symmetry	
N	gravitons	gravitinos	vectors	fermions	scalars	group	d	d_A	group	d'_A
$8=7$	1	8	28	56	70	E_7	56	133	SU(8)	63
6	1	6	16	26	30	SO(12)	32	66	U(6)	36
5	1	5	10	11	10	SU(6)	20	35	U(5)	25
4	1	4	$6+\underline{1}$	$4+\underline{4}$	$2+\underline{6}$	Sp(6)	14	21	Sp(4)×SU(2)	13
3	1	3	$3+\underline{1}$	$1+\underline{4}$	$\underline{6}$	$SU(2)^3$	8	9	SO(3)	3
2	1	2	$1+\underline{1}$	$\underline{2}$	$\underline{2}$	SU(2)	4	3	SO(2)	1
1	1	1	$\underline{1}$	$\underline{1}$	0	SO(2)	2	1	SO(2)	1

$N=5, 6, 7$ symmetries have been verified explicitely by Cremmer and Julia, ref. [4]. $N=1, 2, 3, 4$ symmetries are a conjecture based on Cremmer–Julia counting rules (d = number of vectors and dual vectors, $d_A = d'_A$ + number of scalars) and the present classification, table 3.

chains. In this context the Diophantine condition (8.6) is one of a family of Diophantine conditions of form

$$d_A = \tfrac{1}{3}(k-6)(l-6) - 72 + 360\left(\frac{1}{k} + \frac{1}{l}\right). \tag{11.1}$$

[(8.6) is recovered by taking $k = 24$, $d = 2l - 16$.] Further Diophantine conditions, analogous to (9.1), reduce the solutions to k, $l = 8$, 9, 10, 12, 15, 18, 24, 36. The corresponding Lie algebras form the magic triangle, table 6.

As we have seen above, extended supergravities in $D = 4$ dimensions are related to the E_7 row of the magic triangle. Could it be that supergravities in other dimensions are related to the remainder of the magic triangle? There is some evidence to support such claim. Dimensional reduction of $D = 11$ supergravity yields a sequence of

TABLE 6

The magic triangle for Lie algebras

						0 0	3 A_1 2
					0 0	1 U(1) 1	8 A_2 3
				0 0	0 1	3 A_1 3	14 G_2 7
			0 0	0 1	2 2U(1) 2	9 $3A_1$ 4	28 D_4 8
		0 0	0 2	3 A_1 5	8 A_2 8	21 C_3 14	52 F_4 26
	0 0	0 1	2 2U(1) 3	8 A_2 6	16 $2A_2$ 9	35 A_5 15	78 E_6 27
0 0	1 U(1) 2	3 A_1 4	9 $3A_1$ 8	21 C_3 14	35 A_5 20	66 D_6 32	133 E_7 56
3 A_1 3	8 A_2 8	14 G_2 14	28 D_4 28	52 F_4 52	78 E_6 78	133 E_7 133	248 E_8 248

The Freudenthal magic square is marked by the dotted line. Within each entry the number in the upper left corner is d_A, the dimension of the algebra, and the number in the lower left corner is d, the dimension of the defining representation. Defining representation dimensions for the top four rows are guesses, while for the bottom four rows they have been derived in ref. [2]. The Diophantine conditions also allow for trivial slutions indicated by $d = d_A = 0$ along the hypotenuse.

extended supergravities in $D = 10, 9, \ldots, 5, 4, 3$ with E_{11-D} global symmetry [20]. For $D = 5, 4, 3$ this coincides with the magic triangle E_6, E_7, E_8 algebras. This suggests that different rows of the magic triangle should correspond to different spacetime dimensions, while the columns should correspond to the number of physical degrees of freedom (for example, both $D = 11$ supergravity and $D = 4$, $N = 7$ extended supergravity have 128 Fermi and 128 Bose physical degrees of freedom [21]). It is straightforward to construct the analogue of table 5 for $D = 5$ which corresponds to E_6 row of the magic triangle and which reduces [21] correctly to $D = 4$ row. Proceeding this way one can fill up the magic triangle with conjectured particle content of all supergravities; however, in the absence of explicit equations of motion this scheme is too ambiguous to be of much value.

12. Conclusion

This article consists of two parts. The first part is a non-trivial example of how Grassmann dimensions can be interpreted as negative bosonic dimensions. To recapitulate; replacement of bosonic dimensions by Grassmann dimensions interchanges symmetrizations and antisymmetrizations. The Diophantine conditions and the projection operators in d bosonic dimensions are the same as the corresponding conditions and projection operators in $-d$ Grassmann dimensions. In particular, while the bosonic solutions include SO(4), fermionic solutions are $E_7(56)$, $D_6(32)$, etc.

This motivates the second part of the paper, the conjectured connection to extended supergravities. Two conjectures are made: (i) Extended supergravities in four spacetime dimensions exhaust the classification. This implies that an extra vector multiplet must be added to $N = 1, 2, 3, 4$ extended supergravities. (ii) All possible extended supergravities in $3 \leqslant D \leqslant 11$ spacetime dimensions exhaust all (global) symmetry groups listed in the magic triangle.

Finally, it should be noted that the Grassmann dimensions of the supersymmetric invariants (x, y), (x, y, x, w) are not the customary superspace dimensions θ^α. Indeed, as the classification arose from the consideration of Grassmann–Grassmann sector alone, supersymmetry was inessential to the argument, and it is possible that $D = 4$ extended supergravities just happen to have a symplectic quadratic and symmetric quartic invariant on the mass-shell without any deep connection to the underlying superspace. In that sense conjecture 2 stands on much weaker ground than the conjecture 1. However, if conjecture 2 were true, it would be very intriguing because it suggests an octonionic formulation of supergravities. Similar possibility has been considered by Günaydin [22].

I am grateful to P. Howe and L. Brink for teaching me how to count, E. Cremmer for hospitality at Ecole Normale Superieure and W. Siegel for many helpful criticisms.

Appendix A

DYNKIN INDICES

In the Lie algebra tables [10] a representation is identified by a pair of integers; the dimension and the Dynkin index. Computation of Dynkin indices requires some more group-theoretic formalism.

A linear transformation g acts on the tensor space $\mathrm{V}\otimes\mathrm{V}\otimes\bar{\mathrm{V}}\otimes\cdots$ by acting on all components as

$$g(\mathrm{V}\otimes\mathrm{V}\otimes\bar{\mathrm{V}}\otimes\cdots)=G\mathrm{V}\otimes G\mathrm{V}\otimes\bar{\mathrm{V}}G^{-1}\otimes\cdots. \tag{A.1}$$

For the infinitesimal transformation, $G=1+\varepsilon^{\alpha}_{\beta}T^{\beta}_{\alpha}$, this implies

$$\begin{aligned}T^{\alpha}_{\beta}(\mathrm{V}\otimes\mathrm{V}\otimes\bar{\mathrm{V}}\otimes\cdots)&=(T^{\alpha}_{\beta}\mathrm{V})\otimes\bar{\mathrm{V}}\otimes\cdots+\mathrm{V}\otimes(T^{\alpha}_{\beta}\mathrm{V})\otimes\bar{\mathrm{V}}\otimes\cdots\\&\quad-\mathrm{V}\otimes\mathrm{V}\otimes(\bar{\mathrm{V}}T^{\alpha}_{\beta})\otimes\cdots-\cdots.\end{aligned} \tag{A.2}$$

Hence the $[d^m\times d^m]$ matrix representation of T^{α}_{β}, the generators of the Lie algebra, can be built from the $[d\times d]$ representation by

$$\begin{aligned}(T^{\alpha}_{\beta})^{\alpha_1\alpha_2}_{\alpha_3\cdots\alpha_m,\beta_1\beta_2}{}^{\beta_3\cdots\beta_m}&=(T^{\alpha}_{\beta})^{\alpha_1}{}_{,\beta_1}\delta^{\alpha_2}_{\beta_2}\delta^{\beta_3}_{\alpha_3}\cdots\delta^{\beta_m}_{\alpha_m}+\delta^{\alpha_1}_{\beta_1}(T^{\alpha}_{\beta})^{\alpha_2}{}_{,\beta_2}\delta^{\beta_3}_{\alpha_3}\cdots\delta^{\beta_m}_{\alpha_m}\\&\quad+\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}(T^{\alpha}_{\beta})_{\alpha_3,}{}^{\beta_3}\delta^{\beta_m}_{\alpha_m}+\cdots,\end{aligned} \tag{A.3}$$

$$(T^{\beta}_{\alpha})^{\gamma}{}_{,\,\delta}=-(T^{\beta}_{\alpha})_{\delta,}{}^{\gamma}=(P_A)^{\beta\gamma}_{\alpha\delta}. \tag{A.4}$$

In this matrix notation a tensor $x^{\alpha_1\alpha_2}_{\alpha_3\cdots\alpha_m}$ (or equivalently, a vector x^{α}) is invariant if it is annihilated by the generators

$$T^{\beta}_{\alpha}x=0. \tag{A.5}$$

(Remember that T^{β}_{α} is a $[d^m\times d^m]$ matrix with the two extra labels indexing it as one of the algebra generators.) A matrix $A^{\mu}{}_{,\nu}$ is an invariant matrix if it commutes with the generators (the invariance condition (2.5) is an example):

$$T^{\alpha}_{\beta}A-AT^{\alpha}_{\beta}=0. \tag{A.6}$$

In particular, matrices $T^2=T^{\alpha}_{\beta}T^{\beta}_{\alpha}$, T^3, $T^4,\cdots$ are invariant matrices. If a representation V_i is irreducible the quadratic Casimir operator T^2 has a single eigenvalue

$$P_iT^2=C_iP_i, \tag{A.7}$$

which characterizes the representation. For the defining representation V, which is by definition assumed irreducible,

$$(T^2)^{\alpha}{}_{,\,\beta}=\frac{d_{\mathrm{A}}}{d}\delta^{\alpha}_{\beta}. \tag{A.8}$$

The dimension of the algebra d_A arises because by (A.4) Tr $T^2 = \text{Tr}\, P_A = d_A$. This is my normalization convention for the generators T^α_β.

Instead of computing C_i, which is defined only for the irreducible representation, it is more convenient to compute tr $P_i T^2$, which is a pure number for any subspace V_i. As an example consider T^2 acting on the $V \otimes \bar{V}$ space:

$$(T^2)^{\alpha}_{\beta,}{}^{\gamma}_{\delta} = (T^{\sigma}_{\rho})^{\alpha}_{\beta,}{}^{\varepsilon}_{\chi}(T^{\rho}_{\sigma})^{\chi}_{\varepsilon,}{}^{\gamma}_{\delta}$$

$$= (T^2)^{\alpha},{}_{\delta}\delta^{\gamma}_{\beta} + \delta^{\gamma}_{\beta}(T^2)_{\beta},{}^{\gamma} - 2\sigma(P_A)^{\alpha\gamma}_{\beta\delta}\,.$$

Here $\sigma(\;)$ is the index permutation (4.9). Substituting (A.8) one obtains

$$T^2 = \frac{2d_A}{d}\,1 - 2\sigma(P_A)\,. \tag{A.9}$$

To get more specific, consider the adjoint representation $P_A = P_6$ from (4.7):

$$P_6\sigma(P_6) = \frac{1}{\alpha_6 - \alpha_7}\, P_6(\sigma(E) - \alpha_7\sigma(P_4)) = -\frac{\alpha_6 + \frac{1}{2}\alpha_7}{\alpha_6 - \alpha_7}\, P_6\,. \tag{A.10}$$

Collecting (4.15), (4.17) and (A.9) one gets

$$\text{Tr}\, P_6 T^2 = d_6\,\frac{4(d+2)}{16-d}\,. \tag{A.11}$$

(In another words Tr $P_A T^2 = \sum_{ijk} (C_{ijk})^2$, where C_{ijk} are the structure constants of the Lie algebra.) Tr $P_i T^2$ for any other V_i can be computed in this fashion. However, as the normalization conventions differ [the present one is fixed by (A.8)], it is natural to define a normalization independent Dynkin index by

$$l_i = \frac{\text{Tr}\, P_i T^2}{\text{Tr}\, P_A T^2}\,. \tag{A.12}$$

Like the dimensions, Dynkin indices satisfy simple sum rules. If a representation is reducible, $V_i = V_e \oplus V_m \oplus \cdots V_n$, then

$$l_i = l_e + l_m + \cdots l_n\,. \tag{A.13}$$

In the particular example we are considering, one can use (A.8) and (A.12) to compute the Dynkin index for the defining representation of sect. 4:

$$l = \frac{16-d}{4(d+2)}\,. \tag{A.14}$$

Indices for the representations constructed in this paper are listed in table 2. Again the corresponding indices for the E_7 family of solutions are obtained by replacing $d \to -d$. Their numerical values are listed in table 4.

Appendix B

NEGATIVE DIMENSIONS AND CLASSICAL GROUPS

In the introduction I have alluded to a general relation between SO(d) and Sp($-d$). It is a relation that is implicit in the representation theory of Lie supergroups [23, 24]. This type of relation is best illustrated by the representations of SU(n). Let λ stand for a Young tableau, and $\bar{\lambda}$ for the Young tableau obtained by flipping the original tableau across the diagonal. Then the dimensions of the two tableaux, expressed as polynomials in d, are related by

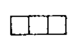
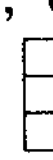
$$d_\lambda(d) = (-1)^p d_{\bar{\lambda}}(-d)\,. \tag{B.1}$$

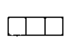
This is evident from the standard recipe for computing dimensions of SU(d) representations [25]:

d	$d+1$	$d+2$	$d+3$
$d-1$	d	$d+1$	
$d-2$			
$d-3$			

(B.2)

Clearly, flipping the tableau across the diagonal (exchanging symmetrizations and antisymmetrizations) amounts to the replacement $d \to -d$ in the dimension formula.

Irreducible tensorial representations of SO(d) are obtained from the irreducible tensors of SU(d) by subtracting all possible traces formed with the invariant $g^{\mu\nu}$ from (3.1). This procedure is described in sect. 10-5 of Hamermesh [26], and the dimension formulas are given by King and Murtaza and Rashid [27]. As $g^{\mu\nu}$ is symmetric, it can be used to contract indices in the same row of a Young tableau, but not the indices in the same column. For example, the dimension of the SO(d) representation corresponding to  is $\frac{1}{6}d(d-1)(d+4)$, different from the corresponding SU(d) representation, but the dimension of  representation is $\frac{1}{6}d(d-1)(d-2)$ for both SU(d) and SO(d).

Irreducible tensorial representations of Sp(d) are obtained from the irreducible tensors of SU(d) by subtracting all possible traces formed with the skew-symmetric invariant $g^{\mu\nu}$ of (7.1), as described by Hamermesh [26]. The dimension formulas are given in ref. [27]. Due to the skew symmetry of $g^{\mu\nu}$, indices in the same column of Young tableau can be contracted, but not the indices in the same row. For example, the dimension of  representation of Sp(d) is $\frac{1}{6}d(d+1)(d-4)$, different from SU(d), but the dimension of  representation is $\frac{1}{6}d(d+1)(d+2)$ for both SU(d) and Sp(d). We note that SO(d) and Sp(d) dimensions are related by $d \to -d$.

In general, flipping an SO(d) tableau across the diagonal means not only that we have exchanged antisymmetrizations and symmetrizations, but also that we have traded in a symmetric $g^{\mu\nu}$ for an antisymmetric $g^{\mu\nu}$, i.e. the flipped tableau corresponds to an Sp(d) representation. Indeed, King [27] has proven that (B.1) also holds when d_λ is the dimension of the SO(d) representation associated with the tableau λ, and $d_{\bar{\lambda}}$ is the dimension of the Sp(d) representation associated with the flipped tableau $\bar{\lambda}$. $d \to -d$ relations also apply to Casimir operators (cf. tables 2 and 4); this shall be described elsewhere [2].

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