Closed complex rays in scattering from elastic voids

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Abstract. The scattering determinant for the scattering of waves from several obstacles is considered in the case of elastic solids with voids. The scattering determinant displays contributions from closed ray splitting orbits. A discussion of the weights of such orbits is presented.

Keywords: semiclassics, zeta function, scattering determinant, elastodynamics
PACS: 03.65.Sq, 05.45.Mt, 46.40.Cd, 62.30.+d

1. INTRODUCTION

Studies of helmholtz scatterers have shown effects of trapped periodic orbits [1, 2]. We shall discuss the influence of closed trapped rays on scattering determinants in a medium with several polarizations each with their own velocity. The example treated will be the case of elastic wave propagation in a solid filled with a finite number of voids. There a ray hitting a boundary can either reflect or refract. In that case ray splitting occurs when the polarization changes. This leads to a ray dynamics which no longer is unique, where in general a single polarized ray evolves into a tree of rays. A similar behaviour is observed in microwave resonators with dielectrica. There rays can either reflect or transmit at the boundaries of the dielectrica.

2. ELASTODYNAMICS

In isotropic elasticity the wave equation in the frequency domain is

$$\mu\Delta(\mathbf{u}) + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \rho\omega^2\mathbf{u} = 0\,, \tag{1}$$

where $\mathbf{u}(\mathbf{x})$ is the displacement field in the body, λ, μ are the material dependent Lamé coefficients and ρ is the density [3, 4]. This wave equation admits two different polarizations: longitudinal and transverse with velocities

$$c_L = \sqrt{\frac{\lambda+2\mu}{\rho}} \qquad \text{and} \qquad c_T = \sqrt{\frac{\mu}{\rho}}\,. \tag{2}$$

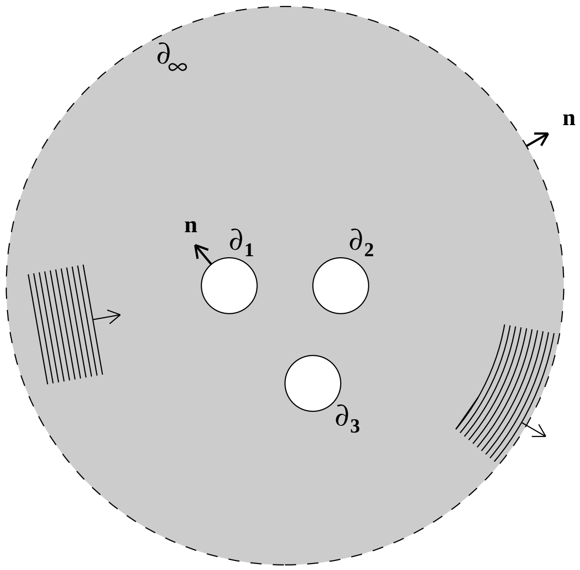

FIGURE 1. General zone of two–dimensional cavity scattering.

The longitudinal and transverse waves correspond to pressure respective shear deformations. This leads to the law of refraction for incoming plane waves

$$\frac{c_L}{c_T} = \frac{\sin\theta_L}{\sin\theta_T}, \tag{3}$$

where θ_L, θ_T denote the angle of incident or reflection of the longitudinal and transverse wave, respectively, measured with respect to the normal to the surface.

The stress tensor in elasticity has the form

$$\sigma_{ij} = \lambda\, \partial_k u_k \delta_{ij} + \mu \left(\partial_i u_j + \partial_j u_i\right) . \tag{4}$$

The boundary conditions considered here are *free*. Hence

$$\mathbf{t}(\mathbf{u}) = \sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{0} \tag{5}$$

for the displacement field at the boundary where **n** denotes the normal to the boundary. The operator **t** refers to the traction.

3. EXACT SCATTERING DETERMINANT

Via the null–field method a class of multi–scattering problems can be solved exactly. In particular exact scattering matrix elements are known for the case of several scatterers of analytically seperable shapes for various types of media [1, 5, 6]. As mentioned, in this treatment the medium corresponds to an elastic solid which is assumed isotropic for simplicity. The scattering geometry consists of parallel cylindrical voids, see fig. 1. Using line sources parallel to the voids respects this symmetry. This leads to two–dimensional elasticity referred to as plane strain.

The scattering determinant may be factorized into a part containing single scattering determinants and the whole set of scatterers [7]

$$\det \mathsf{S}(\omega) = \left\{ \prod_{j \in \text{Cavities}} \det \left[\mathsf{S}^{(1)j} \right](\omega) \right\} \frac{\text{Det}\left[\mathsf{M}(\omega^*)^\dagger\right]}{\text{Det}\left[\mathsf{M}(\omega)\right]}. \tag{6}$$

As the scatterers are moved around only the latter factor, the *cluster* determinant, changes:

$$\mathsf{M} = \mathbf{1} + \mathsf{A} \tag{7}$$

$$\mathsf{A}_{ll'}^{jj'} = (1-\delta_{jj'}) \frac{a_j}{a_{j'}} \left[\mathsf{t}_l^{(J)j}\right] \cdot \left[\mathsf{T}_{ll'}^{(+)jj'}\right] \cdot \left[\mathsf{t}_{l'}^{(+)j'}\right]^{-1},$$

$$\left[\mathsf{T}_{ll'}^{(+)jj'}\right]_{\sigma\sigma'} = \delta_{\sigma\sigma'} H_{l-l'}^{(+)}(k_\sigma R_{jj'}) e^{il\alpha_{j'}^{(j)} - il'(\alpha_j^{(j')} - \pi)}.$$

The matrix $\left[\mathsf{T}_{ll'}^{(+)jj'}\right]_{\sigma\sigma'}$ may interpreted as a translation matrix acting on the scattering states [5, 6]. $\alpha_i^{(j)}$ is the angle to the center of cavity i in the coordinate system of cavity j.

The single cavity scattering matrices decompose over angular momentum l due to the rotational symmetry. They have the general form:

$$\left[\mathsf{S}_l^{(1)j}\right] = -\left[\mathsf{t}_l^{(+)j}\right]^{-1} \cdot \left[\mathsf{t}_l^{(-)j}\right] \tag{8}$$

with the boundary conditions occurring in two-by-two matrices $\left[\mathsf{t}_l^{(Z)j}\right]$ with l the angular momentum, "type" $Z \in \{+,-,J\}$ and j the cavity index. The type refers to outgoing, incoming or regular scattering states and involves $H_l^{(1)}, H_l^{(2)}$ or J_l Bessel functions. Thus, for the outgoing case

$$\left[\mathsf{t}_l^{(+)j}\right] = \frac{2\mu}{a_j^2} \begin{pmatrix} (l^2 - z_T^2/2) H_l^{(1)}(z_L) - z_L \frac{dH_l^{(1)}(z_L)}{dz_L} & il(H_l^{(1)}(z_T) - z_T \frac{dH_l^{(1)}(z_T)}{dz_T}) \\ il(H_l^{(1)}(z_L) - z_L \frac{dH_l^{(1)}(z_L)}{dz_L}) & -(l^2 - z_T^2/2) H_l^{(1)}(z_T) + z_T \frac{dH_l^{(1)}(z_T)}{dz_T} \end{pmatrix}. \tag{9}$$

The row index $i \in \{r, \phi\}$ is a geometric index for polar coordinates and the column index is a polarization index $\sigma \in \{L, T\}$. Hence, the single cavity scattering matrix connects different polarizations. For a full discussion, see [8, 9]. The connection to the interior problem of a single disc is described in [10].

The poles of the cluster determinant $\text{Det}\,\mathsf{M}(\omega)$ cancel by construction the poles of the single determinants. Likewise for the poles $\text{Det}\,\mathsf{M}(\omega^*)$ which cancel the zeros of the single determinants. Thus all scattering resonances corresponding to poles of the scattering determinant can be found from the zeros of the cluster determinant. As an example consider the resonances in fig. 2 of a two–cavity system for a material with $c_L = 1950$ m/s, $c_T = 540$ m/s, cavity radii equal to 1 m and intercavity separation as measured from the centers equal to 6 m. The regular spaced horizontal set of resonances in fig. 2 is placed below an irregular set. This is opposite to the scalar Helmholtz case

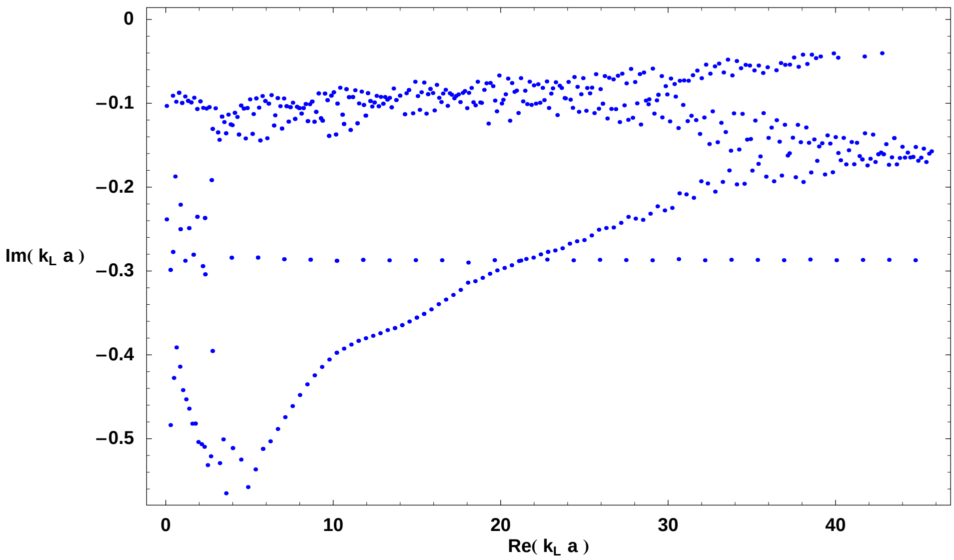

FIGURE 2. Elastodynamic scattering resonances for two cavities (A_1-representation).

for the same geometry where the regular spaced resonances are above the irregular [11, 12]. The regular resonances particular to the fundamental A_1-representation are well described by the following condition

$$0 = 1 - \exp(ik_L L)/\sqrt{\Lambda} \tag{10}$$

with the length $L = 4a$ and instability $\Lambda = 5 + 2\sqrt{6}$. L corresponds to that of the shortest orbit moving in a symmetry reduced domain bouncing between a cavity and the center of mass of the two cavities. Λ is obtained from the product of ray matrices as the leading eigenvalue of the corresponding (geometric acoustic) ray system. This next raises the question about the effect of the remaining set of orbits.

4. ORBITS IN TIME–DELAY

For real frequencies the total scattering phase Θ is seen to be a sum over cluster phase Θ_c and single cavity phases Θ_j, see (6). Likewise the derivative with respect to frequency $d\Theta/d\omega$, the Wigner-Smith time delay, decomposes. The numerics of the cluster time–delay $d\Theta_c/d\omega$ show fluctuations which are related to trapped orbits in the scattering geometry, see fig. 3, where the results for two identical cavities (same as those of fig. 2) are presented [1]. Some of these orbits are diffractive including segments of surface propagation of Rayleigh type, ie. earthquakes [13]. For these proceedings we discuss the purely non-diffractive contributions, called geometrical ray splitting orbits. Although

[1] Due to symmetries the cluster delay decomposes further into a sum over four ireps, two of which are shown.

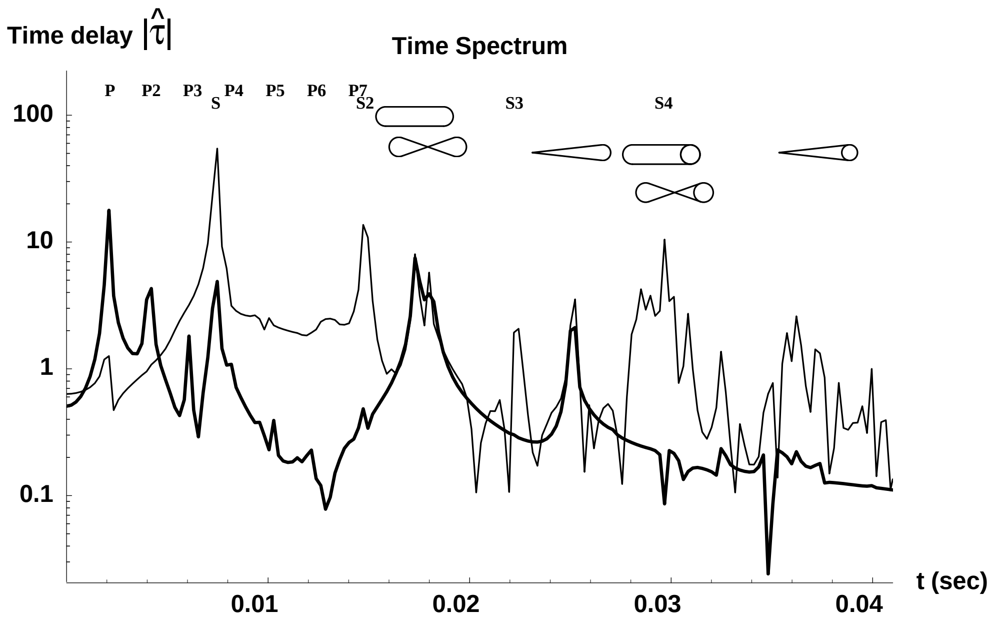

FIGURE 3. Power spectrum of cluster fluctuations in time–delay for two cavities. Symbols Pi and Si denote orbits bouncing i times of pressure respective shear polarization. The circular arcs indicate Rayleigh–surface waves.

these orbits are essential in most geometries, our work done on scattering has shown the importance of the Rayleigh orbits at intermediate frequencies. This is indicated in the title which refers to that such orbits are complex. A short discussion of the diffractive orbits is given in [14].

5. EXPANDING THE CLUSTER DETERMINANT

A first step towards orbits is to consider the definition of the infinite cluster determinant in terms of traces:

$$F(z) = \det(\mathbf{1} + z\mathsf{A}) \equiv \exp\left(-\sum_{n=1}^{\infty} \frac{z^n \operatorname{tr}(-\mathsf{A})^n}{n}\right) \tag{11}$$

which at $z = 1$ evaluates to the desired. This holds if M is Fredholm and (11) is called a Fredholm expansion. This translates into well behaved numerical properties. For example the determinant converges as the dimension of the truncation of M increases. To obtain M with such properties a regularization has been performed. This regularization may be thought of as a Jacobi preconditioner to the original problem in which a matrix is replaced by the one divided by the diagonal. Presently the Fredholm properties of M has only been proved in detail in the scalar case [2]. Nevertheless, we proceed as if this is true also in our more general case. As numerical evidence we mention, that the resonances of the Fredholm expansion of the determinant to fourth order z^4 agree well with the exact resonances, the latter plotted in fig. 2.

6. RAY LIMIT AND ORBITS

The expansion (11) indicates that the cluster determinant can be obtained from the knowledge of an increasing number of traces. In the saddle point approximation closed orbits bouncing n times are seen to contribute to traces of powers of the kernel $\mathrm{tr}\,\mathsf{A}^n$ [15]. These orbits fulfill the laws of reflection and refraction and have phases corresponding to their time periods of revolution T_p.

In the calculation of a trace, the single cavity scattering matrices (8) are inserted as operators between the translation operators. To do so, use the identity

$$J_l(z) = \frac{1}{2}\left(H_l^{(1)}(z) + H_l^{(2)}(z)\right)$$

giving a similar identity for the single cavity matrices that encode the boundary condition:

$$\left[\mathsf{t}_l^{(J)}\right] = \frac{1}{2}\left(\left[\mathsf{t}_l^{(+)}\right] + \left[\mathsf{t}_l^{(-)}\right]\right), \tag{12}$$

which is finally is substituted into A in (7). The ray limit of the single cavity scattering matrix gives unitary reflection coefficients similar to those of scattering from an infinite half–plane [3, 10]. This leads to an overall amplitude α_p defined as a product over all reflection coefficients along the orbit. This amplitude describes the leakage from the orbit due to ray splitting.

The calculation of their geometric amplitudes requires more work. For a general reference in the interior scalar case, see [16]. Asymptotic wave theory indicates [17–20] that for *open* trajectories in two dimensions the amplitude goes as $(kR)^{-1/2}$ where k is the wave number in question and R is the radius of curvature of the wave front at the observer. This radius is studied in e.g. geometric optics and it is possible to keep track of its evolution during free propagation between the scatterers and at impact with possible refractions using suitable ray matrices. Indeed, it can be shown that for our problem open segments, in which end points are fixed and intermediate variables are integrated by the saddle point method, has such an amplitude evolution. This comes about by calculating the accompanying sparse hessian of this restricted integration.

For a full saddle point integration over all variables, a *closed* orbit p, the amplitude turns out to be expressible as yet a sparse hessian and can be expanded into hessians corresponding to the previous considered open pieces as in [2]. Using the previous information then allows the full calculation with the amplitude going as

$$\mathscr{A}_p = \frac{\alpha_p}{|\mathrm{Det}\,(\mathbf{1} - \mathbf{J}_p)|^{1/2}}\, z^{n_p} \tag{13}$$

with $\mathbf{J}_p$ and α_p the product of ray matrices respective reflection coefficients along the orbit bouncing n_p times. This form is precisely part of the conventional semiclassical density of states [21–23]. However, the formal parameter z is also present and can be seen as an ordering of the various orbits in the expansion over infinitely many orbits.

Incorporating the results of the geometric ray splitting orbits gives the following factor of the cluster determinant:

$$F_G(z) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r}\, \alpha_p^r \,\frac{e^{ir\,\omega T_p}}{|\mathbf{1}-\mathbf{J}_p^r|^{1/2}}\, z^{rn_p} \right) . \tag{14}$$

The sum over r refers to a sum over repeats of shortest orbits, *prime* cycles p. Note, that if the logarithmic derivative with respect to ω is taken a result very similar to the spectral density for the interior problem is obtained. This yields an agreement with the general result for the density of states for ray splitting systems described in [24]. Similar results for the case of flexural vibrations in the interior case are given in [25]. As the orbits are unstable and $\mathbf{J}_p$ is symplectic it is possible to expand the instability denominator in (14) in the inverse of its leading eigenvalue $1/\Lambda_p$ for each orbit and obtain a so-called Gutzwiller–Voros resummed zeta function similar to those of two-dimensional Hamiltonian flows [26]:

$$F_G(z) = \prod_{k=0}^{\infty} \zeta_k^{-1} , \tag{15}$$

where

$$1/\zeta_k(z) = \prod_p (1 - t_p^{(k)}) \quad \text{with} \quad t_p^{(k)} = \alpha_p \frac{e^{i\omega T_p}}{\sqrt{|\Lambda_p|}\Lambda_p^k} z^{n_p} . \tag{16}$$

7. SUMMARY

Detailed studies of Helmholtz scattering determinants at small wave lengths have shown the influence of closed geometric rays. The case of scattering from voids in two–dimensional elastodynamics was considered here with a discussion of the analytical contribution of closed geometric ray splitting orbits to the scattering determinant.

ACKNOWLEDGMENTS

N.S. acknowledges discussions with J.D. Achenbach and the European Network on *Mathematical aspects of Quantum Chaos* which supported his stay at the Weizmann Institute, the Crafoord Foundation and from Det Svenska Vetenskapsrådet.

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