ISSN 0106-2646

83-11-360

NORDITA preprint

NORDITA-83/39

UNIVERSALITY IN CHAOS

(or, Feigenbaum for cyclists)

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September 1983

(Lectures given at the XXII Cracow School
of Theoretical Physics, Zakopane, June 1982.
Submitted to the Acta Physica Polonica.)

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The often repeated statement, that given the initial conditions we know what a deterministic system will do far into the future, is false. Poincaré (1892) knew it was false, and we know it is false, in the following sense: given infinitesimally different starting points, we often end up with wildly different outcomes. Even with the simplest conceivable equations of motion, almost any non-linear system will exhibit chaotic behaviour. A familiar example is turbulence.

Turbulence is the unsolved problem of classical physics. However, recent developments have greatly increased our understanding of turbulence, and given us new concepts and modes of thought that we hope will have far reaching repercussions in many different fields (solid state physics, hydrodynamics, plasma physics, chemistry, quantum optics, biology, meteorology, acoustics, mechanical engineering, elementary particle physics, mathematics, fishery[2], astrophysics, cosmology, electrical engineering and so on).

The developments that we shall describe here are one of those rare demonstrations of the unity of physics. The key discovery was made by a physicist not trained to work on problems of turbulence. In the fall of 1975 Mitchell Feigenbaum, an elementary particle theorist, discovered a universality in one-dimensional iterations. At the time the physical implications of the discovery were rather unclear. During the next few years, however, numerical and theoretical studies established this universality in a number of models in various dimensions. Finally, in 1980, the universality theory passed its first test in an actual turbulence experiment.

The discovery was that large classes of non-linear systems exhibit transitions to chaos which are universal and quantitatively measurable. This advance can be compared to past advances in the theory of solid state phase transitions; for the first time we can predict and measure "critical exponents" for turbulence. But the breakthrough consists not so much in discovering a new set of scaling numbers, as in developing a new way to do physics. Traditionally we use regular motions (harmonic oscillators, plane waves, free particles, etc.) as zeroth-order approximations to physical systems, and account for weak non-linearities perturbatively. We think of a dynamical system as a smooth system whose evolution we can follow by integrating a set of differential equations. The universality theory seems to tell us that the zeroth-order approximations to strongly non-linear systems should be quite different. They show an amazingly rich structure which is not at all apparent in their formulation in terms of differential

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1. Lectures given at the XXII-nd Cracow School of Theoretical Physics, Zakopane, June 1982. Sections 1 to 4 were written in collaboration with Mogens Høgh Jensen (Cvitanović and Høgh Jensen 1982).

2. Of special interest to our Icelandic colleagues.

equations. However, these systems do show self-similar structures which can be encoded by universality equations of a type which we will describe here. To put it more succinctly, junk your old equations and look for guidance in clouds' repeating patterns.

In these lectures we shall reverse the chronology, describing first an actual turbulence experiment, then a numerical experiment, and finally explain the observations using the universality theory. We will try to be intuitive and concentrate on a few key ideas, referring you to the literature for more detailed expositions[3]. Even though we illustrate it by turbulence, the universality theory is by no means restricted to the problems of fluid dynamics. The key concepts of phase-space trajectories, Poincaré maps, bifurcations, and local universality are common to all non-linear dynamical systems. The essence of this subject is incommunicable in print; intuition is developed by computing. We urge the reader to carry through a few simple numerical experiments on a desktop computer, because that is probably the only way to start perceiving order in chaos.

1. Onset of turbulence

We start by describing schematically the experiment of Libchaber and Maurer (1980) (a nice description has been given by Libchaber and Maurer(1981). In this type of experiment a liquid contained in a small box is heated from the bottom. The salient points are:

1. There is a controllable parameter, the Rayleigh number, which is proportional to the temperature difference between the bottom and the top of the cell. (Rayleigh number describes the stability of a convective flow (see Velarde and Normand 1980).)

2. The system is dissipative. Whenever the Rayleigh number is increased, one waits for the translents to die out.

For small temperature gradients there is a heat flow across the cell, but the liquid is static. At a critical temperature a convective flow sets in. The hot liquid rises in the middle, the cool liquid flows down at the sides, and two convective rolls appear:

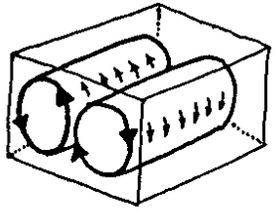

Fig.1.1

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3. The most thorough exposition available is the Collet and Eckmann (1980a) monograph. We also recommend Hu (1982), Crutchfield, Farmer and Huberman (1982), Eckmann (1981) and Ott (1981).

As the temperature difference is increased further, the rolls become unstable in a very specific way - a wave starts running along the roll:

Fig.1.2

As the warm liquid is rising on one side of the roll, while cool liquid is descending down the other side, the position and the sideways velocity of the ridge can be measured with a thermometer:

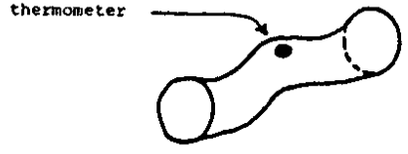

Fig.1.3

One observes a sinusoid:

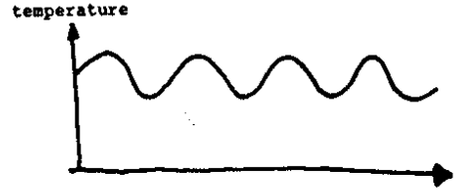

Fig.1.4

The periodicity of this instability suggests two other ways of displaying the measurement:

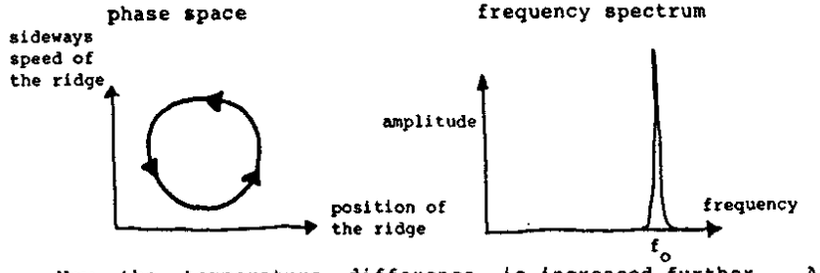

Fig.1.5

Now the temperature difference is increased further. After the stabilisation of the phase-space trajectory, a new wave is observed superimposed on the original sinusoidal instability. The three ways of looking at it (real time, phase space, frequency spectrum) are:

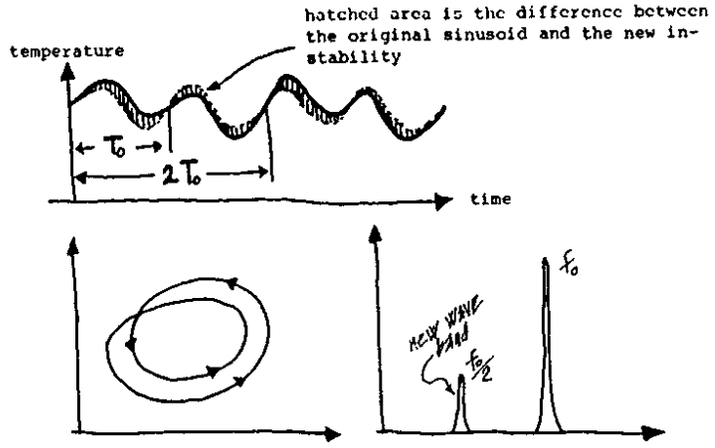

Fig.1.6

A coarse measurement would make us believe that T_o is the periodicity; however, a closer look reveals that the phase-space trajectory misses the starting point at T_o, and closes on itself only after $2T_o$. If we look at the frequency spectrum, a new wave band has appeared at half the original frequency. Its amplitude is small, because the phase-space trajectory is still approximately a circle with periodicity T_o.

Now, as one increases the temperature very slightly, a fascinating thing happens - the phase-space trajectory undergoes a very fine splitting:

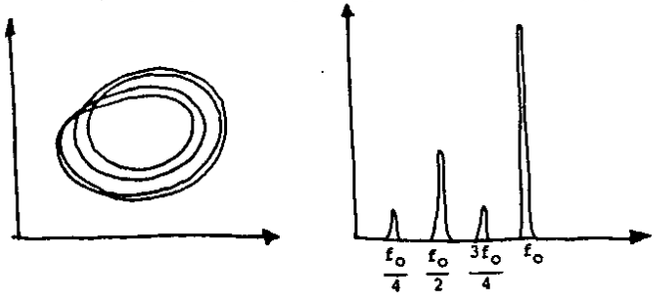

Fig.1.7

We see that there are three scales involved here. Looking casually, we see a circle with period T_o; looking a little closer, we see a pretzel ⊗ with period $2T_o$; and looking very closely, we see that the trajectory closes on itself only after $4T_o$. The same information can be read off the frequency spectrum; the dominant frequency is f_o (the circle), then $f_o/2$ (the pretzel), and finally, much weaker $f_o/4$ and $3f_o/4$.

The experiment now becomes very difficult. A minute increase in the temperature gradient causes the phase-space trajectory to split on an even finer scale, with the periodicity 2^3T_o. If the noise were not killing us, we would expect these splittings to continue, yielding a trajectory with finer and finer detail, and a frequency spectrum

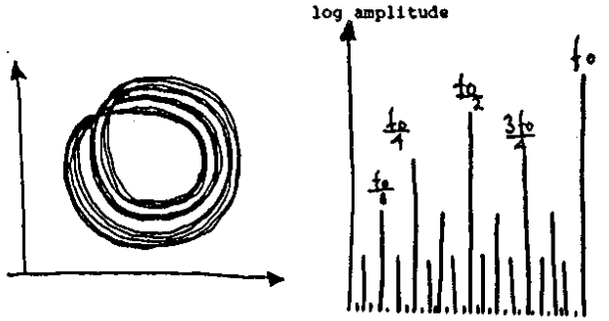

Fig.1.8

with families of ever weaker frequency components. For a critical value of the Rayleigh number, the periodicity of the system is $2^\infty T$, and the convective rolls have become turbulent (this is weak turbulence - the rolls persist, wiggling irregularly). The ripples which are running along them show no periodicity, and the spectrum of idealized, noise-free experiment contains infinitely many subharmonics:

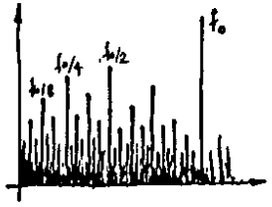

Fig.1.9

If one increases the temperature gradient beyond this critical value, there are further surprises: we refer you to Libchaber and Maurer (1981). We now turn to a numerical simulation of a simple non-linear oscillator in order to start understanding why the phase-space trajectory splits in this peculiar fashion.

2. Onset of chaos in a numerical experiment

In the experiment that we have just described, limited experimental resolution makes it impossible to observe more than a few bifurcations. Much longer sequences can be measured in numerical experiments; the non-linear oscillator studied by Arecchi and Lisi (1982) is a typical example:

$$\ddot{x} + k\dot{x} - x + 4x^3 = A\cos(\omega t) \qquad (2.1)$$

The oscillator is driven by an external force of frequency ω, with amplitude A and the natural time unit $T_o = 2\pi/\omega$. The dissipation is controlled by the friction coefficient k. Given the initial displacement and velocity one can easily follow numerically (by the Runge-Kutta method for example) the phase-space trajectory of the system. Due to the dissipation it does not matter where one starts in the phase space; for a wide range of initial points the phase-space trajectory converges to a

limit cycle (trajectory loops onto itself) which for some $k = k_o$ looks something like this (fig. 12a in Feigenbaum 1980a):

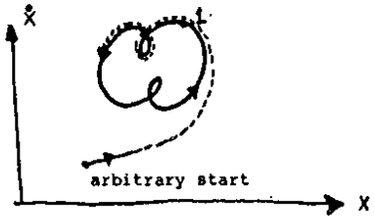

Fig.2.1

If it were not for the external driving force, the oscillator would have simply come to a stop; as it is, it is executing a motion forced on it externally, independent of the initial displacement and velocity. You can easily visualise this non-linear pendulum executing little backward jerks as it swings back and forth. Starting at the point marked 1, the pendulum returns to it after the unit period T_o.

However, as one decreases the friction, the same phenomenon is observed[4] as in the turbulence experiment; the limit cycle undergoes a series of period-doublings

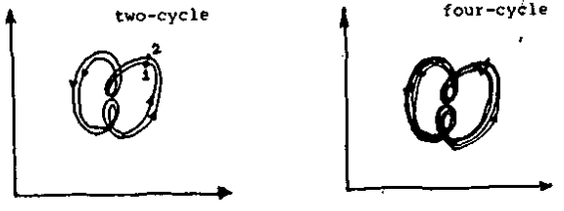

Fig.2.2

The trajectory keeps on nearly missing the starting point, until it hits it after exactly $2^n T_o$. The phase-space trajectory is getting increasingly hard to draw; however, the sequence of points 1, 2, ..., 2^n, which corresponds to the state of the oscillator at times T_o, $2T_o$, ..., $2^n T_o$, sits in a small region of the phase space, so we enlarge it for a closer look:

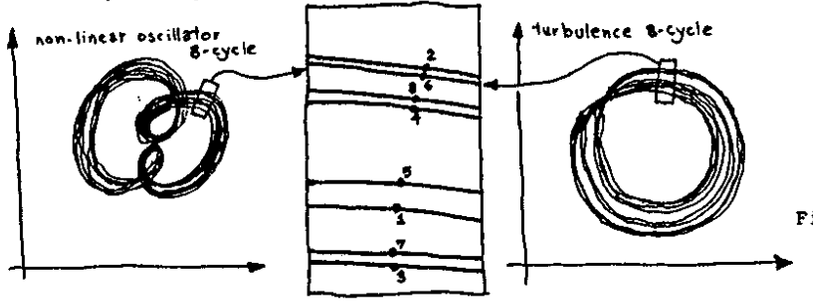

Fig.2.3

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4. If you have a desktop computer with graphics, you can easily do this experiment yourself. For example, if you take $k = 0.154$, $\omega = 1.2199778$ and $A = 0.1$, 0.11, 0.114, 0.11437, ..., you will observe bifurcations. There is nothing special about these parameter values; we give them just to help you with finding your first bifurcation sequence.

Globally the phase-space trajectories of the turbulence experiment and of the non-linear oscillator numerical experiment look very different. However, the above sequence of near misses is local, and looks roughly the same for both systems. Furthermore, this sequence of points lies approximately on a straight line

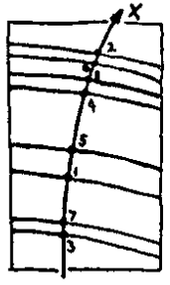

Fig.2.4

Let us concentrate on this line: this way of reducing the dimensionality of the phase space is often called a Poincaré map. Instead of staring at the entire phase-space trajectory, one looks at its points of intersection with a given surface. The Poincaré map contains all the information we need; from it we can read off when an instability occurs, and how large it is. One varies continuously the non-linearity parameter (friction, Rayleigh number, etc.) and plots the location of the intersection points; in the present case, the Poincaré surface is a line, and the result is a bifurcation tree:

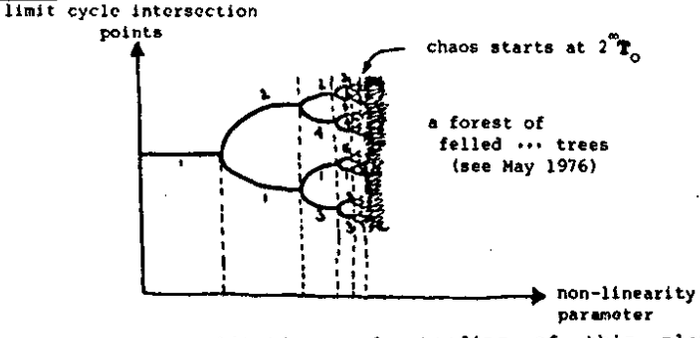

Fig.2.5

We already have some qualitative understanding of this plot. The phase-space trajectories we have drawn are localised (the energy of the oscillator is bounded) so the tree has a finite span. Bifurcations occur simultaneously because we are cutting a single trajectory; when it splits, it does so everywhere along its length. Finer and finer scales characterise both the branch separations and the branch lengths.

Feigenbaum's discovery[5] consists of the following quantitative observations:

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5. The geometric parameter convergence was first noted by Myrberg (1958), and independently of Feigenbaum, by Grossmann and Thomae (1977). However, these authors have not emphasised the universality of δ.

1. The parameter convergence is universal (i.e., independent of the particular physical system):

$$\Delta_i/\Delta_{i+1} \to \delta = 4.6692... \quad \text{for i large}$$

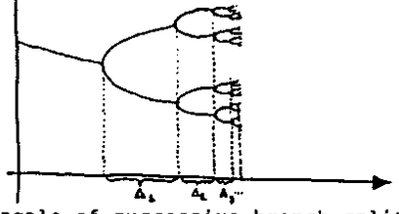

Fig.2.6

2. The relative scale of successive branch splittings is universal:

$$\varepsilon_i/\varepsilon_{i+1} \to \alpha = 2.5029... \quad \text{for i large}$$

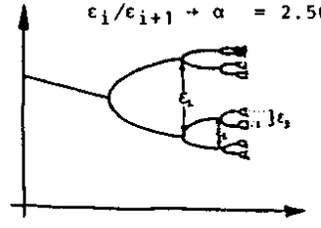

Fig.2.7

The beauty of this discovery is that if turbulence (chaos) is arrived at through an infinite sequence of bifurcations, we have two quantitative predictions:

1. The convergence of the critical Rayleigh numbers corresponding to the cycles of length 2, 4, 8, 16, ... is controlled by the universal convergence parameter $\delta = 4.6692016...$.

2. The splitting of the phase-space trajectory is controlled by the universal scaling parameter $\alpha = 2.50290787...$. As we have indicated in our discussion of the turbulence experiment, the relative heights of successive subharmonics measure this splitting and hence α.

These universal numbers are measured in a variety of experiments: we shall summarise the experimental situation in section 9.

You might think that this universality applies only to very simple, essentially one-dimensional systems (single pendulum, oscillations along a convective roll), but that is not true. For example, Franceschini and Tebaldi (1979) have investigated numerically the following system:

$$\begin{aligned}
\dot{x}_1 &= -2x_1 + 4x_2 x_3 + 4x_4 x_5 \\
\dot{x}_2 &= -9x_2 + 3x_1 x_3 \\
\dot{x}_3 &= -5x_3 - 7x_1 x_2 + r \\
\dot{x}_4 &= -5x_4 - x_1 x_5 \\
\dot{x}_5 &= -x_5 - 3x_1 x_4
\end{aligned}$$

where r is the controllable external parameter. Within certain intervals of values of the parameter r ("Reynolds number" for the system) they found infinite sequences of period-doublings. Moreover, even though the phase space was ten-dimensional, any Poincaré map they tried yielded period-doublings along a one-dimensional line. The convergence parameter δ and the scaling number α they obtained agree with Feigenbaum's universal numbers. This particular system of equations is a truncation of the Navier-Stokes equations, and in the literature there are innumerable other examples of period-doublings in many-dimensional systems. A wonderful thing about this universality is that it does not matter much how close our equations are to the ones chosen by nature; as long as the model is in the same universality class (in practice this means that it can be modelled by a mapping of form (3.2)) as the real system, both will undergo a period-doubling sequence. That means that we can get the right physics out of very crude models, and this is precisely what we will do next. (Of course, we have no clue whether we are really in the same universality class.)

The reason why multidimensional dissipative systems become effectively one-dimensional is roughly this: for a dissipative system phase-space volumes shrink. They shrink at different rates in different directions; the direction of the slowest convergence defines a one-dimensional line which will contain the attractor (the region of the phase space to which the trajectory is confined at asymptotic times):

Fig.2.8

The real story is both more subtle and more interesting; the phase-space volume shrinks in some directions and grows and folds onto itself in others - so the attractor, while thin, can be very complicated. Nearby points on this attractor can have very different histories. You should read Collet and Eckmann (1980a) and Collet, Eckmann and Koch (1980) for a detailed description of how a dissipative system becomes one-dimensional.

What we have presented so far are a few experimental facts; we now have to convince you that they are universal. To do this, we shall have to talk about fish.

3. What does all this have to do with fishing?

Looking at the phase-space trajectories shown earlier, we observe that the trajectory bounces within a restricted region of the phase space. How does this happen? One way to describe this bouncing is to plot the (n+1)th intersection of the trajectory with the Poincaré surface as a function of the preceding intersection. Referring to fig.2.4 we find:

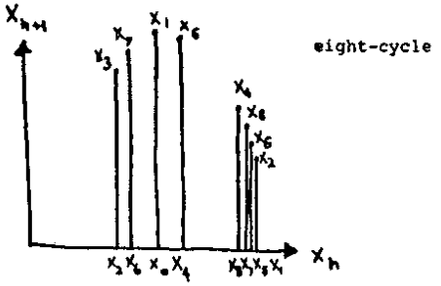

Fig.3.1

This is a Poincaré map (or return map) for the limit cycle. If we start at various points in the phase space (keeping the non-linearity parameter fixed) and mark all passes as the trajectory converges to the limit cycle, we trace an approximately continuous curve f(x)

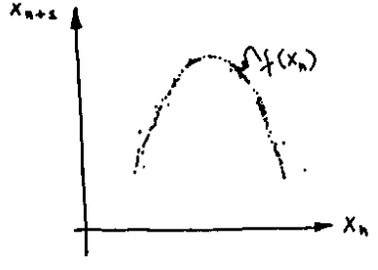

Fig.3.2

which gives the location of the trajectory at time $t+T_o$ as a function of its location at time t:

$$x_{n+1} = f(x_n) \qquad (3.1)$$

The trajectory bounces within a trough in the phase space, and f(x) gives a local description of the way the trajectories converge to the limit cycle. In principle we know f(x), as we can measure it (see Simoyi, Wolf and Swinney (1982) for a construction of a return map in a chemical turbulence experiment) or compute it from the equations of motion. The form of f(x) depends on the choice of Poincaré map, and obtaining an analytic expression for f(x) is difficult (see Gonzales and Piro (1983) for an example of an explicit return map), but we know what f(x) should look like; it has to fall on both sides (to confine the trajectory), so it has a maximum. Around the maximum it looks like a parabola

$$f(x) = a_o + a_2(x-x_c)^2 + \ldots \qquad (3.2)$$

like any sensible polynomial approximation to a function with a hump [6]

---

6. If f(x) is not quadratic around the maximum, the universal numbers will be different - see Villela Mendés (1981) and Hu and Mao (1982b) for their values. According to Kuramoto and Koga (1982) such mappings can arise in chemical turbulence.

This brings us to the problem of a rational approach to fishery. By means of a Poincaré map we have reduced a continuous trajectory in phase space to one-dimensional iteration. This one-dimensional iteration has an analog in population biology, where f(x) is interpreted as a population curve (the number of fish x_{n+1} in the given year as a function of the number of fish x_n the preceding year), and the bifurcation tree fig.2.5 has been studied in considerable detail. We recommend reviews by May (1976) and Hoppensteadt (1978) for further reading.

The first thing we need to understand is the way in which a trajectory converges to a limit cycle. A numerical experiment will give us something like:

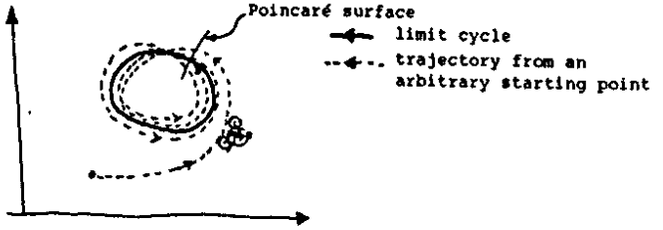

Fig.3.3

In the Poincaré map the limit trajectory maps onto itself

$$x^* = f(x^*)$$

Hence a limit trajectory corresponds to a fixed point of f(x). Take a programmable pocket calculator and try to determine x^*. Type in a simple approximation [7] to f(x), such as

$$f(x) = 1-Rx^2 \qquad (3.3)$$

Here R is the non-linear parameter. Enter your guess x_1 and press the button. The number x_2 appears on the display. Is it a fixed point? Press the button again, and again, until $x_{n+1} \approx x_n$ to desired accuracy. Diagrammaticaly

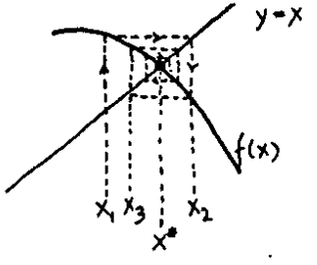

Fig.3.4

Note the tremendous simplification gained by the use of the Poincaré map. Instead of computing the entire phase-space trajectory by a numerical integration of the equations of motion, we are merely pressing a button on a pocket calculator.

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7. This way of modelling dynamical systems was introduced by Lorenz (1964).

This little calculation confirms one's intuition about fishery. Given a fishpond, and sufficient time, one expects the number of fish to stabilise. However, no such luck - a rational fishery manager soon discovers that anything can happen from year to year. The reason is that the fixed point x* need not be attractive, and our pocket calculator computation need not converge.

4. A universal equation

Why is the naive fishery manager wrong in concluding that the number of fish will eventually stabilise? He is right when he says that $x^* = f(x^*)$ corresponds to the same number of fish every year. However, this is not necessarily a stable situation. Reconsider how we got to the fixed point in fig.3.4. Starting with a sufficiently good guess, the iterates converge to the fixed point. Now start increasing gently the non-linearity parameter (Rayleigh number, the nutritional value of the pond, etc.). f(x) will slowly change shape, getting steeper and steeper at the fixed point,

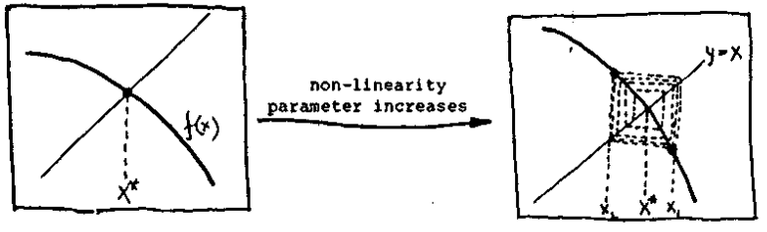

Fig.4.1

until the fixed point becomes unstable and gives birth to a cycle of two points [8]. This is precisely the first bifurcation observed in our experiments. This is the only gentle way in which our trajectory can become unstable (cycles of other lengths can be created, but that requires delicate fiddling with parameters; they are not generic). Now we return to the same point after every second iteration

$$x_i = f(f(x_i)), \quad i = 1, 2,$$

so the cycle points of f(x) are the fixed points of f(f(x)).

To study <u>their</u> stability, we plot f(f(x)) alongside f(x):

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8. Program your pocket calculator to evaluate (3.3). Chose some value of R between 0 and 2, and x between -1 and 1. Type in the initial x, press the start button and read off the next x. Now press the start button again. The game consists in staring at the display, and looking for regularities in the sequences of iterates. Try also the following values of R: 1, 1.31070274134, 1.38154748443, 1.3979453597. Compute the next number in this series.

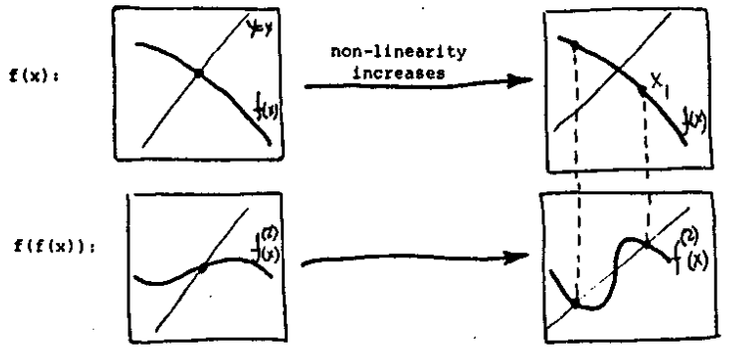

Fig.4.2

What happens as we continue to increase the "Rayleigh number"? f(x) becomes steeper at its fixed point, and so does f(f(x)). Eventually the magnitude of the slope at the fixed points of f(f(x)) exceeds one, and they bifurcate. Now the cycle is of length four, and we can study the stability of the fixed points of the fourth iterate. They too will bifurcate, and so forth. This is why the phase-space trajectories keep on splitting $2 \to 4 \to 8 \to 16 \to 32 \ldots$ in our experiments. The argument does not depend on the precise form of f(x), and therefore the phenomenon of successive period-doublings is <u>universal</u> (Metropolis,Stein and Stein, 1973).

More amazingly, this universality is not only qualitative. In our analysis of the stability of fixed points we kept on magnifying the neighbourhood of the fixed point:

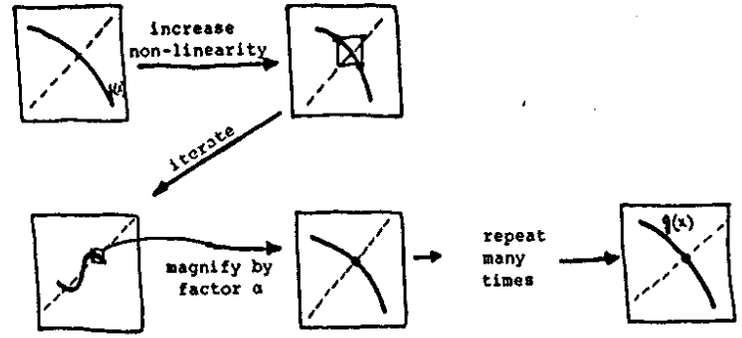

Fig.4.3

The neighbourhoods of successive fixed points look very much the same after iteration and rescaling. After we have magnified the neighbourhoods of fixed points many times, practically all information about the global shape of the starting function f(x) is lost, and we are left with a <u>universal function</u> g(x). Denote by T the operation indicated in fig.4.3: iterate <u>twice</u> and rescale by α (without changing the non-linearity parameter),

$$Tf(x) = -\alpha f(f(-x/\alpha)), \qquad (4.1)$$

g(x) is self-reproducing under rescaling and iteration:

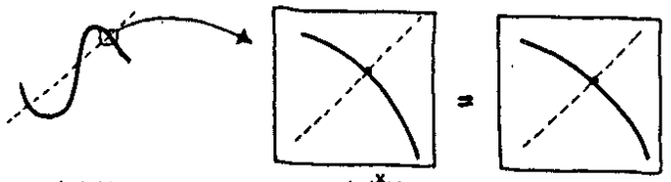

Fig.4.4

More precisely, this can be stated as the universal equation

$$g(x) = -\alpha g(g(-x/\alpha)), \qquad (4.2)$$

which determines both the universal function g(x) and $\alpha = -1/g(1) = 2.50290787...$ (with normalisation convention $g(0) = 1$).

If you arrive at g(x) the way we have, by successive bifurcations and rescalings, you can hardly doubt its existence. However, if you start with (4.2) as an equation to solve, it is not obvious what its solutions should look like. The simplest thing to do is to approximate g(x) by a finite polynomial and solve the universal equation numerically, by Newton's method (Feigenbaum 1979a). This way you can compute α and δ to much higher accuracy than you can ever hope to measure them to experimentally.

There is much pretty mathematics in universality theory. Despite its simplicity, nobody seems to have written down the universal equation [9] before 1976, so the subject is still young. We do not have a series expansion for α, or an analytic expression for g(x); the numbers that we have are obtained by boring numerical methods. So far, all we know is that g(x) exists (Lanford 1982). What is proved is that the Newton iteration converges, so we are no wiser for the result. In some situations the universal equation (4.2) has analytic solutions; we shall return to this in the discussion of intermittency (section 10). The universality theory has also been extended to iterations of complex polynomials (section 11).

To see why the universal function must be a rather crazy function, consider high iterates of f(x) for parameter values corresponding to 2-, 4- and 8-cycles:

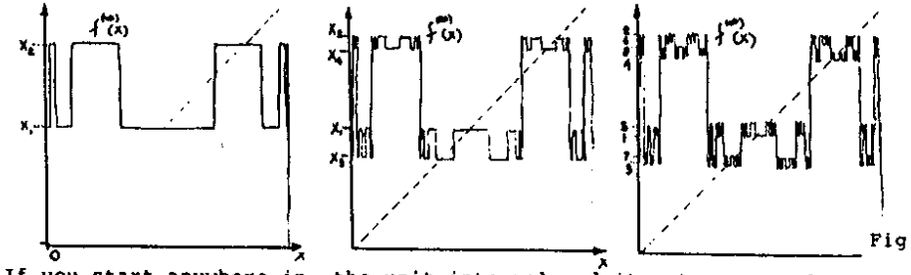

Fig.4.5

If you start anywhere in the unit interval and iterate a very large number of times, you end up in one of the cycle points. For the 2-cycle there are two possible limit values, so f(f(...f(x))) resembles a castle battlement. Note the infinitely many intervals accumulating at the unstable x = 0 fixed point. In a bifurcation of the 2-cycle into the 4-cycle each of these intervals gets replaced by a smaller battlement. After infinitely many bifurcations this becomes a fractal (i.e., looks the same under any enlargement), with battlements within battlements on every scale. Our universal function g(x) does not look like that close to the origin, because we have enlarged that region by the factor $\alpha = 2.5029...$ after each period-doubling, but all the wiggles are still there; you can see them in Feigenbaum's (1978) plot of g(x). For example, (4.2) implies that if x^* is a fixed point of g(x), so is αx^*. Hence g(x) must cross the lines $y = x$ and $y = -x$ infinitely many times. It is clear that while around the origin g(x) is roughly a parabola and well approximated by a finite polynomial, something more clever is needed to describe the infinity of g(x)'s wiggles further along the real axis and in the complex plane.

All this is fun, but not essential for understanding the physics of the onset of chaos. The main thing is that we now understand where the universality comes from. We start with a complicated many-dimensional dynamical system. A Poincaré map reduces the problem from a study of differential equations to a study of discrete iterations, and dissipation reduces this further to a study of one-dimensional iterations (now we finally understand why the phase-space trajectory in the turbulence experiment undergoes a series of bifurcations as we turn the heat up!). The successive bifurcations take place in smaller and smaller regions of the phase space. After n bifurcations the trajectory splittings are of order $\alpha^{-n} = (0.399...)^n$ and practically all memory of the global structure of the original dynamical system is lost:

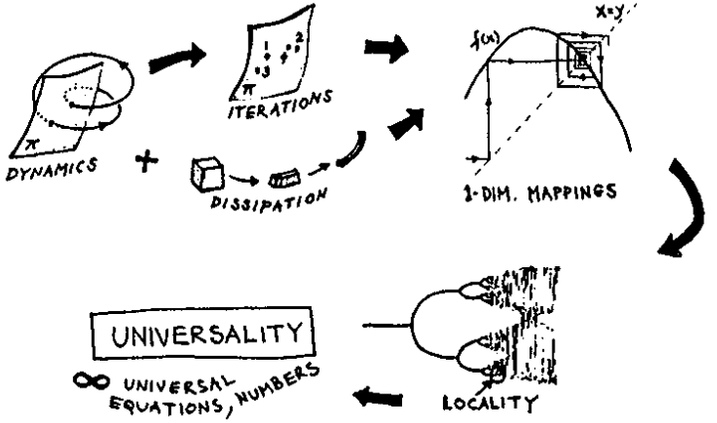

Fig.4.6

The asymptotic self-similarities can be encoded by universal equations. The physically interesting scaling numbers can be quickly estimated by simple truncations of the universal equations (May and Oster 1980, Derrida and Pomeau 1980, Helleman 1980a, Hu 1981). The full universal equations are designed for accurate determinations of universal numbers; as they have built-in rescaling, the round-off errors do not accumulate, and the only limit on the precision of the calculation is the machine precision of the computer.

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9. The universal equation was introduced by the author, in collaboration with M. J. Feigenbaum (1978). Coullet and Tresser (1978a,b) have proposed similar equations.

Anything that can be extracted from the asymptotic period-doubling regime is universal[10]; the trick is to identify those universal features that have a chance of being experimentally measurable. We will discuss several such extensions of the universality theory in the remainder of these lectures.

5. The unstable manifold

Feigenbaum's delta:

$$\delta = \lim_{n\to\infty} \frac{r_{n-1} - r_n}{r_n - r_{n+1}} = 4.6692016... \qquad (5.1)$$

is the universal number of the most immediate experimental import: it tells us that in order to reach the next bifurcation we should increase the Rayleigh number (or friction, or whatever the controllable parameter is in the given experiment) by about one fifth of the preceding increment. Which particular parameter is being varied is largely a question of experimental expedience; if r is replaced by another parameter R = R(r), then the Taylor expansion

$$R(r) = R(r_\infty)+(r-r_\infty)R'(r_\infty)+(r-r_\infty)^2R''(r_\infty)/2 + \ldots$$

yields the same asymptotic delta

$$\delta \propto \frac{R(r_{n-1}) - R(r_n)}{R(r_n) - R(r_{n+1})} = \frac{r_{n-1} - r_n}{r_n - r_{n+1}} + O(\delta^{-n}) \qquad (5.2)$$

providing, of course, that $R'(r_\infty)$ is non-vanishing (the chance that a physical system just happens to be parametrised in such a way that $R'(r_\infty) = 0$ is nil).

In deriving the universal equation (4.2) we were intentionally sloppy, because we wanted to introduce the notion of encoding self-similarity by universal equations without getting entangled in too much detail. We obtained a universal equation which describes the self-similarity in the x-space, under iteration and rescaling by α. However, the bifurcation tree fig.2.5 is self-similar both in the x-space and the parameter space: each branch looks like the entire bifurcation tree. We will exploit this fact to construct a universal equation which determines both α and δ.

Let T* denote the operation of iterating twice, rescaling x by α, shifting the non-linearity parameter to the corresponding value at the next

---

10. Derrida, Gervois and Pomeau (1979) have extracted a great many metric universalities from the asymptotic regime. Grassberger (1981) has computed the Hausdorff dimension of the asymptotic attractor. Lorenz (1980) and Daido (1981b) have found a universal ratio relating bifurcations and reverse bifurcations.

bifurcation[11], and rescaling it by δ:

$$T^* f_{R_n + \Delta_n p}(x) = -\alpha_n f^{(2)}_{R_n + \Delta_n (1 + p/\delta_n)}(-x/\alpha_n) \qquad (5.3)$$

Here R_n is a value of the non-linearity parameter for which the limit cycle is of length 2^n, Δ_n is the distance to R_{n+1}, $\delta_n = \Delta_n/\Delta_{n+1}$, p provides a continuous parametrisation, and we appologise that there are so many subscripts. T* operation encodes the self-similarity of the bifurcation tree[12] (fig.2.5):

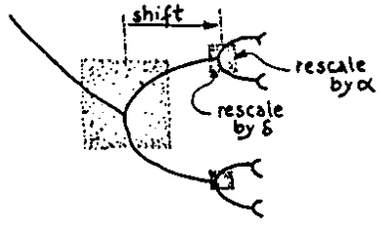

Fig.5.1

For example, if we take the fish population curve f(x) (3.3) with R value corresponding to a cycle of length 2^n, and act with T*, the result will be a similar cycle of length 2^n, but on a scale α times smaller. If we apply T* infinitely many times, the result will be a universal function with a cycle of length 2^n:

$$g_p(x) = (T^*)^\infty f_{R+p\Delta}(x) \qquad (5.4)$$

If you can visualise a space of all functions with quadratic maximum, you will find the following picture helpful:

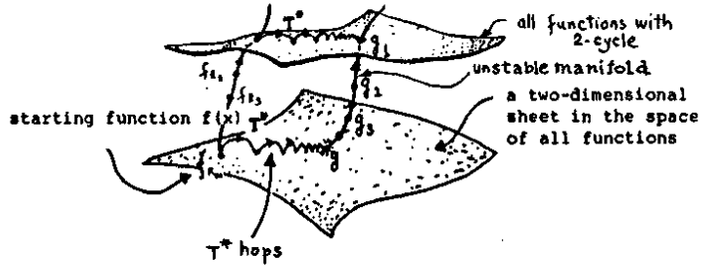

Fig.5.2

Each transverse sheet is a manifold consisting of functions with 2^n-cycle of given stability. T* moves us across this transverse manifold toward g_p.

$g_p(x)$ is invariant under the self-similarity operation T*, so it satisfies a universal equation:

$$g_p(x) = -\alpha g_{1+p/\delta}(g_{1+p/\delta}(-x/\alpha)) \qquad (5.5)$$

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11. More precisely, the value of the nonlinearity parameter with the same stability, i.e. the same slope at the cycle points.

12. Collet and Eckmann (1980a) give a very nice illustration of this self-similarity in Fig.I.28 of their monograph.

p parametrises a one-dimensional continuum family of universal functions[13]. Our first universal equation (4.2) is the fixed point of the above equation:

$$p^* = 1+p^*/\delta \qquad (5.6)$$

and corresponds to the asymptotic 2^∞-cycle[14].

You have probably forgotten by now, but we started this section promising a computation of δ. Feigenbaum (1979a) solved this problem by linearising the equations (5.5) around the fixed point p^*. Close to the fixed point $g_p(x)$ does not differ much from $g(x)$, so one can treat it as a small deviation from $g(x)$:

$$g_p(x) = g(x) + (p-p^*)h(x)$$

Substitute this into (5.5), keep the leading term in $p-p^*$, and use the universal equation (4.2). This yields a universal equation for δ:

$$g'(g(x))h(x) + h(g(x)) = -(\delta/\alpha)h(\alpha x) \qquad (5.7)$$

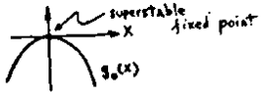
We already know $g(x)$ and α, so this can be solved numerically by polynomial approximations, yielding $\delta = 4.6692016...$ (plus a whole spectrum of eigenvalues and eigenvectors $h(x)$ (see Feigenbaum 1979a).

Actually, one can do better with less work; T*-operation treats the coordinate x and the parameter p on the same footing, which suggests that we should approximate the entire unstable manifold by a double power series (Vul and Khanin 1982, Golberg, Sinai and Khanin 1983)

$$g_p(x) = \sum_{j=0}^{N}\sum_{k=0}^{M} c_{jk}x^{2j}p^k \qquad (5.8)$$

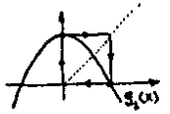
The scale of x and p is arbitrary. We will fix it by the normalisation conditions

$$g_0(0) = 0 \qquad (5.9)$$



$$g_1(0) = 1 \qquad (5.10)$$

$$g_1(1) = 0$$



The first condition means that the origin of p corresponds to the superstable fixed point. The second condition sets the scale of x and p by the superstable 2-cycle. (Superstable cycles are the cycles which have the

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13. This elegant formulation of universality is due to Vul and Khanin (1982) and Golberg, Sinai and Khanin (1983). I have learned it from M.J. Feigenbaum. The family of universal functions parametrised by p is called the unstable manifold because T-operation (4.1) drives p away from the fixed point value $g(x) = g_{p^*}(x)$.

14. This is not necessarily the only way to formulate universality; for example, Daido (1981a) has introduced a different set of universal equations.

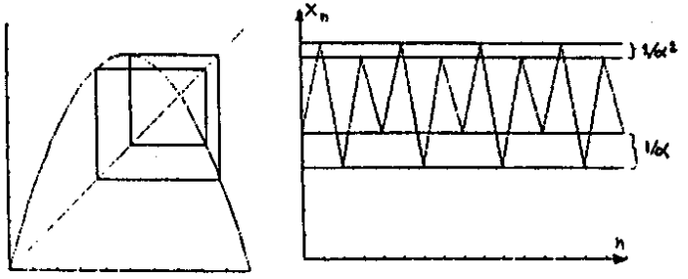
maximum of the map as one of the cycle points.) Start with any simple approximation to $g_p(x)$ which satisfies the above conditions (for example, $g_p(x) = p-x^2$). Apply the T*-operation (5.3) to it. This involves polynomial expansions in which terms of order higher than M and N in (5.8) are dropped. Now find by Newton's method the value of δ which satisfies normalisation (5.10). This is the only numerical calculation you have to do; the condition (5.10) automatically yields the value of α. The result is a new approximation to g_p. Keep applying T* until the coefficients in (5.8) repeat; this has moved the approximate g_p toward the unstable manifold along the transverse sheets indicated in fig.5.2. Computationally this is straightforward, the accuracy of the computation is limited only by computer precision, and at the end you will have α, δ and a polynomial approximation to the unstable manifold $g_p(x)$.

As δ controls the convergence of the high iterates of the initial mapping toward their universal limit $g(x)$, it also controls the convergence of most other numbers toward their universal limits, such as the scaling number $\alpha_n = \alpha+O(\delta^{-n})$, or even δ itself, $\delta_n = \delta+O(\delta^{-n})$. As $1/\delta = 0.2141...$ the convergence is very rapid, and already after a few bifurcations the universality theory is good to a few per cent. This rapid convergence is both a blessing and a curse. It is a theorist's blessing because the asymptotic theory applies already after a few bifurcations; but it is an experimentalist's curse because a measurement of every successive bifurcation requires a fivefold increase in the experimental accuracy of the determination of the non-linearity parameter r.

6. Power spectra

We have stated that the physical significance of the scaling number is that it sets the scale of trajectory splitting. That is true, but not good enough to make connection with experiments; our theory describes the splitting of the phase-space trajectories, while experimentalists (as we have discussed in section 1) usually measure the power spectrum. To construct the asymptotic spectrum, we need not only α (the splitting at the maximum of the return map), but also the splitting everywhere along the trajectory. This splitting is described by Feigenbaum's (1979b, 1980a) scaling function $\sigma(t)$.

To estimate the shape of the power spectrum, consider the 1-dimensional iteration analogue of the phase-space and real time outputs of the turbulence experiment (fig.1.6):



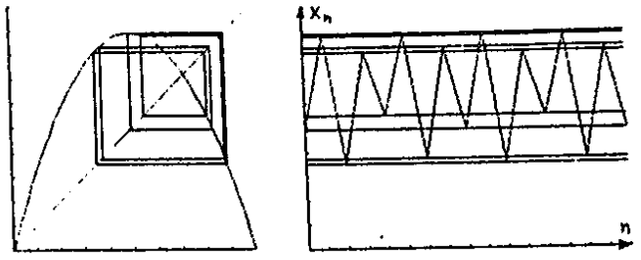


Fig.6.1

(these were obtained by taking the non-linearity parameter in (3.3) corresponding to 4- and 8-cycles and iterating). With each period-doubling a new instability comes in, at twice the period, or half the frequency of the preceding instability. Its amplitude can be read off the above figures; very crudely, it is just $1/\alpha$ (the width of the trajectory splitting) times the previous amplitude. We can easily do better; it is obvious from the above figures that the trajectory splitting is dominated by two scales; half of the time the splitting is of order $1/\alpha$, and the other half is determined by the projection through the quadratic maximum, $1/\alpha^2$. The successive subharmonics in the power spectrum (amplitude squared) fall off like the mean-square average of the two dominant trajectory splitting scales:

$$\mu \simeq 2(1/\alpha^2 + 1/\alpha^4)^{-\frac{1}{2}} = 4.648... \qquad (6.1)$$

This rough estimate tells us that the scaling function $\sigma(t)$ is basically either α^{-1} or α^{-2}. (A plot of $\sigma(t)$ is given in Feigenbaum 1980b.) The splitting is taking place on a smaller and smaller scale, and we expect universality. A numerical calculation (Nauenberg and Rudnick 1981) yields $\mu = 4.578...$. The actual power spectrum (a trivial calculation to do on a desktop computer) looks something like this:

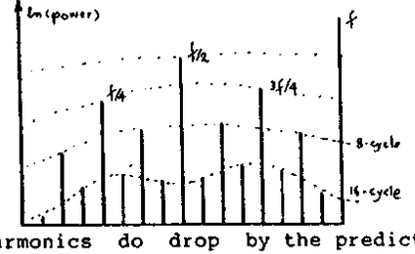


Fig.6.2

On average the subharmonics do drop by the predicted ratio, but they are strongly modulated.

Comparison with experiments (see section 9) requires some care. The problem is that one wants to use all subharmonics and higher harmonics seen in an experiment in computing the average (6.1), but the envelope of higher harmonics is not universal; it represents the deviation of the global phase-space trajectory (fig.1.6) from a perfect circle. A careful analysis of an experiment would utilise the spectrum observed in the first few bifurcations as an input for the calculation of the spectrum of the subsequent subharmonics.

7. Reverse bifurcations

So far we have concentrated on the sequence of bifurcations which arises as we smoothly destabilise a fixed point into a 2-cycle, 2-cycle into 4-cycle, and so on, until 2^∞-cycle. One can say that the system has become turbulent in the sense that the time period has become infinite, but the trajectory is anything but chaotic; it follows a very strict itinerary. Chaotic motion would usually mean that nearby trajectories diverge exponentially (positive Lyapunov coefficients) or that the power spectrum is characterised by broad-band noise (we somehow forgot to mention what we mean by chaos in this introduction to chaos). What happens if we keep on increasing the non-linearity parameter?

The most surprising thing that happens is that the system does not necessarily get more chaotic (remember, we are turning up the heat in the turbulence experiment, so one would expect more and more turbulence). Instead, one observes an infinite number of parameter ranges (windows) for which the system becomes periodic again. The largest such window is the 3-window, whose emergence we can detect by searching for fixed points of $f(f(f(x)))$. For the parameter value $R = 1.750...$ in the mapping (3.3), $f(f(f(x)))$ acquires three attractive fixed points, and $f(x)$ acquires a 3-cycle:

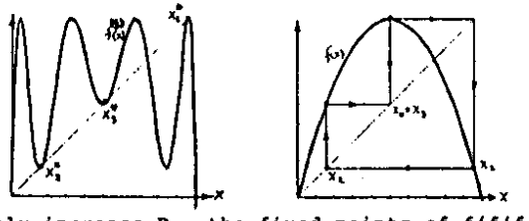


Fig.7.1

If we now smoothly increase R, the fixed points of $f(f(f(x)))$ bifurcate as before: the 3-cycle goes into a 6-cycle which then bifurcates into a 12-cycle, etc., with the same asymptotic universal period-doubling numbers[15].

Metropolis, Stein and Stein (1973) have discovered a qualitative universality in the relative ordering of the windows of different periods; the order does not depend on the map $f(x)$, as long as $f(x)$ has a differentiable maximum and falls off monotonically on both sides (see Collet and Eckmann (1980a) for a detailed discussion of universal ordering and itineraries of periodic orbits). Observation of the universal ordering in an experiment (such as Simoyi, Wolf and Swinney 1982) is strong evidence that the return map is one-dimensional; there is no ordering in

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15. There is a crisis ahead, though - see Grebogi, Ott and Yorke (1982).

higher dimensions where one can have co-existing cycles of different lengths and with different basins of attraction.

Periodic windows notwithstanding, the regime beyond 2^∞ parameter value is very chaotic. You can easily map out both the periodic windows and the chaotic bands if you increase R in (3.3) in small steps and plot a thousand or so iterates for each parameter increment:

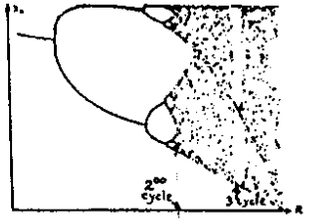

Fig.7.2

Ulam and von Neumann (1947) have actually used mapping (3.3) with R = 2 as a random number generator. At R = 2 the critical point (maximum of f(x)) is mapped into an unstable fixed point, which causes infinitesimally different initial conditions to result in wildly different sequences of iterates:

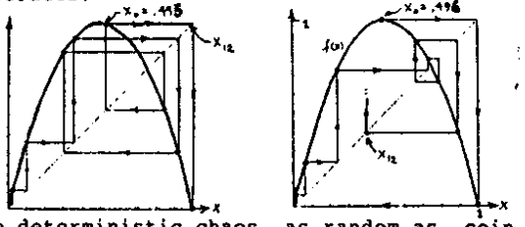

Fig.7.3

This is genuine deterministic chaos, as random as coin flipping (Grossmann and Thomae 1977, Ott 1981); iterates of any initial point (except for a set of measure zero) fill out the interval with a probability density independent of the initial point (see the plots of probability distributions in Collet and Eckmann 1980a). Another striking feature apparent in fig.7.2 is the existence of reverse bifurcations: the sequence of chaotic band-doublings which joins onto the bifurcation tree at R_∞. We can explain this sequence by the same tricks that we have used in our study of period-doublings. Plot f(f(x)) and look for the parameter value for which the critical point maps into the unstable fixed point:

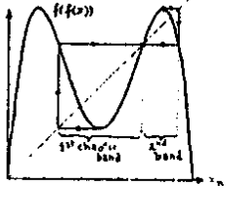

Fig.7.4

There are two chaotic bands, both similar to the original one. Now look at the second iterate of f(f(x)) and find the parameter value for which each of these bands has been split into a pair of chaotic bands. This yields a

Misiurewicz (1981) sequence of chaotic band-doublings:

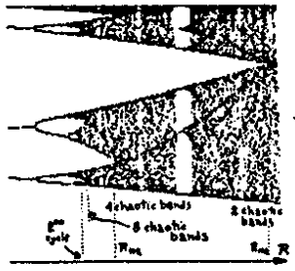

Fig.7.5

The Misiurewicz sequence is defined by the same self-similarity rule as the bifurcation sequence, and obeys the same universal equations (5.5), with the same universal convergence and scaling numbers α and δ. As the motion within the chaotic bands is aperiodic, the associated power spectrum will be characterised by broad-band noise. As was already observed by Lorenz (1964, 1979), the chaotic bands are a fat Cantor set in the parameter space, so this noise is appreciable. The universality of the asymptotic chaotic band-doublings suggests that we look for universal features in the noisy power spectrum. Using the self-similarity of the chaotic bands, Wolf and Swift (1981), Huberman and Zisook (1981), and Farmer (1981) have shown that the total power of the broad-band spectrum scales as

$$N(R) = \text{const.}(R-R_\infty)^{1.5247..} \qquad (7.5)$$

The exponent can be estimated (Wolf and Swift 1981) in terms of α by the same kind of argument as the one that lead to (6.1).

Periodic windows and broad-band noise are seen in all experiments on period doublings. The above universal exponent $\beta = 1.5247...$ has been measured in several non-linear electronic circuit experiments (Testa, Pérez and Jeffries 1982, Yeh and Kao 1982a).

8. External noise

You must have asked yourself by now: "But what about the noise in a real physical experiment? Do bifurcations survive noise?" Indeed, one might worry that at the asymptotic times to which our theory applies, the noise might build up without bound and wipe out everything. However, as we are considering dissipative physical systems, the noise gets damped as well. You can see that by iterating a quadratic return map - every time a noisy

trajectory goes through the maximum, the noise width σ is reduced to σ^2:

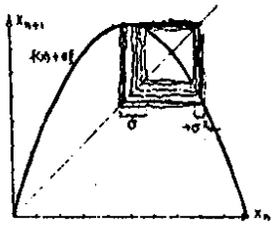

Fig.8.1

The noise does not get out of hand, but it does truncate the bifurcation sequences. As soon as the scale of trajectory splitting is smaller than the noise width, the bifurcations cannot be resolved any longer (Crutchfield and Huberman 1980). If α^{-n} (the scale of trajectory splittings) is of the order of the noise width, one can hope for at most n bifurcations. Actually, the bifurcations stop sooner than that, because the trajectory gets noisier with each bifurcation; after each period-doubling there is twice as much time for noise to accumulate.

As we know by now, anything that can be extracted from the aymptotic n regime is universal. Adding a weak noise term to mapping (3.1) is a good example of such universality

$$x_{n+1} = f(x_n)+\sigma\xi$$
$$\langle\xi\rangle = 0, \quad \langle\xi^2\rangle = 1 \qquad (8.1)$$

If σ is very small, the bifurcations proceed as before, until the scale of the nth bifurcation is comparable to the noise level. Hence we can iterate, rescale, etc. and the result will be close to the universal function (4.2):

$$g_n(x) = g(x) + \varepsilon_n\sigma\xi h(x)$$

Here n refers to a parameter p value corresponding to a 2^n-cycle, and ε_n is the noise at the nth level. As in (5.7) we can linearise around the fixed point, obtaining

$$\xi g'(g(x))h(x) + \xi' h(g(x)) = -(\kappa/\alpha)\xi' h(x) \qquad (8.2)$$

where ξ is the stochastic noise at the nth level, ξ′ the noise at the next level, and $\kappa = \varepsilon_{n+1}/\varepsilon_n$ is the factor by which the noise increases from one level to the next. Performing the average we obtain [16]:

$$\sqrt{[g'(g(x))h(x)]^2+h(g(x))^2} = \kappa/\alpha h(\alpha x) \qquad (8.3)$$

(A more careful derivation can be found in Shraiman, Wayne and Martin 1981 and Feigenbaum and Hasslacher 1982). You can think of noise as the uncertainty in determining the non-linearity parameter; it measures the distance from the fixed point, just like δ. That is why the eigenvalue equation looks very much like (5.7), and can be solved by the same

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16. Crutchfield, Nauenberg and Rudnick (1981)

methods. As $\kappa^n\sigma = 1$ is the last resolvable bifurcation, the new universal number $\kappa = 6.61903...$ can be interpreted as the factor by which the noise should be increased to wipe out one bifurcation. This means that demands on an experimentalist eager to observe still another bifurcation are even greater than intimated hitherto; not only does she have to have higher phase-space and parameter resolution, but also lower noise.

Experimentalists are usually not in a position to vary noise over many decades, so κ is not as interesting physically as α and δ. Still, it has been measured in a few experiments on non-linear electronic circuits (Testa, Pérez and Jeffries 1982[40], Yeh and Kao 1982a).

9. Experiments

Sequences of bifurcations, reverse bifurcations, return maps, periodic windows and universal scaling numbers have been measured in many experiments in a wide variety of physical systems: hydrodynamic (water, helium, liquid mercury), optical (lasers), acoustical, electronic, biological (heart muscles), chemical (Belousov-Zhabotinsky) and so on. Numerical experiments are even more numerous, and they are not necessarily trivial; we include some of them in the tabulation of the experimental results to give you a feeling for what kind of precision one can hope for in measurements of universal numbers.

Table 9.1 summarises the experimental situation; the experimental evidence that sequences of bifurcations are common and characterised by the universal numbers is firm. However, accurate measurement of universal numbers is difficult, if not impossible, because each successive bifurcation requires a fivefold increase in the precision of determination of the controllable external parameter, a threefold increase in the experimental resolution of the phase-space trajectory, and twice the time needed for the transients to damp out. Inevitably the noise inherent in the system, long time drifts, and other experimental problems interrupt the bifurcation sequence - about the best that one can hope for is 4 to 5 bifurcations. There are also theoretical problems; in most of these experiments we do not have sufficient understanding of the underlying dynamics to be able to estimate how well they are approximated by the one-dimensional theory. Because of all this it is hard to know what to make of the errors quoted by experimentalists; in some cases the universal numbers measured from the first three bifurcations are so uncannily close to the predicted asymptotic values that one wishes that publication of theoretical predictions were outlawed.

To summarise, the one-dimensional universality is in good shape. It is theoretically sound and it has been tested in many experiments. We lack (and will lack for a long time) a global theory which would tell us what physical systems, and which parameter ranges exhibit transitions to chaos via infinite sequences of bifurcations.

TABLE 9.1. Period doublings, experimental. A summary of the experimental observations of period doublings. The numbers in brackets are estimates of experimental errors; 4.3(8) means 4.3±.8. A few numerical "experiments" are included to indicate the precision of measurement of universal numbers attainable in numerical simulations.

experiment	no. period doublings	δ	α	μ	σ	κ
hydrodynamic:						
water[1]	2					
water[2]	4	4.3(8)		4(1)		
helium[3]	4	3.5(1.5)		4(?)		
mercury[4]	4	4.4(1)		5(1)		
Electronic:						
diode[5]	4	4.5(6)		6(?)		
diode[6]	5	4.3(1)	2.4(1)	O.K.		6.3(3)
transistor[7]	4	4.7(3)		O.K.		
Josephson simul.[8]	3	4.5(3)	2.7(2)		1.5(1)	5(2)
Laser:						
laser feedback[9]	3	4.3(3)	O.K.			
laser[10]	2					
laser[11]	3					
Acoustic:						
helium[12]	3					
helium[13]	3	4.8(6)		6(?)		
Chemical:						
B-Zh reaction[14]	3					
Computer:						
N-S truncation[15]	5	4.6(2)	2.5(1)			
Brusselator[16]	7	4.6(2)		4.77(3)	1.5(?)	
Theory:	∞	4.669..	2.503..	4.58..	1.52..	6.55..
equation no.		(5.1)	(4.2)	(6.1)	(7.5)	(8.3)

1. Gollub and Benson (1980).
2. Giglio, Musazzi and Perini (1981).
3. Libchaber and Maurer (1981).
4. Libchaber, Laroche and Fauve (1982).
5. Linsay (1982).
6. Testa, Pérez and Jefferies (1982).
7. Arecchi and Lisi (1982).
8. Yeh and Kao (1982).
9. Hopf, Kaplan, Gibbs and Shoemaker (1981).
10. Arecchi, Meucci, Puccioni and Tredicce (1982).
11. Weiss, Godone and Olafsson (1983).
12. Lauterborn and Cramer (1981).
13. Smith, Tejwani and Farris (1982).
14. Simoyi, Wolf and Swinney (1982).
15. Franceschini and Tebaldi (1979).
16. Kai (1981).

10. Intermittency

Period-doublings are rather common, but they are by no means the only way in which a deterministic system can reach chaos. Intermittency is another type of chaotic behaviour commonly observed in deterministic systems. It is characterised by long periods of regular motion interrupted by short chaotic bursts. For dissipative systems it too can be modelled by one-dimensional iterations (3.1). Consider the neighborhood of a fixed point of f(f(f(x))) as the non-linearity parameter sweeps through the critical value for the creation of the 3-cycle (fig.7.1):

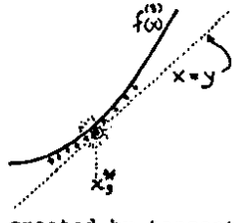


Fig.10.1

The new fixed point is created by tangent bifurcation. If you iterate f(f(f(x))) for a value of the non-linearity parameter just below the birth of the 3-cycle, you will note that the iterates accumulate in the neighbourhood of the 3-cycle points-to-be:

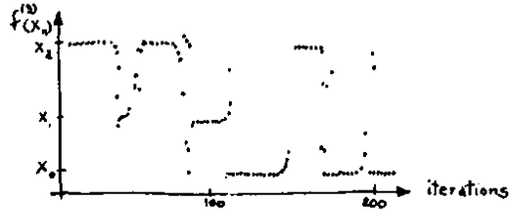


Fig.10.2

For many iterations the system is being fooled into believing that it is converging toward a fixed point, only to discover that the fixed point is not there after all; it then wanders away again, in the hope of finding a true fixed point:

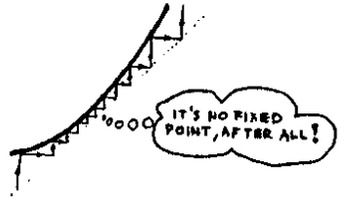


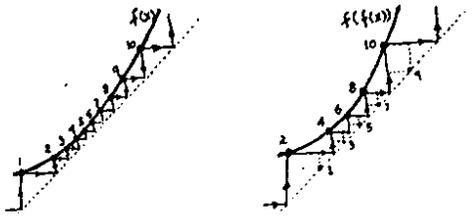
Fig.10.3

This is typical intermittent behaviour. Arbitrarily small differences in initial conditions will result in totally different sequences of iterates, so the intermittent motion is chaotic. Can we make any quantitative predictions about it?

Pomeau and Manneville (1980) have discovered a scaling law which relates the average duration of regular motion to the parameter deviation from its tangent bifurcation value. Somewhat suprisingly, not only can this scaling can be described by universal equations of the same form as for the period-doubling sequences, but in this case the universal equations

have simple analytic solutions (Cosnard 1981, Hirsch, Nauenberg and Scalapino 1982).

The self-similarity in this case is almost trivial; close to the tangent fixed point, f(x) and f(f(x)) look very much the same; the only difference is that for f(f(x)) the steps in the iteration staircase are twice as long



Pig.10.4

so one writes the self-similarity equation (4.2)

$$g(x) = \alpha g(g(x/\alpha)) \tag{10.1}$$

imposes tangency conditions

$$g(0) = 0 \tag{10.2}$$

$$g'(0) = 1 \tag{10.3}$$

and guesses $\alpha = 2$. As you can easily check, $g(x) = x/(1-ax)$, a arbitrary, solves the above equation. The equation is universal for the same reason as in the period-doubling case; it describes the neighbourhood of the tangent fixed point enlarged infinitely many times with the scaling factor $\alpha = 2$, so almost all memory of the original mapping is lost. The unstable manifold can be studied the same way as in the period-doubling case, by linearising as in (5.7); this time one obtains $\delta = 4$.

It is cute to see that the universal equations can be solved explicitly. More importantly, the above universal numbers α and δ have direct physical significance. The length of the periodic motion depends on how close f(x) is to the (incipient) tangent fixed point. The universal equation (10.1) says that if we move $1/\delta = 1/4$ times closer to the tangent fixed point, the iteration hangs $\alpha = 2$ times as long in the neighbourhood of the fixed point. Hence L, the average length of periodic motion, and R_c-R, the amount by which the parameter differs from the tangency value, are related by the Manneville and Pomeau (1979) scaling law

$$L = \text{const.}(R_c-R)^{-1/2} \tag{10.4}$$

Beyond this, one can play with everything we tried for period-doublings; add noise (Hirsch, Nauenberg and Scalapino 1982, Hirsch, Huberman and Scalapino 1982), change the power of the mapping (Hu and Rudnick 1982), and so on. Intermittency has been observed in chemical (Pomeau, Roux, Rossi, Bachelart and Vidal 1981) and electronic (Jeffries and Pérez 1982, Yeh and Kao 1982a) experiments.

11. Universality in higher dimensions

The theory of the period-doubling route to chaos, as discussed above, is essentialy complete. A pretty mathematical theory is lacking, but we understand when and why period-doublings can occur, and we have solid experimental evidence that they are common in many physical systems. Where do we go from here?

Period-doubling is a one-dimensional theory; it describes dynamical systems for which the dissipation has effectively damped out all but one of the dynamical degrees of freedom. Sooner or later we have to face the real world: chaos in many-dimensional systems. The simplest way to postpone this unpleasant moment is to turn to the study of iterations of two-dimensional maps. Three types of two-dimensional mappings have been extensively studied: contractive, complex and conservative.

For some regions of parameter values the contractive mappings reduce to the one-dimensional theory, as argued at the end of section 2, and shown by numerical calculations by Derrida, Gervois and Pomeau (1979), Franceschini (1979), Franceschini and Tebaldi (1979) and many others. Beyond this there is much more (such as Hénon's 1976 strange attractor) that is chaotic, but so far not covered by the universality theory, and therefore beyond the scope of these lectures.

Iterations of functions of one complex variable can be viewed as mappings in two real dimensions. They have been extensively studied (Julia 1918, Fatou 1919, 1920, Myrberg 1958-1962, Brolin 1965, Mandelbrot 1980, 1982, Douady and Hubbard 1982, Douady 1982, Sullivan 1982, Manton and Nauenberg 1983). The universal function (4.2) has rich structure in the complex plane (Epstein and Lascoux 1981). In the complex plane the theory of period-doublings has been generalised to the theory of period n-tuplings (Golberg, Sinai and Khanin 1983, Cvitanović and Myrheim 1983). The universality theory for complex mappings is very beautiful, but we do not know whether it can be used to model any physical systems.

Conservative (Hamiltonian) mappings appear in many physical problems. In solid state physics conservative mappings arise in the study of commensurate-incommensurate transitions, this time not as Poincaré maps, but as iterations on physical lattices (Bak 1981). An effective theory of conservative chaos would have many important practical applications in a variety of problems, such as the design of plasma fusion devices and intersecting storage rings (Helleman 1981).

In a dissipative system transients die out and the trajectory settles into some low-dimensional attractor. In a conservative system transients never die out, and the trajectory (for example, an ion in a tokomak) keeps spiralling on forever. We have seen that an ordinary fish population curve leads to an amazingly rich structure. The phase-space structure of conservative systems is truly bewildering, and has fascinated physicists and mathematicians for many generations (see for example Berry 1978 and Helleman 1980a).

The discovery of the universality for one-dimensional iterations has prompted a search for period-doublings in two-dimensional conservative mappings. They have been discovered (Benettin, Cercignani, Galgani and Giorgilli 1980, Bountis 1981, Bak and Høgh Jensen 1982) and look something like this:

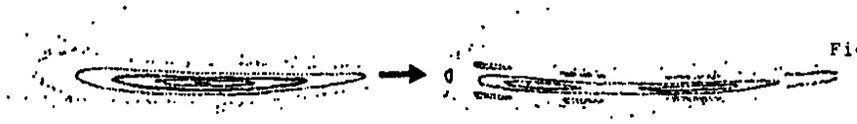


Fig.11.1

An elliptic fixed point turns hyperbolic and gives birth to a pair of new elliptic fixed points. Variation of the external parameter yields an infinite sequence of such period-doublings, ending in chaos. The universal scaling and convergence numbers can be computed as before; they are different for conservative and dissipative systems. Their size has been predicted by simple renormalisation group arguments (Derrida and Pomeau 1980), and they can be related smoothly to the one-dimensional universal numbers by varying the amount of dissipation in two-dimensional mappings (Helleman 1980a, 1980b, Hu 1981, Zisook 1981). The universal equations for two-dimensional conservative period-doublings have been formulated and investigated by Greene, MacKay, Vivaldi and Feigenbaum (1981), Collet, Eckmann and Koch (1981), Eckmann, Koch and Wittwer (1982), MacKay (1983) and for two-dimensional intermittency by Zisook (1982) and Zisook and Shenker (1982). As far as we know, there have been no experiments which probe this type of universality.

Another problem in which the universality ideas have had some success is the transitions to chaos for diffeomorphisms on the circle. Maps of this type arise in a variety of physical problems.

Hamiltonian mappings are one class of such problems. According to the KAM theorem (see Arnold 1978, Moser 1968), the phase-space trajectories in two dimensions are confined within KAM tori, and large-scale chaos sets in with the dissolution of these tori. Greene (1979) has shown that the winding number of the last surviving KAM is the golden mean. Shenker and Kadanoff (1982) have investigated the precise manner in which the last KAM dissolves and discovered that it does so by turning into a fractal. A way to study this transition is to neglect the radial variation of the torus, and model the angular variable by a map on the circle;

$$x_{n+1} = x_n + \Omega + \frac{k}{2\pi} \sin(2\pi x_n) \qquad (\mathrm{mod}\ 1) \qquad (11.1)$$

The circle mappings also arise in the study of dissipative systems, such as the cylindrical Couette flow. After two Hopf bifurcations (a fixed point with inward spiralling stability has become unstable and outward spirals to a limit cycle) a system lives on a two-torus, executing quasi-periodic motion. The Poincaré map of this system can again be modelled by a mapping of type (11.1). The question is what happens next. In an influential paper Ruelle and Takens (1971) have proposed that chaos

is reached via a three-torus (see Eckmann 1981 for a discussion). However, chaos can arise already on the two-torus, again just as the map is losing its invertibility.

Yet another situation in which the circle maps arise naturally is for periodically driven non-linear oscillators, such as the Duffing oscillator and models of the Josephson junction (Kautz 1981, Crutchfield and Huberman 1980, Levinsen 1982, D'Humiéres et al. 1982, Tomita 1982). Periodicity is imposed by the driving frequency, and the phase-lockings between the driving frequency and the intrinsic frequency of the oscillator can again be modelled by the circle map (11.1).

The transition to chaos occurs as the mapping (11.1) starts to lose invertibility, $k = 1$. The iterates for Ω corresponding to the golden mean winding number show self-similarity (Shenker 1982). It is not hard to see what this self-similarity is and to write down the corresponding universality equations (Rand et al. 1982, Ostlund et al. 1983, Feigenbaum, Kadanoff and Shenker 1982) - the method is the same as in the examples discussed above, but getting into this here would take too much space.

It is rather unlikely that the predictions of the universality theory for the golden mean winding number can be tested experimentally; the scaling numbers for the critical case differ by only few per cent from the scaling numbers for the trivial (invertible mapping) case.

Beyond circle maps and two dimensions, there is no end to the problems waiting for us. We have to understand how the attractors grow as more and more degrees of freedom go chaotic on us (Farmer 1982), how to do the quantum mechanics of classically chaotic systems (Berry 1978, Zaslavsky 1981, Pullen and Edmonds 1981), prove quark confinement as an effect of Yang-Mills turbulence (Matinyan, Savvidy and Ter-Arutyunyan-Savvidy 1981), write the universal equation for the brain (Feigenbaum, unpublished), and finally, understand why the clouds are the way they are.

ACKNOWLEDGEMENTS

These lectures have their origin in a Nordita lecture prepared together with Mogens Høgh Jensen (Cvitanović and Høgh Jensen 1982), to whom I am very much indebted. I would also like to thank Ulla Selmer, Jan Myrheim, Peter Scharbach, Jim Revill, Nils Robert Nilsson, Olivia Kaypro, and Mitchell Feigenbaum for their help and encouragement, and David Pritchard for (unknowingly) naming these lectures.

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