Spectra of stochastic evolution operators: Beyond all orders

Predrag Cvitanović†, C P Dettmann‡, Gergely Palla§, Niels Søndergaard†and Gábor Vattay§

† Northwestern University, Department of Physics & Astronomy
2145 Sheridan Road, Evanston, Illinois 60208 USA
‡ Center for Chaos and Turbulence Studies, Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø
§ Dept. Solid State Physics, Eötvös University
Muzeum krt. 6-8, H-1088 Budapest

Abstract. The asymptotics of weak noise corrections to the spectrum of the evolution operator associated with a nonlinear stochastic flow with additive noise is evaluated. The method works for arbitrary noise strength. The method also yields estimates for the late terms in the asymptotic saddlepoint expansions.
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1. Outline

(i) $\det(1 - z\mathcal{L})$ spectrum by diagonalization in polynomial basis around $x = 0.5$ or similar, compared to PD lattice calculation, or the small σ perturbative results.
  (a) for $0 < \sigma < 1$ Fokker-Planck kernel integrated (GP)

(ii) late terms in the asymptotic saddle expansion
  (a) for parabola fixed point (repeats dominated by $(f'')^r$ term?)
  (b) for the test Cantor set repeller?
  (c) understand the significance of the singulant [1]

(iii) find crossover in the local spectrum from the detrministic to the pure Fokker-Planck noise eigenvalues?

(iv) find a finite-dimensional representation of the noisy evolution kernel

(v) have perhaps the best beer on Østerbro

‡ **file** alf.nbi.dk:predrag/articles/asym/asym.tex
version of 4 April 99 (first draft 4 April 99) - printed 5 April 1999

2. Matrix representation of Perron-Frobenius operator

Here we take a different approach and calculate the contribution of a single periodic orbits to the integrals by constructing a matrix representation for the operator valid globally.

As in physical applications one studies smooth dynamical observables, we restrict the space that $\mathcal{L}$ acts on to smooth functions. In practice "real analytic" means that all expansions are polynomial expansions.

The main point is that the spectral determinants are entire functions in any dimension, provided that

1. the evolution operator is *multiplicative* along the flow,
2. the symbolic dynamics is a *finite subshift*,
3. all cycle eigenvalues are *hyperbolic* (sufficiently bounded away from 1),
4. the map (or the flow) is *real analytic*, *ie.* it has a piecewise analytic continuation to a complex extension of the phase space.

Finally we note that for 1-d repellers a diagonalization of an explicit truncated $\mathbf{L}_{mn}$ matrix yields many more eigenvalues than the cycle expansions [17, 13]. The reasons why one persists anyway in using the periodic orbit theory are partially aesthetic, and partially pragmatic. Explicit $\mathbf{L}_{mn}$ demands explicit choice of a basis and is thus non-invariant, in contrast to cycle expansions which utilize only the invariant information about the flow. In addition, we do not know how to construct $\mathbf{L}_{mn}$ for a realistic flow, such as the 3-disk problem, while the periodic orbit formulas are general and straightforward to apply. The difficulty lies in the existence of a stable manifold, in which the invariant measure is smooth only below the level of the noise. It remains to be seen whether a basis that assumes smoothness only in the unstable direction, and uses the periodic orbit covering for the stable direction can work.

Why noise actually is supposed to be good or the real motivation:

In semi-classical theories one has to include several periodic orbits in the calcualations. At first it seems that one has to use all periodic orbits, however, a cut-off is introduced in ref. [25] corresponding to the Heisenberg time. A similar feature should be expected in systems with noise for the following physical reasons: Noise itself should wash out the finer details of phase space making longer orbits irrelevant for the description. Also noise will smear out a singular wave function representing say a particle and thereby regularize the theory [8].

3. Asymptotia

Another way of obtaining these results is to create a global matrix represention of the evolution operator using the ideas of ref. [17, 13]. Here one expands the operator in a polynomial basis around a suitable point. Because of poor convergence one has to use a large basis - much larger than a typical local basis for a given orbit. The virtue of this latter approach is that it gives the non-perturbative result with which we can compare. The drawback of the method presented here is that we do not yet know how to optimize the truncation of the polynomials used. In practice, however, the method converges.

3.1. *Dettmannism I (Jan 26, '99)*

With Gergely's more precise ν_{12}, I return to my original formula,

$$\nu_{2n} = 0.0527 \frac{n! 32^n}{2n-1}$$

which gives $\nu_{12} = 3.70382 \times 10^9$, ie about 30 parts per million over Gergely's result. Prediction:

$$\nu_{14} = 7.02017 \times 10^{11}.$$

3.2. *Vattayismo I (Jan 29, '99)*

I discussed with Andre some of the basics: (we skipped Berry-Howls) etc.. The recommended reading for tonight are refs. [2, 3, 4, 5]. There is a common feature in all this vast literature: in saddle point integrals of our kind the tail behaviour is

$$n! a^n, a_n + 1/a_n$$

should form a line for large n.

Andre explained how to calculate the late terms in an expansion. He gave quantum mechanical examples to illustrate it. He even made a fit to our numbers based on the assumption that our expansion is not much different from a quantum one. The result is

$$\nu_2 n = (n-1)! 32^n (0.0269 + 0.00956/n + ...) \tag{1}$$

which fitts well except to order 12 ($n = 6$). So, I assume that order 12 is still wrong, at least Gergely's order 12. I have found that order 12 is negative while the rest is positive. This is wrong as you will see later. Andre explained me that assymptotic expansions cannot behave differently than

$$C\Gamma(n+\mu)a^n.$$

For example $\Gamma(2n+\mu)$ cannot happen! So, the right way to present result is to plot the ratio of consequtive terms and see if it is linear and fit $(n+\mu)a$.

I have a contour integral representation of $\mathrm{tr}\,\mathcal{L}$ for the corrections. The trick is that instead of a contour integral you compute this integral via saddle point method such that your large parameter is now the order of the expansion! OK. Then you find that there is a single saddle in this case. Interestingly, in our problems complex or real periodic orbits do not give saddle point contributions to this (like it happens typically in QM). The saddle of interest in our case is the point where $f'(x) = 1$! We neglected this saddle so far. It is a solution of the saddle point equation

$$\frac{d}{dx}(x - f(x))^2 = 2(x - f(x))(1 - f'(x)) = 0. \tag{2}$$

Another thing is that the factorial-like term in our expansion is due to the factorials in the moments of the Gaussian. Combining these things you end up with the formula

$$C(n - 1/2)! \frac{2^n}{(x - f(x))^{2n+1}} \tag{3}$$

where x is the point $f'(x) = 1$. It is very likely that this formula, calculated for $\mathrm{tr}\,\mathcal{L}$ is basically valid for the leading eigenvalue. I still have to check this in detail.

One thing is now clear: It is reasonable to expect that within a few months we will be able to control the late terms. The strategy now is:

n	$\mathrm{tr}\,\mathcal{L}_0$
0	0.11111111
1	0.07315957
2	0.20196660
3	1.00936323
4	7.34673301
5	70.3826169
6	837.075709

Table 1. $\sigma^2, \cdots, \sigma^6$ perturbative coefficients for the fixed point 0.

(i) predict the behaviour of the late terms, technology is available

(ii) push the calculation of corrections until the terms become late ones

(iii) summ up the late terms and append (add) this as a correction after the polinomial formed by known terms.

We are probably lucky in this case, since the subleading stuff might be so much supressed that we can reconstruct the measured curve quite accurately. An indication of this is that the convergence of our series is very rapid to the asymptotic form. Andre sees such a fast convergence the first time. Usually tens or more terms are needed.

I learned a lot from Andre and he seems to be interested, although he finds this model ad-hoc. Also, I had enormous amount of discussions with him on corrections and polynomial basis and local calculations since 1996. I would put him on the list of authors at an appropriate time. He would put a quality stamp on the next paper.

3.3. Vattayismo II (Jan 31, '99)

Following the explanations of Andre Voros I managed to derive a formula for the asymptotics of the trace $\mathrm{tr}\,\mathcal{L}$ of the fixed points. Gergely produced for me the data for this quantity.

The numbers are given in table 1. The formula I derived is

$$C\frac{2^n}{(x-f(x))^{2n}}\frac{\Gamma(n+1/2)}{\sqrt{n+1/2}} \tag{4}$$

where x is the point where $f'(x)=1$. C also can be determined, I just had no time yet. When you evaluate this numerically you get for the exponential

$$2/(x-f(x))^2 = 2.35... \tag{5}$$

When I numerically fit the series of table 1 with the formula

$$cb^n\Gamma(n+a)/\sqrt{n+a} \tag{6}$$

I get

$$\begin{aligned} a &= 0.497675 \pm 0.00320107(0.643203\%) \\ b &= 2.35514 \pm 0.00191123(0.0811514\%) \\ c &= 0.0436761 \pm 5.14877e-05(0.117885\%) \end{aligned} \tag{7}$$

So, the theory works perfectly. A similar result should come out for the other fixpoint of the map. It seems that Gergelys numbers for this other fixpoint are not OK. For example, he got a negative number for $n = 6$, which can be excluded. Also the numbers after $n = 4$ are not convincing, since the ratio of consequtive terms does not converge as it should. So, I suggest not to publish ν_{10} and ν_{12} yet. I hope to get a formula for the eigenvalues very soon and write a note.

3.4. Vattayismo III (Feb 2, '99)

It seems that I get a "hyper trace formula" for the asymptotics.

The k-th σ^2 correction to $\mathrm{tr}\,\mathcal{L}^n$ is a sum over noisy periodic orbits of length n and has the form (only the k dependence is shown)

$$\sum_p C_p \frac{\Gamma(k+1/2)(2/S_p)^k}{\sqrt{k+1/2}} \tag{8}$$

where S_p are the noise actions of the orbits. Noisy periodic orbits are the periodic orbits of the 2D map

$$\begin{aligned} x' &= f(x) + p \\ p' &= p/f'(x) \end{aligned} \tag{9}$$

such that there is at least one nonzero p_n of the periodic orbit (x_n, p_n). The noise action is $S_p = \sum_n p_n^2 = \sum_n (x_{n+1} - f(x_n))^2$. (For $n = 1$ we just get back the result I posted earlier.) For ordinary periodic orbits $p_n = 0$. These do not qualify.

For the global eigenvalues 32^k means that we will find an orbit with action $S_p = 1/16$.

We are really close to something great! For the first few k we can compute the corrections based on normal periodic orbits and then terminate the sequance with this new trace formula, the larger k is the formula gets more accurate.

4. Pallatables: matrix rep for arbitrary σ, Nov 19, 1998

Let's investigate the concrete form of the following matrix:

$$L_{l,k} = \left\langle \frac{\partial^l}{\partial y'^l} \middle| e^{-\frac{y'-f(y)}{2\sigma^2}} \middle| \frac{y^k}{k!} \right\rangle \tag{10}$$

First, let's the act with the operator on $\frac{y^k}{k!}$:

$$\begin{aligned} &\int dy e^{-\frac{y'-f(y)}{2\sigma^2}} \frac{y^k}{k!} = \int dz \frac{e^{-\frac{z}{2\sigma^2}}}{k!} \frac{\left(f^{-1}(y'-z)\right)^k}{|\frac{dz}{dy}|} = * \\ &\left(y(z) = f^{-1}(y'-z) \qquad \frac{dz}{dy} = -f'(y) = -f'(f^{-1}(y'-z)) \right) \\ * &= \int dz \frac{e^{-\frac{z}{2\sigma^2}}}{k!} \frac{\left(f^{-1}(y'-z)\right)^k}{|f'(f^{-1}(y'-z))|} = \int dz \frac{e^{-\frac{z}{2\sigma^2}}}{k!} \left(f^{-1}(y'-z)\right)^k (f^{-1})'(y'-z) \\ &= \int dz \frac{e^{-\frac{z}{2\sigma^2}}}{(k+1)!} \frac{d}{dy'} \left[\left(f^{-1}(y'-z)\right)^{k+1} \right] \end{aligned} \tag{11}$$

So the matrix element can be written:

$$L_{l,k} = \frac{\partial^{l+1}}{\partial y'^{l+1}} \left[\int dz \frac{e^{-\frac{z}{2\sigma^2}}}{(k+1)!} \left(f^{-1}(y'-z)\right)^{k+1} \right] \Bigg|_{y'=0} \tag{12}$$

4.1. Pallatables: matrix rep for parabola fixed point, Nov 17, 1998

The trace of the Perron-Frobenius operator at fixed point:

$$\operatorname{tr}\mathcal{L} = \int dx\delta_\sigma(f(x)-x) = \int dx \sum_{m=0}^{\infty} a_m \sigma^m \delta^{(m)}(f(x)-x) \tag{13}$$

can be evaluated using

$$\int dx\delta^{(n)}(y) = \int dy \frac{1}{|y'(x)|}\delta^{(n)} = \sum_{x:y(x)=0} (-1)^n \frac{d^n}{dy^n}\frac{1}{|y'(x)|} \,. \tag{14}$$

In general [**?**]:

$$(-1)^n \frac{d^n}{dy^n}\frac{1}{|y'|} = \frac{1}{|y'|}\sum_{\{k_l\}} \frac{(2n-k_1)!}{(-y')^{2n-k_1}} \prod_{l\geq 2} \frac{{f^{(l)}}^{k_l}}{(l!)^{k_l} k_l!}, \tag{15}$$

where the sum runs over all sets $\{k_l\}$ satisfying $l \geq 1, k_l \geq 0, \sum k_l = n$ and $\sum l k_l = 2n$. If f is a parabola, then f''' and higher derivatives are zero, so only thoose terms count in the expression above, where $k_2 \neq 0, k_i = 0, i > 2$. This means, that we have to look for the set satisfying $k_1 + k_2 = n$ and $k_1 + 2k_2 = 2n$, that is $k_2 = n, k_1 = 0$. So in case of parabola:

$$(-1)^n \frac{d^n}{dy^n}\frac{1}{|y'|} = \frac{1}{|y'|}\frac{(2n)!}{(y')^{2n}}\frac{(f'')^n}{2^n n!} = \frac{1}{|y'|}\frac{(2n)!}{n!}\left[\frac{f''}{2(y')^2}\right]^n \tag{16}$$

Let us suppose that the noise is a Gaussian noise (**??**). The trace is then

$$\operatorname{tr}\mathcal{L} = \frac{1}{|y'|} + \frac{1}{|y'|}\sum_{m=1}^{\infty} \frac{(4m)!}{m!(2m)!}\left[\frac{\sigma f''}{2\sqrt{2}(y')^2}\right]^{2m} \tag{17}$$

For concreteness, consider a parabola with given parameter values:

$$f(x) = 6x(1-x) \tag{18}$$

The fixed points are $x_0 = 0$ and $x_1 = \frac{5}{6}$. At x_1:

$$y'(x_1) = -5\,, \qquad f''(x_1) = -12 \tag{19}$$

and the σ^{2m} coefficients in (17) are

$$A_m = \frac{1}{5}\frac{(4m)!}{m!(2m)!}\left[\frac{12}{\sqrt{2}\cdot 50}\right]^{2m} \tag{20}$$

The value of the first few A_m is:

$$\begin{array}{lll} A_1 = 0.06911999 & A_2 = 0.13934592 & A_3 = 0.52973745 \\ A_4 = 2.97500552 & A_5 = 22.1397531 & A_6 = 205.315214 \\ A_7 = 2280.75872 & A_8 = 29525.7901 & A_9 = 436509.280 \end{array} \tag{21}$$

At the other fixed point, we will get $f'(x_0) = 6 \rightarrow y'(x_0) = 5$, so everything is the same.

4.2. Keating speaketh

The idea you outline would appear to be the one employed by Bleistein and Handelsman [?] - except that rather than mapping onto a quadratic function, they map onto a linear one (in the case of an isolated saddle). The difference is, of course, trivial.

They do the isolated saddle case this way in section 7.2. The mapping is given by 7.2.2 and the resulting integral with a Jacobian and an exactly linear exponent is the result (7.2.3).

In the case of many saddles, the corresponding mapping is derived in section 9.2. The Chester, Friedman and Ursell [?] paper referenced at the end of that chapter, together with other papers by Ursell should be relevant.

As I said, this mapping idea leads to what is usually called an integral in the Borel plane [?]. There have been a lot of papers by C.J. Howls on this - especially the one [?] on multiple integrals might be useful to you.

Abstract of ref. [?]: The method of steepest descents for single dimensional Laplace-type integrals involving an asymptotic parameter k was extended by Berry and Howls [3] to provide exact remainder terms for truncated asymptotic expansions in terms of contributions from certain non-local saddlepoints. This led to an improved asymptotic expansion (hyperasymptotics) which gave exponentially accurate numerical and analytic results, based on the topography of the saddle distribution in the single complex plane of the integrand. In this paper we generalize these results to similar well-behaved multidimensional integrands with quadratic critical points, integrated over infinite complex domains. As previously pointed out the extra complex dimensions give rise to interesting problems and phenomena. First, the conventionally defined surfaces of steepest descent are no longer unique. Second, the Stokes's phenomenon (whereby contributions from subdominant saddles enter the asymptotic representation) is of codimension one. Third, we can collapse the representation of the integral onto a single complex plane with branch cuts at the images of critical points. The new results here demonstrate that dimensionality only trivially affects the form of the exact multidimensional remainder. Thus the growth of the late terms in the expansion can be identified, and a hyperasymptotic scheme implemented. We show by a purely algebraic method how to determine which critical points contribute to the remainder and hence resolve the global connection problem, Riemann sheet structure and homology associated with the multidimensional topography of the integrand.

4.3. Words of the wise

Leo Kadanoff suggests that we should check *nonlinear scaling fields*, perhaps in Wegner, *Encyclopedia of Phase Transitions*. D. Nelson [?] did this for his PhD thesis, and was dsappointed that the method never caught on.

Viviane Baladi suggests checking Milnor's Stony Brook lecture notes
rational maps → Köning's linearization
results on domain of convergence $h^{-1}(\Lambda h(x))$. There might be only 2 global branches in the test model we consider.

To Ezra Getzler http://athos.math.nwu.edu this is reminiscent of the flatening of genus 0 manifolds in Dubrovin's work on geometry of 2d topological field theories [?]. Predrag has been unable to read that opus magnum.

5. Summary

Intuitively, the noise inherent in any realistic system washes out fine details and makes chaotic averages more robust. Quantum mechanical h-bar resolution of phase space implies that in semi-classical approaches no orbits longer than the Heisenberg time need be taken into account. We explore these ideas in some detail by casting stochastic dynamics into path integral form and developing perturbative and nonperturbative methods for evaluating such such integrals. In the weak noise case the standard perturbation theory is expansion in terms of Feynman diagrams. Now the surprise; we can compute the same corrections faster and to a higher order in perturbation theory by integrating over the neighborhood of a given saddlepoint exactly by means of a nonlinear change of field variables. The new perturbative expansion appears more compact than the standard Feynman diagram perturbation theory; whether it is better than traditional loop expansions for computing field-theoretic saddlepoint expansions remains to be seen, but for a simple system we study the result is a stochastic analog of the Gutzwiller trace formula with the $\hbar$ corrections so far computed to five orders higher than what has been attainable in the quantum-mechanical applications.

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