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Exploring Chaotic Motion Through Periodic Orbits

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The fractal invariant measure of chaotic strange attractors can be approximated systematically by the set of unstable n-periodic orbits of increasing n. Algorithms for extracting the periodic orbits from a chaotic time series and for calculating their stabilities are presented. With this information alone, important properties like the topological entropy and the Hausdorff dimension can be calculated.

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It can be said with some confidence that many attractors that belong to the borderline of chaos are presently well understood. Now-classical examples are infinitely period-doubled attractors,[1–3] critical orbits on the two-torus with quadratic irrational winding numbers, etc.[2–5] We know how to construct scaling functions for these sets and how to calculate their[2] $f(\alpha)$ or generalized dimensions[6] D_q from first principles, and the results agree extremely well with experimental realizations.[5,7] Quite in contrast, once we enter the chaotic regime it appears that we understand very little. We have no quantitative description of the apparent self-similarity of chaotic, strange attractors.

The reason for the success with borderline fractal sets, and for the failure with chaotic fractal sets, is that we know how to approach the former hierarchically via a series of nonfractal sets of increasing complexity. We approach an infinitely doubled orbit with 2^n-periodic orbits.[1] We approach a golden-mean orbit on the two-torus with orbits that have rational winding numbers[4] (i.e., ratios of Fibonacci numbers). This allows for the renormalization and understanding of these sets. In contradistinction, it appears that we do not know how hierarchically to approach chaotic sets with nonfractal sets.

The aim of this Letter is to point towards a possible remedy of this situation. The basic idea is not new; it is well known that chaotic orbits are closures of the set of unstable periodic orbits.[8] The two new statements of this Letter are the following: (a) We can extract *all* the periodic orbits of order n, for n not too large, straight from the chaotic orbit, and calculate their stabilities (Lyapunov exponents). (b) This information can be used to describe important properties of general chaotic sets.

In particular we shall use just the periodic orbits to calculate the fractal dimension and the topological entropy of the strange attractor. We shall argue that the number of the periodic orbits, their distribution, and their properties unfold the structure of chaotic orbits. If we want to know whether two experimental attractors are the same (up to possible smooth change of coordinates), or if we want to assert that a theoretical model faithfully reproduces an experimental attractor, this information is crucial for a meaningful comparison.

We first explain how we extract the periodic orbits from an experimental chaotic time series, and how to calculate their eigenvalues. Assume knowledge of a time series $\{\mathbf{X}_i\}_{i=1}^{N}$, with $\mathbf{X}_i$ being points in R^2. Although all the methods generalize immediately to R^n, the examples that we studied so far are in R^2, and so we focus on those. If N is sufficiently large, the time series will visit the neighborhood of an arbitrary period-n cycle point at some time i. At time $i+1$ the time series will be in the vicinity of another cycle-n point, and so on. After n iterations the time series will again visit near the initial cycle-n point, under the assumption that n time steps previously the sequence was sufficiently close to it. This idea is used to locate the periodic orbits by scanning of the time series for pairs of points separated by n time steps that are within a small preassigned spatial distance (r_1) of one another.

At this stage all points in the time series which return after n steps are located and must now be grouped into periodic cycles. Throughout the length of the run, the vicinity of a particular periodic point may have been visited many times. In order to decide whether two nearly periodic orbits in the time series correspond to distinct periodic orbits, their positions relative to each other are checked. If all the corresponding pairs of points of the two orbits are less than a preassigned distance apart (r_2), then they are grouped into the same unstable periodic cycle. Otherwise, they represent distinct periodic cycles. The position of a point belonging to a true unstable n-cycle is estimated by our finding the center of mass of all the points in the time series which were found to correspond to it.

The two externally assigned parameters r_1 and r_2, used in the process of determining the almost periodic sequences and grouping them into cycles, are chosen by the following criteria: r_1 is chosen large enough to include several sequences corresponding to a particular periodic orbit. The distance between cycles, r_2, is set small enough so as to distinguish between distinct periodic orbits under the condition that $r_2 > r_1$. The correct grouping should not change with an increase of the length of the time series.

In order to calculate the eigenvalues and eigenvectors of the periodic cycles, a mean-squares procedure[9,10] is used to fit a 2×2 Jacobian matrix to each point of a periodic cycle. In order to calculate this matrix for a particular cycle point, all M points in the time series which were found to correspond to it are used. Their deviations $\{\mathbf{u}_i\}$ from the cycle point and the deviations $\{\mathbf{v}_i\}$ of their iterates from the consecutive cycle point are used in order to fit the matrix J which minimizes the norm $\|UJ-V\|$ where U and V are both $M\times 2$ matrices consisting of $\{\mathbf{u}_i\}$ and $\{\mathbf{v}_i\}$, respectively. In order to obtain the eigenvalues and eigenvectors for a cycle point, the Jacobians are multiplied around the cycle in reverse order and then diagonalized.

As an example of this procedure consider the paradigmatic Hénon map $(x,y)\rightarrow(1-ax^2+y,bx)$ with $a=1.4$, $b=0.3$. We find that in order to pinpoint all the cycles of order $n\leq 10$ we need a time series of the order of 10^5 points. r_1 and r_2 are typically of size 10^{-4}–10^{-5}. The numbers of cycle points of orders 1–10 are 1, 3, 1, 7, 1, 15, 29, 63, 55, 103, respectively. The Lyapunov numbers of all these cycles were calculated. To check the validity and accuracy of our algorithms we compared the results with "exact" calculations which use the explicit knowledge of the map. (In extracting the data from the chaotic signal as above no knowledge of the underlying map is assumed.) In this calculation one first lays a grid of points covering the attractor. Typically the number of grid points is at least 5 times larger than the number of periodic points that one expects to find. Then starting at each point, one solves for a periodic orbit using the map and a Newton-Raphson iteration scheme. The different periodic points are recorded, and then the number of grid points is increased by a factor of 2 or 3 to check that no new orbit appears. Since the map is known, the Lyapunov exponents can be calculated exactly.

It turns out that the algorithm described above works very well. With a data file of 2×10^5 points all the cycles of length ≤ 10 were captured, and the eigenvalues are close to the exact values. In fact, most of the eigenvalues differ from their exact counterparts by less than 1–2%. In the worse cases, which appear only when two cycles are very close together and hard to resolve, the errors in the eigenvalues may reach a factor of 2.

The knowledge of the number of periodic orbits of order n allows an estimate of the topological entropy K_0.[11] The topological entropy of a dynamical system can be defined as

$$K_0=\lim_{n\to\infty}\left[\frac{1}{n}\ln N_n\right] \tag{1}$$

where N_n is the number of cycle points of order n. Accordingly, we can define nth-order approximants

$$K_0^{(n)}=n^{-1}\ln N_n. \tag{2}$$

In the case considered above it is meaningful to use data for $n\geq 6$ only. The results for $K_0^{(n)}$ are as follows: $K_0^{(6)}=0.451$, $K_0^{(7)}=0.481$, $K_0^{(8)}=0.517$, $K_0^{(9)}=0.445$, and $K_0^{(10)}=0.463$. Evidently, the number of eight-cycles is anomalously large. Disregarding it we estimate $K_0=0.46\pm 0.02$, in excellent agreement with previous estimates.[11] The validity of this conclusion can be supported by the results of the "exact" calculations. For example, there we find 155 points of order 11 and 247 points of order 12, leading to $K_0^{(11)}=0.458$ and $K_0^{(12)}=0.459$.

The calculation of the Hausdorff dimension needs further discussion. The basic idea is that we want to consider the n-order cycle points with increasing n as a better and better approximation to the fractal measure that is obtained from a very long chaotic run. To calculate the Hausdorff dimension D_0 of this measure we have to know how to cover it with balls, and how many balls there are of radii l_i. Given this information we evaluate D_0 using the formula (see for example Ref. 6)

$$\sum_i l_i^{D_0}=1. \tag{3}$$

In other words, we have to know the local scales that characterize the set at the nth level of refinement. For the purpose of calculating D_0 it sufficient to focus on refinement in the contracting (stable) direction only. Each cycle point has a local stable direction with an eigenvalue λ_i^s that estimates the ith local scale. In the unstable direction we do not refine and we say that all the scales are of ~ 1. The picture is therefore that of a coverage with "slabs" of length 1 and of width λ_i^s. Each such slab can be covered with $1/\lambda_i^s$ balls of radius λ_i^s, and therefore we estimate the nth-order approximants $D_0^{(n)}$ from

$$\sum_{i=1}^{N_n} \frac{1}{\lambda_i^s}(\lambda_i^s)^{D_0^{(n)}} = \sum_{i=1}^{N_n} (\lambda_i^s)^{D_0^{(n)}-1} = 1. \quad (4)$$

The results of this calculation, with the eigenvalues extracted from the "experimental" chaotic run, for $n=6$–10, are $D_0^{(6)}=1.26$, $D_0^{(7)}=1.29$, $D_0^{(8)}=1.30$, $D_0^{(9)}=1.26$, and $D_0^{(10)}=1.27$. Remembering that the eight-cycles seem anomalous we conclude that $D_0=1.27\pm0.02$, again in excellent agreement with previous estimates.[11] This conclusion can again be supported by the "exact" calculations which show good convergence at higher order cycles (we have all the cycles up to $n=21$) to $D_0=1.274\pm0.001$.

Can we proceed to calculate[11,12] D_q or K_q for $q\neq0$? Not at this point. These quantities depend on the invariant measure. The invariant measure is expected to be nonuniform along the strips of the attractor, and to capture this nonuniformity one has to refine the description in the unstable direction in addition to the refinement done above in the stable direction. It appears that doing so calls for developing algorithms to equip the cycles found with symbolic dynamics. This is already beyond the scope of this Letter and will be discussed elsewhere.

In summary, we pointed out that the unstable periodic orbits can be extracted straight from the chaotic signal, for an order n that depends on the amount of data available. The theoretical message is that the unstable orbits reveal the skeleton of the chaotic set and appear to be crucial for the understanding of strange attractors. Further analysis of the scaling properties of strange attractors as revealed by the underlying periodic orbits will be published later.

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