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Invariant Measurement of Strange Sets in Terms of Cycles

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We argue that extraction of unstable cycles and their eigenvalues is not only experimentally feasible, but is also a theoretically optimal measurement of the invariant properties of a dynamical system.

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The same dynamical system can be presented to us in many guises: as a phase-space trajectory, as a time-delay plot, and as a Poincaré section.[1] With data presented in such form it is not always easy to determine whether the theory and the experiment (or different experiments) indeed refer to the same system. Clearly, the question of *invariant* characterization is of paramount importance for the theory of dynamical systems. Here we shall argue that for the deterministic dynamical systems of low intrinsic dimension, the cycles (periodic orbits) provide a detailed invariant characterization, whose virtues are the following: (1) Cycle symbol sequences are *topological* invariants—they give the spatial layout of a strange set. (2) Cycle eigenvalues are *metric* invariants—they give the scale of each piece of a strange set. (3) Cycles are ordered *hierarchically*—short cycles give good approximations to a strange set and the errors due to neglect of long cycles can be *bounded.* (4) Cycles are *robust*—eigenvalues of short cycles vary slowly with smooth parameter changes. (5) Short cycles can be accurately extracted from the *experimental data.*

The cycles were introduced into the theory of the dynamical systems by Poincaré[2] and have played a central role in the mathematical work on the subject ever since.[3,4] What is new here is the realization that with the recent advances in the techniques of experimental data analysis,[5,6] and a deeper theoretical understanding of dynamically generated strange sets,[7-9] the cycles are now not only experimentally accessible, but they indeed appear to be the optimal practical tool for the description of strange sets. The new experimental ingredient is point (5): Future investigations of experimental strange sets will use deterministic noise-smoothening techniques[6] —with those, the cycles are available at little extra effort. The new theoretical ingredient is point (3): We now know how to control the errors due to neglect of longer cycles.

An important virtue of cycles is their robustness [point (4)]: regardless whether a time series is a long transient, a long cycle, or truly ergodic, the eigenvalues of short cycles are stable and equally easy to extract. For example, under a change of the parameter a of the Hènon map[10] ($b=0.3$ fixed) from 1.4 to 1.399452219, the asymptotic attractor undergoes a dramatic change but the eigenvalues of short unstable cycles vary gently.

When and if the cycles suffice for the complete characterization (and reconstruction) of a dynamical system is not clear, but they do go further toward detailed low-dimensional modeling of transitions to turbulence than the dimensions, and we hope that in the future the data will be presented in terms of cycles rather than the "thermodynamic" averages.

That the cycle topology and eigenvalues are invariant properties of dynamical systems follows from elementary considerations. If the same dynamics is given by a map f in one set of coordinates, and a map g in the next, then f and g (or any other good representation) are related by a reparametrization and a coordinate transformation $f=h^{-1}\circ g\circ h$. As both f and g are arbitrary representations of the dynamical system, the explicit form of h is of no interest, only the properties invariant under any transformation h are of general importance. The most obvious invariant properties are topological; a fixed point must be a fixed point in any representation; a trajectory which exactly returns to the initial point (a cycle) must do so in any representation. Furthermore, a good representation should not mutilate the data; h must be a smooth transformation which maps nearby cycle points of f into nearby cycle points of g. This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metrically invariant: As is well known, the eigenvalues of the Jacobians $df^{(n)}(x_k)/dx$ of periodic orbits $f^{(n)}(x_k)=x_k$ are invariant. What is perhaps not widely appreciated is the fact that with the modern data acquisition techniques, the cycle eigenvalues are measurable with accuracy. Though these methods have by now been successfully applied to a variety of nontrivial higher-dimensional strange sets,[5,6,11] the simplest of strange attractors suffices to illustrate the essential ideas.

Consider a unimodal 1D map which sends the critical point x_c into the unstable fixed point x_0, Fig. 1(a). At the nth level of coarse graining, a "neighborhood" i consists of all points x which follow the itineary $i=\epsilon_1\epsilon_2\epsilon_3 \cdots \epsilon_n$, with $\epsilon_k=0$ if $f^{(k)}(x)<x_c$, and $\epsilon_k=1$ if $f^{(k)}(x)>x_c$. n iterations of this map thus resolves the strange attractor in 2^n distinct neighborhoods. Conversely, given a time series $x_0,x_1,x_2,\ldots,x_k=f^{(k)}(x_0)$, we aim at *reconstructing* the dynamical system by partitioning the data into such neighborhoods.

In the first step of such reconstruction, all returns after one iteration are used to locate the fixed points; fitting, *inter alia,* yields their eigenvalues. We include into the neighborhood of a fixed (or periodic) point all close-by points whose Jacobian is qualitatively similar (for example, flips or does not flip along the unstable eigendirection). The reconstruction now proceeds by partitioning the second iterates into those that remain close to the fixed points, and those that jump from the neighborhood of one periodic point to the neighborhood of another periodic point [intervals 01 and 10 in Fig. 1(a)]. The two-cycles and their eigenvalues are obtained by our fitting returns after two iterations, then the third iterates are partitioned, and so forth. The reconstruction stops when the density of points becomes insufficient to fit a neighborhood,[12] and we are left with a finite list of cycle itineraries i together with their eigenvalues Λ_i.

Here we are not concerned so much with *how* the cycles are to be extracted (the requisite numerical methods are available in the literature[5,6]), but *why* they should be extracted. The central problem here is our ability to control the errors arising from approximating the dynamics by a finite number of short cycles. As the error estimates depend on the particular quantity that is being computed, we shall illustrate the general method by a simple example, a computation of the escape rate from a one-dimensional repeller.

Consider a unimodal map like the one of Fig. 1(a), but with $f(x_c)>1$. In such a map an interval around x_c escapes in the first iteration, its two preimages escape in two iterations, and so on. Let l_i be the length of the interval of all points which follow the itineary $i=\epsilon_1\epsilon_2\epsilon_3 \cdots \epsilon_n$ without escaping in n steps. The measure

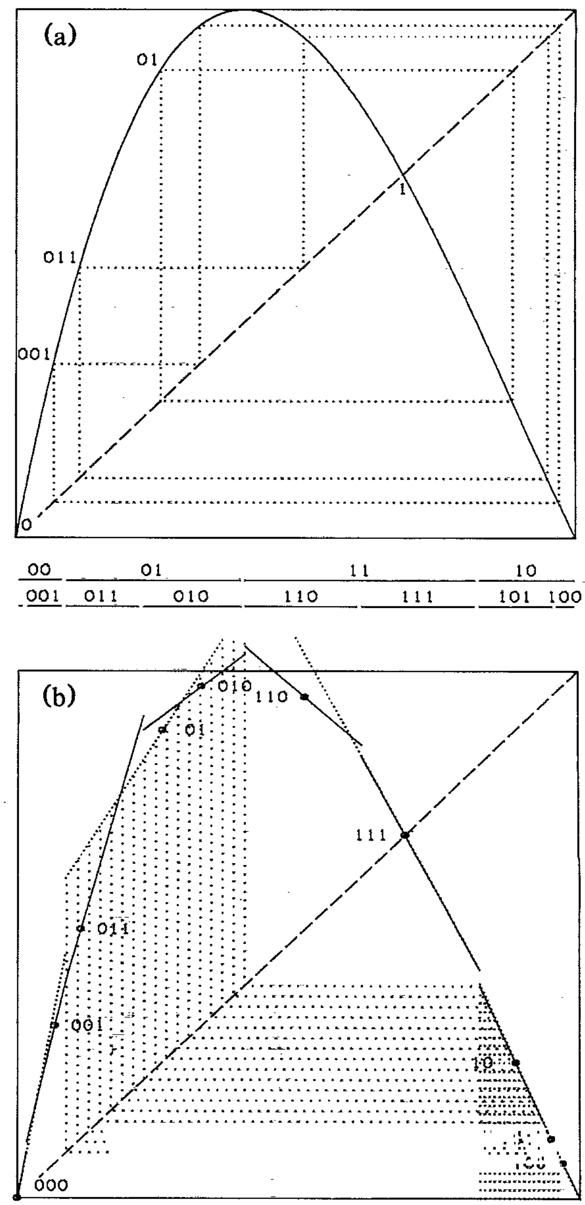

FIG. 1. A "skew Ulam" map $f(x)=cx(1-x)(1-bx)$, $1/c=x_c(1-x_c)(1-bx_c)$, is a simple example of a strange attractor. Here $b=0.6$. (a) Indicated are the partitionings of the attractor into four neighborhoods on the second level, the eight neighborhoods on the third level, the fixed points 0 and 1, the two-cycle 01, and the three-cycles 001 and 011. (b) The polygonal approximation to the attractor in terms of two-cycles (dotted segments) and three-cycles (full segments). A neighborhood (here the interval l_{101}) expands at a rate well approximated by the cycle eigenvalue [here $df^{(3)}(x_{101})/dx$].

of initial x which survive n iterations is given by

$$e^{-n/T_n} \equiv \sum_i^{(n)} l_i \,. \tag{1}$$

The mean lifetime of a random initial x is given by $T = \lim_{n\to\infty} T_n$. Each interval l_i contains a periodic point x_i, and the expansion of l_i onto the unit interval in n iterations is well approximated the stability of x_i: $l_i = a_i/|\Lambda_i|$. For large n the prefactors $a_i \approx O(1)$ are overwhelmed by the exponential growth of Λ_i and can be neglected. Intuitively, as $e^{-n/T_n} \to e^{-n/T}$, the formal sum over all orbits of all lengths

$$Z(z) = \sum_{n=1}^{\infty} z^n \sum_i^{(n)} |\Lambda_i|^{-1} = \sum_p n_p \sum_{r=1}^{\infty} (z^{n_p}/|\Lambda_i|)^r \tag{2}$$

should diverge for $z = e^{1/T}$. Furthermore, this sum, unlike (1), is both asymptotic (evaluated in the $n \to \infty$ limit) and $f \to h^{-1} \circ f \circ h$ invariant, as it depends only on the cycle eigenvalues. The second equality comes from the observation that if the trajectory retraces itself r times, its stability is Λ_p^r, where is p is the primitive cycle (single traversal of the orbit), and that each primitive cycle of length n_p contributes n_p terms to the above sum. The sum (2) is a logarithmic derivative $Z(z) = \zeta^{-1} z\, d\zeta/dz$ of the dynamical ζ function[13]

$$1/\zeta = \prod_p (1 - t_p) \,. \tag{3}$$

The product is over all primitive cycles p (nonrepeating symbol strings): for example, $p = \overline{011} = \overline{101} = \overline{110} = \ldots 011011\ldots$ is primitive, but $\overline{0101} = 010101\ldots = \overline{01}$ is not. In the present example $t_p = |\Lambda_p|^{-1} z^{n_p}$; however, the technique is by no means restricted to escape rates, and it is a simple exercise to repeat the above derivation for a class of "thermodynamical" averages used in the extraction of generalized dimensions,[4,14] with (1) replaced by

$$1 = \sum_i^{(n)} p_i^q / l_i^\tau \,. \tag{4}$$

In this case t_p, the weight associated with the cycle p in (3), is given by $t_p = e^{\mu_p \tau - \nu_p q}$, where $\mu_p = \ln|\Lambda_p|$ is the stability exponent, and ν_p is proportional to the cycle length n_p. For example, the cycles are counted by our setting $\tau = 0$ and $\nu_p q = n_p h$: h is the topological entropy. For a τ value such that $q(\tau) = 0$, (4) is the classical definition of the Hausdorff-Besicovitch dimension $D_H = -\tau$, and so on. The stability Λ_i need not refer to motion in the dynamical space; in more general settings it can be the renormalization scaling function[15] of trajectory splitting, or even a scaling function describing a strange set in the parameter space.[7,16] In the escape rate computations, the cycle "probability" p_i might depend on the particular cycle; or p_i might have altogether different interpretations.[14,17]

How are formulas such as (3) used? Once a set of the shortest cycles has been extracted, the $q = q(\tau)$ function can be evaluated by substituting the available cycle eigenvalues into (3) and determining the zeros of $1/\zeta$. For example, if the strange set is labeled by binary symbol sequences, as in Fig. 1, the cycle expansion is given by

$$1/\zeta(\tau,q) = 1 - t_0 - t_1 - (t_{10} - t_1 t_0) - (t_{100} - t_{10} t_0) - (t_{101} - t_{10} t_1) - (t_{1001} - t_1 t_{001} - t_{101} t_0 + t_{10} t_0 t_1) - \cdots \,. \tag{5}$$

A fit to a dynamical system $f(x)$ in terms of finite cycles and their eigenvalues is a polygonal fit [see Fig. 1(b)]. It provides estimates for eigenvalues of all longer cycles. For example, a four-segment fit in Fig. 1(b) expresses the eigenvalue of the $\overline{001}$ cycle, $\Lambda_{001} = f'(x_{001}) \times f'(x_{010}) f'(x_{100})$, in terms of shorter cycles: $\Lambda_{001} \approx \Lambda_0 \Lambda_{01}$. More generally, the quantity such as $t_{001} - t_0 t_{01}$ is a measure of the quality of the fit of a dynamical system by a finite polygonalization. For a finite polygonal approximation to the system, such as Fig. 1(b), the cycle expansion (5) is a finite polynomial; it generates the set in the same sense in which the original Cantor rule (remove $\frac{1}{3}$ of each interval) generates the Cantor set (for the simplest of fractals $1/\zeta = 1 - t_0 - t_1$, $t_0 = t_1 = 3^\tau 2^{-q}$). For smooth dynamics, such deviations of n-cycle eigenvalues from their estimates in terms of $(n-1)$-cycles are intuitively expected to be of order N^{-2}, where N is the number of polygon sections (distinct cycle points of length n). Thus the smoothness of a dynamical flow implies that the errors due to neglect of long cycles in cycle expansions such as (5) should fall off *exponentially fast* with the cutoff cycle length.

For the "skew Ulam" example of Fig. 1, this works very well; a few cycles suffice to estimate the Hausdorff dimension to many digits.[11] However, as this is an exceptionally well behaved model, we have tested the technique on a variety of generic strange sets which differ from it in two essential aspects: (1) the topology of such sets is highly irregular, as many symbol sequences are *pruned* (there are no physical cycles corresponding to such sequences); (2) the sets are *nonhyperbolic,* i.e., generically, one expects a mixture of sinks and repelling orbits with large variations in the expansion rates.

For the Hènon-type maps we solve the pruning problem[8] by approximating the physical set by a sequence of regular self-similar Cantor sets. For each such regular Cantor set, a *minimal set* of cycles suffices to account correctly for the topology of the set. Cycles longer than the minimal set contribute exponentially small refinements. Applied to the Hènon-type maps, the cycle expansions exhibit rather impressive improvement of convergence[11] relative to the standard methods such as those of Refs. 8 and 14. For example, for $a = 1.81258$, $b = 0.022864$ Hènon repeller introduced in Ref. 8, the eigenvalues of only fourteen unstable cycles of lengths up to seven yield the transverse partial dimension $D_s = 0.1205\ldots$ with 0.5% accuracy.

The *nonhyperbolicity* does not, *per se,* prevent the averaging, but the intuitive expectation that the smoothness of the dynamical system implies exponential falloff of the long cycle contributions has to be checked case by case; in certain cases, such as at the phase transitions[17] (eigenvalue crossovers, $\Lambda_i \rightarrow 1$ marginal scaling situations), the above arguments about exponential convergence fail, and the contributions of long cycles must be carefully controlled.

We have used the thermodynamic averages here only as an illustration of the quality of the finite cycle approximations to dynamical systems. Our main point is that the theory and experiment now can and should be compared cycle by cycle. The cycle eigenvalues are not only available from the experimental data, but in the future they should be extracted; both from the experimental and the theoretical point of view a list of the cycle itineraries, the cycle eigenvalues, and the densities of points used in the fitting of each cycle measured in an experiment is possibly the optimal invariant description of the topology and the scale structure of a given dynamical system.

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