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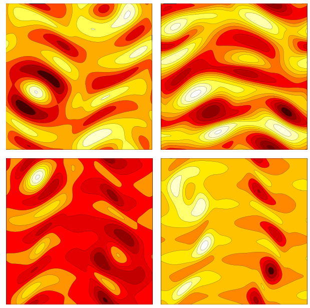
Recurrent flows: the clockwork behind turbulence



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The understanding of chaotic dynamics in high-dimensional systems that has emerged in the last decade offers a promising dynamical framework to study turbulence. Here turbulence is viewed as a walk through a forest of exact solutions in the infinite-dimensional state space of the governing equations. Recently, Chandler & Kerswell (*J. Fluid Mech.*, vol. 722, 2013, pp. 554–595) carry out the most exhaustive study of this programme undertaken so far in fluid dynamics, a feat that requires every tool in the dynamicist's toolbox: numerical searches for recurrent flows, computation of their stability, their symmetry classification, and estimating from these solutions statistical averages over the turbulent flow. In the long run this research promises to develop a quantitative, predictive description of moderate-Reynolds-number turbulence, and to use this description to control flows and explain their statistics.

Key words: instability, transition to turbulence, turbulence theory

1. Introduction

Turbulent weather patterns never settle down, yet we can recognize a cloud by its transitory shapes. What are these fleeting shadows? The discovery of unstable steady and travelling solutions of Navier–Stokes equations, together with glimpses of them in experiments (Hof *et al.* 2004), carries a promise of obtaining a description of a turbulent flow in terms of the dynamics of a handful of exact solutions. Intriguing as they are, these solutions are shapes that do not change in time. However, turbulence does, and to capture it one needs dynamics.

Moore & Spiegel (1966) have proposed that time-periodic solutions should do the job, and, in fact, Poincaré had already speculated that periodic orbits offer 'a breach into a domain hitherto reputed unreachable', but it took 80 years before one could compute such orbits. The answer is that one has to evaluate a determinant (zeta function) and take a logarithm (trace formula), but a determinant that is fashioned out of infinitely many infinitely small pieces. With his trace formula, Gutzwiller (1990) was the first to demonstrate that chaotic dynamics is built upon unstable periodic

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orbits. Ruelle (1978) derived the dynamical zeta function, in which the natural measure is a weighted sum over the infinite set of unstable periodic orbits. The Copenhagen school (Cvitanović *et al.* 2013) turned these formulas into computation, and by 1991 the group of Wintgen was able to obtain a surprisingly accurate helium spectrum from a small set of periodic orbits (Ezra *et al.* 1991) – 50 years after old quantum theory had failed to do so. A *tour de force*, but three ordinary differential equations are still a far cry from the infinite-dimensional state space of fluid dynamics.

The possibility that one could adjust the myriad of initial velocity fields of a three-dimensional turbulent fluid state in such a way that the flow would *exactly* recur after a finite time had always seemed utterly out of reach. That is, until the Christmas of 2000, when Kawahara & Kida (2001) announced that they had found an 'unstable periodic solution of the Navier–Stokes equation', in an 8448-dimensional discretization of the plane Couette flow! This solution was not only periodic in time, but lay smack in the turbulent sea and predicted turbulent averages surprisingly well. Once the first transistor is built, the impossible becomes possible, and heroic feat that it was, it was eventually followed by Viswanath (2007), Cvitanović & Gibson (2010), Kreilos & Eckhardt (2012) and Willis, Cvitanović & Avila (2013), who together determined some hundred such solutions for three-dimensional flows. Today, an undergraduate can download Channelflow.org code and find even more. A single solution makes for a pretty journal cover. But an infinity of orbits ....

2. Overview

Recently, Chandler & Kerswell (2013) (C&K) undertake the most exhaustive study so far of periodic orbit theory (POT) applied to fluid dynamics. Periodic orbits are one class of invariant solutions (the acronym UPOs for unstable periodic orbits is also used, which is a bit like calling all bicycles 'unstable bicycles'). C&K's use of the term 'recurrent flow' is broader and more descriptive of the plethora of invariant solutions encountered in fluid dynamics: 'coherent structures', 'recurrent patterns', 'modulated amplitude waves', and so on. C&K investigate two-dimensional Kolmogorov flow on a doubly periodic domain at several moderate Reynolds numbers, resolved to 22 428 degrees of freedom. An individual turbulent state tends to break all symmetries, so searches are undertaken in the full state space, with no symmetry restrictions. First, direct numerical simulations (DNS) are run for long times. Near recurrences are flagged and used as starting guesses for recurrent flows, which are then determined by an iterative Newton–Raphson root search; their linear stability is computed by the Arnoldi method. After CPU months of trawling, some hundred recurrent flows are determined, typically with four or five unstable eigendirections. The title-page figure shows one such solution as four snapshots of its vorticity contours. Recurrent flows, however, are more profitably viewed as movies; see http://dx.doi.org/10.1017/jfm.2013.122 for two solutions of Chandler & Kerswell (2013) and also the Gibson (2013) database. Embedded in the turbulent flow, these solutions are likely to be dynamically important.

What is one to do with all these solutions? The prophecies of Hopf (1948) are now coming to fruition: 'The ultimate goal, however, must be a rational theory of statistical hydrodynamics where [...] properties of turbulent flow can be mathematically deduced from the fundamental equations of hydromechanics.' C&K's calculations vindicate Hopf's vision; the (power, dissipation) plots of sets of their recurrent flows trace out the high probability density regions of the turbulent attractor, and go a long way towards giving us the 'geometrical picture' of the turbulent state space flow.

Hopf goes even further: 'The geometrical picture of the phase flow is, however, not the most important problem of the theory of turbulence. Of greater importance is the determination of the probability distributions associated with the phase flow.'

That is what POT does. In describing the statistics of turbulence, one is interested in the average over all fluid states of a property $a(x)$, such as the dissipation rate. In POT (for details, see ChaosBook.org) this average is given by the *exact cycle averaging formula*: $\langle a \rangle = \sum_\pi A_\pi t_\pi / \sum_\pi T_\pi t_\pi$, where the sums are over all sets of distinct prime periodic orbits, or 'pseudo-cycles' $\pi = \{p_1, p_2, \ldots, p_k\}$. Here the weight of a pseudo-cycle $t_\pi = (-1)^{k+1}/\Lambda_\pi$ is given by the product of the expanding eigenvalues of the Jacobian matrices of $\{p_k\}$, T_π is the pseudo-cycle period, and A_π is the integral of the property $a(x)$ over the pseudo-cycle. Geometrically a pseudo-cycle is a sequence of shorter periodic orbits that shadow a longer periodic orbit along segments p_1, $p_2, \ldots, p_k$. For hyperbolic flows, the Λ_π grow exponentially with period T_π, so these infinite sums are exponentially convergent, provided the terms are ordered by the decreasing pseudo-cycle weights t_π.

How does POT do compared with DNS long-time averaging? C&K find that a more complete set of recurrent flows will be needed for a meaningful test. The single, least unstable of all solutions, recurrent flow $R19$, the only one with only two unstable directions, lumps up the averages. As we shall explain, that POT averaging is not working well yet is not surprising. C&K then do the simplest, 'agnostic' test, and compare DNS to the average with equal weight given to all solutions. For them this actually works somewhat better than POT. They also compare POT to so-called 'escape-time weighting' averages. These formulas are wrong. So they work just as well as the equally wrong simple 'agnostic' averages.

3. Future

What's to be done? First, Hopf (1948) visualized the function space of allowable Navier–Stokes velocity fields as an infinite-dimensional state space, with the instantaneous state of the fluid being a point in this space. Sixty-five years later, the prevailing custom is still to project the dynamics onto two or three physical global averages, such as the Reynolds (1883) and Orr (1907) energy equation, where the exchange of kinetic energy with the base flow is balanced by the viscous energy dissipation rate. This is a bit like hoping to land 'Curiosity' on another planet by tracking the *sum* of the kinetic energies of all planets versus the sum of their angular momenta squared. If two fluid states are clearly separated in such projection, they are also separated in the full, infinite-dimensional state space, but the converse does not hold. The implementation of Hopf's vision by Gibson, Halcrow & Cvitanović (2008) is a promising alternative: the dynamics of a moderate-Reynolds-number turbulent flow is profitably visualized in dynamically intrinsic coordinate frames constructed from key solutions of the Navier–Stokes equations (see ChaosBook.org/tutorial).

Second, continuous symmetries (rotations and translations) induce drifts that are the dominant (and obscuring) feature of turbulence, with recurrent flows predominantly of the relative (equivariant) kind. Willis *et al.* (2013) have shown that only after *symmetry reduction* are the state space relations between important recurrent flows revealed; in particular, it is the geometry of flows in the symmetry-reduced state space that enabled them to find, for the first time for pipe flows, relative periodic orbits that are embedded within the chaotic saddle.

Third, once periodic orbits form the skeleton (symbolic dynamics) underpinning the chaotic dynamics, and their unstable manifolds trace out connections between neighbourhoods of important recurrent flows (Markov graphs), the recurrent flows are organized hierarchically and the search for the longer-period solutions is systematic, in contrast to the blind trawling for near-recurrences. It is *only at this point* that the exact cycle averaging formulas have a chance of converging.

Fourth, the low numbers of unstable eigendirections for recurrent flows that form the turbulent web (in contrast to the highly unstable and never revisited ghosts of equilibria past) are suggestive of the extremely low dimensionality of the turbulent attractors studied here. In the long run it might be possible to make this dimensionality precise, by carrying out the 'covariant Lyapunov vector' programme of Ginelli *et al.* (2007).

Much progress has been achieved by Chandler & Kerswell (2013), and the stage is set to see whether periodic orbit theory can indeed serve as 'a rational theory of statistical hydrodynamics'. If so, this would profoundly advance our understanding of turbulence, well beyond the current descriptive level.

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