Periodic orbit theory of linear response

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Abstract

[1] cycle expansions.

1 Introduction

This paper is organized as follows: Section 2 is a brief introduction to cycle expansion averaging. In section 3 some known facts are reviewed and section 4 considers linear response in terms of periodic orbits. The summary and conclusion is presented in section 7.

Cycle expansion is a technique for evaluation of averages and eigenvalues of chaotic systems (for an introduction to the periodic orbit theory we refer the reader to ref. [1]) and is based on calculation of derivatives of dynamical zeta functions.

2 Periodic orbit averaging

Expand the dynamical zeta function as a formal power series,

$$\begin{aligned} 1/\zeta &= \prod_p (1-t_p) = 1 - \sum_{p_1+p_2+\ldots+p_k}{}' t_{p_1+p_2+\ldots+p_k}, \\ t_{p_1+p_2+\ldots+p_k} &= (-1)^{k+1} t_{p_1} t_{p_2} \ldots t_{p_k} \end{aligned} \tag{1}$$

[1]**file** predrag/articles/lin_resp/linresp.tex
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where the prime on the sum indicates that the sum is over all distinct non-repeating combinations of prime cycles. For $k > 1$, $t_{p_1+p_2+\ldots+p_k}$ are weights of *pseudocycles*; they are sequences of shorter cycles that shadow a cycle with the symbol sequence $p_1 p_2 \ldots p_k$ along segments p_1, $p_2, \ldots$, p_k.

The simplest example is the cycle expansion for a system described by a complete binary symbolic dynamics. In this case the Euler product (**??**) is given by

$$
\begin{aligned}
1/\zeta \;=\;& (1-t_0)(1-t_1)(1-t_{01})(1-t_{001})(1-t_{011}) \\
& (1-t_{0001})(1-t_{0011})(1-t_{0111})(1-t_{00001})(1-t_{00011}) \\
& (1-t_{00101})(1-t_{00111})(1-t_{01011})(1-t_{01111})\ldots
\end{aligned}
$$

and the first few terms of the expansion (1) ordered by increasing total pseudocycle length are:

$$
\begin{aligned}
1/\zeta \;=\;& 1-t_0-t_1-t_{01}-t_{001}-t_{011}-t_{0001}-t_{0011}-t_{0111}-\ldots \\
& -t_{0+1}-t_{0+01}-t_{01+1}-t_{0+001}-t_{0+011}-t_{001+1}-t_{011+1} \\
& -t_{0+01+1}-\ldots
\end{aligned}
$$

We refer to such series as a *cycle expansion*. A cycle expansion is a series representation of a dynamical zeta function or a spectral determinant, expanded as a sum over pseudocycles, ordered by increasing cycle length and instability.

The next step is the key step: regroup the terms into the dominant *fundamental* contributions t_f and the decreasing *curvature* corrections c_n. For the binary case this regrouping is given by

$$
\begin{aligned}
1/\zeta \;=\;& 1-t_0-t_1-[(t_{01}-t_1t_0)]-[(t_{001}-t_{01}t_0)+(t_{011}-t_{01}t_1)] \\
& -[(t_{0001}-t_0t_{001})+(t_{0111}-t_{011}t_1) \\
& \quad +(t_{0011}-t_{001}t_1-t_0t_{011}+t_0t_{01}t_1)]-\ldots \\
=\;& 1-\sum_f t_f-\sum_n c_n \;. \qquad (2)
\end{aligned}
$$

We refer to such regrouped series as *curvature expansions*. This separation into “fundamental” and “curvature” parts of cycle expansions is possible *only* for dynamical systems whose symbolic dynamics has finite grammar. The fundamental cycles t_0, t_1 have no shorter approximants; they are the “building blocks” of the dynamics in the sense that all longer orbits can be approximately pieced together from them. The terms grouped in brackets are the curvature corrections; the terms grouped in parenthesis are combinations of longer orbits and their shorter "shadowing" approximants. If the flow is continuous and smooth, orbits of similar symbolic dynamics will traverse the same neighborhoods and will have similar weights, and the weights in such combinations will almost cancel. The utility of cycle expansions dynamical zeta functions and spectral determinants lies precisely in this organization into nearly cancelling combinations: cycle expansions are dominated by short cycles, with long cycles giving exponentially decaying corrections.

3 Cycle formulas for dynamical averages

The eigenvalue condition for the dynamical zeta function (1)

$$0 = 1 - \sum{}' t_{p_1+p_2+\ldots+p_k} \,, \qquad t_i = t_i(\beta, s(\beta)) = \frac{1}{|\Lambda_i|} e^{\beta\cdot A_i - s(\beta) T_i} \tag{3}$$

is an implicit equation for $s = s(\beta)$ of form $G(\beta, s(\beta)) = 0$. The cycle averaging formulas for the slope and the curvature of $s(\beta)$ are obtained by taking the derivatives of the eigenvalue condition. The first derivative leads to

$$\begin{aligned} 0 &= \frac{d}{d\beta} G(\beta, s(\beta)) \\ &= \frac{\partial G}{\partial \beta} + \frac{\partial s}{\partial \beta} \frac{\partial G}{\partial s}\Big|_{s=s(\beta)} \quad \Longrightarrow \quad \frac{\partial s}{\partial \beta} = -\frac{\partial G}{\partial \beta} \Big/ \frac{\partial G}{\partial s} \,. \end{aligned} \tag{4}$$

Denoting by

$$\langle A \rangle_G = -\left.\frac{\partial G}{\partial \beta}\right|_{\beta, s=s(\beta)} \,, \qquad \langle T \rangle_G = \left.\frac{\partial G}{\partial s}\right|_{\beta, s=s(\beta)} \tag{5}$$

respectively the mean cycle expectation value of A and the mean cycle period computed from the $G(\beta, s(\beta)) = 0$ condition we obtain the cycle averaging formulas for the expectation value of the observable

$$\langle a \rangle = \frac{\langle A \rangle_G}{\langle T \rangle_G} \tag{6}$$

3.1 Dynamical zeta function cycle expansions

Substituting the cycle expansion (1) for dynamical zeta function we obtain the cycle averaging formulas for the mean cycle A and the mean cycle period

$$\langle A \rangle_\zeta \ := \ -\frac{\partial}{\partial \beta} \frac{1}{\zeta} \ = \sum{}' (A_{p_1} + A_{p_2} \cdots + A_{p_k})\, t_{p_1+p_2+\ldots+p_k}$$

$$\langle T \rangle_\zeta \;:=\; \frac{\partial}{\partial s}\frac{1}{\zeta} = \sum{}' (T_{p_1} + T_{p_2} \cdots + T_{p_k})\, t_{p_1+p_2+\ldots+p_k}\,,$$

where $\langle \cdots \rangle_\zeta$ stands for the dynamical zeta function average over prime cycles, and cycle weights are evaluated at their leading eigenvalue values $t_p = t_p(\beta, s(\beta))$. For bounded flows $s(0) = 0$, so

$$\begin{aligned} \langle A \rangle_\zeta &= \sum{}' (-1)^{k+1} \frac{A_{p_1} + A_{p_2} \cdots + A_{p_k}}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} \\ \langle T \rangle_\zeta &= \sum{}' (-1)^{k+1} \frac{T_{p_1} + T_{p_2} \cdots + T_{p_k}}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|}\,. \end{aligned} \tag{7}$$

For example, for the complete binary symbolic dynamics the mean cycle period $\langle T \rangle_\zeta$ is given by

$$\begin{aligned} \langle T \rangle_\zeta &= \frac{T_0}{|\Lambda_0|} + \frac{T_1}{|\Lambda_1|} + \left(\frac{T_{01}}{|\Lambda_{01}|} - \frac{T_0 + T_1}{|\Lambda_0 \Lambda_1|} \right) \\ &\quad + \left(\frac{T_{001}}{|\Lambda_{001}|} - \frac{T_{01} + T_0}{|\Lambda_{01}\Lambda_0|} \right) + \left(\frac{T_{011}}{|\Lambda_{011}|} - \frac{T_{01} + T_1}{|\Lambda_{01}\Lambda_1|} \right) + \ldots \end{aligned} \tag{8}$$

and similarly for $\langle A \rangle_\zeta$. Note that these cycle expansions are also grouped into shadowing combinations, with nearby pseudoorbits nearly cancelling each other.

The mean cycle period $\langle T \rangle_\zeta$ fixes the normalization of the unit of time; it can be interpreted as the average near recurrence or the average first return time. For example, if we have evaluated a billiard expectation value $\langle a \rangle$ in terms of continuous time, and would like to also have the corresponding average $\langle a \rangle_{\mathrm{dscr}}$ measured in discrete time given by the number of reflections off billiard walls, the two averages are related by

$$\langle a \rangle_{\mathrm{dscr}} = \langle a \rangle \, \langle T \rangle_\zeta \,/\, \langle n \rangle_\zeta \;, \tag{9}$$

where $\langle n \rangle_\zeta$ is the average of the number of bounces n_p along the cycle p.

4 Linear response in terms of periodic orbits

4.1 Eigenvalue shift using dynamical zeta functions

Consider the dynamical system

$$\frac{d}{dt} x_i = v_i(x) \tag{10}$$

Assume hyperbolicity and construct the dynamical zeta function; an eigenvalue s_α is determined by the condition

$$1/\zeta_0(s_\alpha) = 0. \tag{11}$$

Consider now a weakly perturbed system

$$\frac{d}{dt}x_i = v_i(x) + \epsilon\delta v_i(x), \quad |\epsilon| \ll 1 \tag{12}$$

where the perturbation $\delta v_i(x)$ is space, but not time dependent. The eigenvalues of the perturbed system are slightly shifted:

$$s_\alpha \to s_\alpha + \delta s_\alpha, \tag{13}$$

so the condition on the dynamical zeta function becomes

$$1/\zeta(s_\alpha + \delta s_\alpha) = 0. \tag{14}$$

To linear order

$$1/\zeta(s_\alpha) + \delta s_\alpha \frac{\partial}{\partial s}\frac{1}{\zeta(s_\alpha)} = 0 \tag{15}$$

so the eigenvalue shift is

$$\delta s_\alpha = -\frac{1/\zeta(s_\alpha)}{\frac{\partial}{\partial s}\frac{1}{\zeta(s_\alpha)}}. \tag{16}$$

The cycle expansion of $1/\zeta(s)$ is a cycle-by-cycle deformation of $1/\zeta_0(s) \pm$ new/lost cycles. We assume that the dynamics is structurally stable, so the latter contribution will be ignored until further notice. To linear order the cycle expansion is

$$1/\zeta(s_\alpha) = \sum_p t_p(s_\alpha) + \sum_p \delta t_p(s_\alpha). \tag{17}$$

The first sum is the cycle expansion of $1/\zeta_0(s_\alpha)$ which vanishes by (11). Now the denominator can be replaced by $\frac{\partial}{\partial s}\frac{1}{\zeta_0(s_\alpha)}$ to leading order, so

$$\begin{aligned} \frac{\partial}{\partial s}\frac{1}{\zeta(s_\alpha)} &= \frac{\partial}{\partial s}\frac{1}{\zeta_0(s_\alpha)} = \sum_p \frac{\partial}{\partial s} t_p(s_\alpha) \\ &= -\sum_p T_p t_p(s_\alpha) = -\langle T \rangle_\zeta . \end{aligned} \tag{18}$$

The cycle weight variation is a combination of variations of A_p, T_p and Λ_p due to deformation of the cycle. One finds with the orbit weight

$$t_p = \frac{1}{|\Lambda_p|} e^{\beta A_p - sT_p} \tag{19}$$

$$\delta t_p = \left(\beta \delta A_p - s \delta T_p - \frac{\delta \Lambda_p}{\Lambda_p} \right) t_p, \tag{20}$$

so the eigenvalue shift is given by

$$\delta s_\alpha = - \left. \frac{\beta \left\langle \delta A_p \right\rangle_\zeta - s \left\langle \delta T_p \right\rangle_\zeta - \left\langle \frac{\delta \Lambda_p}{\Lambda_p} \right\rangle_\zeta}{\left\langle T \right\rangle_\zeta} \right|_{s=s_\alpha} \tag{21}$$

4.2 Cycle averaging formulas for linear response of observables

The observable, the eigenvalue s_α and the characteristics of the orbits now all depend on the variation of the system $\delta\epsilon$. However, for each fixed value of the external parameter (12) the averaging formula in terms of periodic orbits should be applicable. Thus:

$$\begin{aligned} \frac{\partial \left\langle a \right\rangle}{\partial \epsilon} &= \frac{\partial}{\partial \epsilon} \frac{\left\langle A \right\rangle_\zeta}{\left\langle T \right\rangle_\zeta} = \frac{\left\langle T \right\rangle_\zeta \frac{\partial \langle A \rangle_\zeta}{\partial \epsilon} - \left\langle A \right\rangle_\zeta \frac{\partial \langle T \rangle_\zeta}{\partial \epsilon}}{\left\langle T \right\rangle_\zeta^2} && (22) \\ &= \frac{\frac{\partial \langle A \rangle_\zeta}{\partial \epsilon} - \left\langle a \right\rangle_\zeta \frac{\partial \langle A \rangle_\zeta}{\partial \epsilon}}{\left\langle T \right\rangle_\zeta}. && (23) \end{aligned}$$

Here the zeta averages are calculated with the weight (19).

We remark that in a typical calculation of a physical average β is put to zero. For a bounded system the leading eigenvalue $s_0 = 0$ independent on the perturbation. Let us show how to calculate $\frac{\partial \langle A \rangle_\zeta}{\partial \epsilon}$ (the formula for $\frac{\partial \langle T \rangle_\zeta}{\partial \epsilon}$ is similar). From (20) it follows that:

Figure 1: The truncated dynamical zeta function.

$$\frac{\partial \langle A \rangle_\zeta}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \sum_p t_p A_p = \sum_p \frac{\partial t_p}{\partial \epsilon} A_p + \sum_p \frac{\partial A_p}{\partial \epsilon} t_p \tag{24}$$

$$= \sum_p t_p \left(\left(-s \frac{\partial T_p}{\partial \epsilon} + \beta \frac{\partial A_p}{\partial \epsilon} - \frac{\frac{\partial \Lambda_p}{\partial \epsilon}}{\Lambda_p} \right) A_p + \frac{\partial A_p}{\partial \epsilon} \right). \tag{25}$$

Using this and the formula above for the eigenvalue shift we calculate $\frac{\partial \langle a \rangle}{\partial \epsilon}$.

We find

$$\frac{\partial \langle a \rangle_\zeta}{\partial \epsilon} = \frac{1}{\langle T \rangle_\zeta} \left(\left\langle (\beta (A - \langle a \rangle T) + 1) \frac{\partial A}{\partial \epsilon} \right\rangle - \left\langle (s_\alpha - (1 - s_\alpha) \langle a \rangle) \frac{\partial T}{\partial \epsilon} \right\rangle \right.$$
$$\left. - \left\langle (A - \langle a \rangle T) \frac{\frac{\partial \Lambda}{\partial \epsilon}}{|\Lambda|} \right\rangle \right) \tag{26}$$

5 Numerical tests of the linear response theory

6 Results

The averages we wish to know consist of

7 Conclusions

In conclusion we have demonstrated that exact

References

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