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Symmetry and control: spatially extended chaotic systems

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Abstract

In a recent paper [Phys. Rev. E 57 (1998) 1550] it was demonstrated that the symmetries of the evolution equation and the target state have a profound effect on controlling the chaotic behavior. In the present paper we extend these results to the cases of time-periodic target trajectories and inexact symmetries, and apply the developed formalism to the problem of controlling spatiotemporal chaos. We use the example of a lattice dynamical system in arbitrary spatial dimension to show that there exists an intimate relationship between the geometry of an extended system and the geometry of feedback control.

Keywords: Control; Symmetry; Spatiotemporal chaos

1. Introduction

The desire to improve performance of many practically important systems and devices often calls for shifting their operating range into a highly nonlinear area, which after a series of bifurcations usually leads to irregular chaotic behavior. This kind of behavior, however, is rarely desired, while substantial benefits could be obtained by making the dynamics regular. This goal can typically be achieved by applying small preprogrammed perturbations to steer the system towards a periodic orbit with desired properties, which is broadly referred to as chaos control.

The present paper is devoted to a special, but very interesting, class of systems which possess some kind of symmetry. In fact, many practically important dynamical systems, for instance spatially extended dynamical systems, are intrinsically symmetric and cannot be successfully treated using the formalism developed for the generic case. Indeed, such phenomena as fluid flows, convection or chemical reactions often take place inside symmetric containers — cylinders, spheres, pipes and annuli. As a result, the dynamical equations also show rotational and translational symmetries. Even the dynamics of unbounded systems is often significantly influenced by the symmetries of the physical space. Although the presence of symmetries usually simplifies the analysis of the dynamics, it also makes control more complicated due to the inherent degeneracies of the evolution operators. In particular, the presence of symmetries, explicit or implicit, makes a number of single-control-parameter methods fail [1–3], calling for multi-parameter control [4–8].

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For example, it was found experimentally that in order to successfully suppress the helical traveling wave in a liquid bridge (a drop of fluid suspended between two coaxial cylinders kept at different temperatures) convection experiment [9], at least two sensors and two heating elements were needed. The one-sensor, one-heating element arrangement failed to stabilize the unstable axially symmetric state. In fact, the authors recognized that this was due to the symmetry between the left-and right-going waves which produced the degeneracy.

The difficulties experienced in attempts to control instabilities in several other practically important nonlinear systems and devices with various sorts of symmetries could be traced to the lack of adequate analytical analysis of the control problem. Examples include rotating stall and surge instabilities in turbine engines [10], rough idling of internal combustion engines operating on lean fuel mixture [11], surface roughening during epitaxial growth of thin films [12], random beam steering in high-power wide-aperture semiconductor lasers [13], and even such unlikely phenomenon as brain epilepsy [14]. In what follows we present a theoretical framework for treating the symmetric control problem. It turns out that the symmetry analysis of lattice dynamical systems, which often serve as a prototypical example of spatially extended systems – continuous or discrete – can provide us with valuable insights on controlling spatiotemporally chaotic dynamics.

In order to see how the control problem is affected by symmetries, consider a continuous-time system described by the dynamical equation

$$\dot{\mathbf{x}}(\tau) = \Phi(\mathbf{x}(\tau), \mathbf{u}(\tau)), \tag{1}$$

where $\mathbf{x} \in \mathcal{R}$ is the n_x-dimensional state of the system and $\mathbf{u}$ is the n_u-dimensional vector of system parameters. It is convenient for our purposes to discretize this equation using a sequence of sampling times tT, so that $\mathbf{x}^t = \mathbf{x}(tT)$ and $\mathbf{u}^t = \mathbf{u}(\tau)$ for $tT < \tau < (t+1)T$. Integrating (1) from kT to $(k+1)T$ one obtains a discrete-time system, whose evolution is described by the map $\mathbf{F} : \mathcal{R} \times \mathbb{R}^{n_u} \to \mathcal{R}$ such that

$$\mathbf{x}^{t+1} = \mathbf{F}(\mathbf{x}^t, \mathbf{u}^t). \tag{2}$$

The objective of control is to make the system follow the (possibly unstable) periodic target trajectory $\bar{\mathbf{x}}^t$. Let us linearize Eq. (2) about this target trajectory to obtain

$$\Delta\mathbf{x}^{t+1} = A^t \Delta\mathbf{x}^t + B^t \Delta\mathbf{u}^t, \tag{3}$$

where $\Delta\mathbf{x}^t \in \mathcal{T}$ is the deviation from the target trajectory, $\mathcal{T}$ the space tangent to the phase space $\mathcal{R}$ at $\bar{\mathbf{x}}^t$, $A^t = \mathbf{D_x F}(\bar{\mathbf{x}}^t, \bar{\mathbf{u}})$ the Jacobian matrix which determines the stability properties of the target trajectory, and $B^t = \mathbf{D_u F}(\bar{\mathbf{x}}^t, \bar{\mathbf{u}})$ is the control matrix, which defines the linear response of the system to perturbation of system parameters.

According to linear systems theory [15], if the target trajectory $\bar{\mathbf{x}}^t$ is unstable, it can be stabilized by an appropriate feedback through the time-dependent control perturbation $\Delta\mathbf{u}^t = \mathbf{u}^t - \bar{\mathbf{u}}$, provided the matrices A^t and B^t satisfy certain conditions. In the present study we concentrate on selecting from the complete set of available *system* parameters a minimal set of *control* parameters, whose perturbation allows the stabilization of the target state, i.e., on making an appropriate choice of the control matrix B^t, given the Jacobian A^t. We will see below that the constraints affecting the choice of control parameters can be easily obtained from the symmetry properties of the system and the controlled state. What is more interesting, symmetry allows one to determine the minimal number of control parameters without requiring the knowledge of the Jacobian A^t.

The outline of the paper is as follows. Section 2 contains the symmetry analysis of the linear feedback control problem and discusses the effects of weak symmetry violation. Section 3 illustrates applications of the theory using the canonical example of an inverted pendulum in three dimensions, followed by a thorough analysis of lattice dynamical systems. Finally, we present our conclusions as they relate to control of general spatially extended chaotic systems, using the liquid bridge experiment as our final example, in Section 4.

2. Symmetry and control

2.1. Stabilizability and controllability

Although our analysis is applicable to time-varying systems, for simplicity we assume here that the target state is time-invariant, $\bar{\mathbf{x}}^t = \bar{\mathbf{x}}$. Then the matrices A^t and B^t become constant, and we can drop the time index in (3) to obtain

$$\Delta\mathbf{x}^{t+1} = A\Delta\mathbf{x}^t + B\Delta\mathbf{u}^t. \tag{4}$$

It is useful to introduce and compare two characterizations of the linearized evolution equation (4), which extremely simplify the analysis of feedback control algorithms: *stabilizability* and *controllability*.

The dynamical system (4) or the pair (A, B) is said to be *stabilizable*, if there exists a *state feedback*

$$\Delta\mathbf{u}^t = -K\Delta\mathbf{x}^t, \tag{5}$$

making the system (4) stable, i.e., it is possible to find a *feedback gain matrix* K such that all eigenvalues λ'_k of the matrix $A' = A - BK$ lie within a unit circle of the complex plane, $|\lambda'_k| < 1\,\forall k$. Otherwise the system or the pair (A, B) is called *unstabilizable*. Indeed, substituting the feedback (5) into (4) one obtains the linearized evolution equation for the *closed-loop* system:

$$\Delta\mathbf{x}^{t+1} = (A - BK)\Delta\mathbf{x}^t \tag{6}$$

with $\Delta\mathbf{x} = \mathbf{0}$ becoming the stable fixed point of the map (6) if and only if $A - BK$ is stable.

Alternatively, the n_x-dimensional linear system (4) or the pair (A, B) is said to be *controllable* if, for any initial state $\Delta\mathbf{x}^{t_\mathrm{i}} = \Delta\mathbf{x}_\mathrm{i}$, times $t_\mathrm{f} - t_\mathrm{i} \geq n_x$, and final state $\Delta\mathbf{x}_\mathrm{f}$, there exists a sequence of control perturbations $\Delta\mathbf{u}^{t_\mathrm{i}}, \dots, \Delta\mathbf{u}^{t_\mathrm{f}-1}$ such that the solution of Eq. (4) satisfies $\Delta\mathbf{x}^{t_\mathrm{f}} = \Delta\mathbf{x}_\mathrm{f}$. Otherwise, the system or the pair (A, B) is called *uncontrollable*.

At first sight Eq. (5) seems to impose strict limitations on the allowed form of the feedback law (the feedback gain K is assumed to depend explicitly on system parameters, but not on time). However, this is precisely the form demanded by a number of widely used control algorithms [1,2,16]. Besides, even if the control perturbation $\Delta\mathbf{u}^t$ is chosen as a smooth nonlinear function of the system state $\mathbf{x}^t$, for small deviations we recover (5), which makes the linear analysis presented below equally relevant for nonlinear control.

Stabilizability is a property, which usually sensitively depends on the values of system parameters. In the majority of practical applications, however, it is preferable to have an adaptive control that would stabilize a given steady state $\bar{\mathbf{x}}(\bar{\mathbf{u}})$ for arbitrary values of system parameters. This is especially important, if one is to track the trajectory $\bar{\mathbf{x}}$ as parameters slowly vary, which might be advantageous in many applications, e.g., for moving the operating point of a nonlinear device across a bifurcation, from the stable region to the chaotic region. In this case, similarly to matrices A and B, K acquires implicit dependence on time through the parameters. Such an adaptive control scheme can be obtained, if the more restrictive condition of controllability, which is essentially parameter-independent, is imposed on the matrices A and B. It can be demonstrated [15] that, if the feedback is chosen in the form (5), the controllability condition guarantees that the eigenvalues of the matrix $A - BK$ can be freely assigned (with complex ones in conjugate pairs) by an appropriate choice of the matrix K. Therefore, if the system is controllable, it is stabilizable as well, and by requiring controllability we satisfy both conditions at once.

The controllability condition can be represented in a number of different equivalent forms. To obtain one particularly convenient form, we make the trivial observation that, if it is possible to drive the linear system from an arbitrary initial state $\Delta\mathbf{x}_\mathrm{i}$ to an arbitrary final state $\Delta\mathbf{x}_\mathrm{f}$ in n_x steps, it is possible to do the same in any number of steps n exceeding n_x. Suppose we let the system evolve under control for n_x steps from the initial state $\Delta\mathbf{x}^t$. The final state will be given by [1]

$$\Delta\mathbf{x}^{t+n_x} = (A)^{n_x}\Delta\mathbf{x}^t + \sum_{k=1}^{n_x}(A)^{n_x-k}B\Delta\mathbf{u}^{t+k-1}. \tag{7}$$

Denote $\mathbf{b}_m$ the mth column of the matrix B

$$B = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \dots \quad \mathbf{b}_{n_u}\,]. \tag{8}$$

Regarding the terms $(A)^{n_x-k}\mathbf{b}_m$ as vectors in the n_x-dimensional tangent space $\mathcal{T}$,

$$\mathbf{h}_m^k = (A)^{n_x-k}\mathbf{b}_m, \quad k = 1, \dots, n_x,\ m = 1, \dots, n_u, \tag{9}$$

and the control perturbations Δu_m^{t+k-1} as coordinates, we immediately conclude that Eq. (7) rewritten as

$$\Delta\mathbf{x}_\mathrm{f} - (A)^{n_x}\Delta\mathbf{x}_\mathrm{i} = \sum_{k=1}^{n_x}\sum_{m=1}^{n_u}\Delta u_m^{t+k-1}\mathbf{h}_m^k, \tag{10}$$

can only be satisfied, if and only if there are n_x linearly independent vectors in the set (9), i.e., the set $\{\mathbf{h}_m^k\}$ spans the tangent space $\mathcal{T}$. (In contrast, the stabilizability condition requires that this set spans only the unstable subspace $L^u \subseteq \mathcal{T}$ of the Jacobian A, instead of the whole tangent space $\mathcal{T}$.) This is equivalent to requiring that

$$\mathrm{rank}(\mathcal{C}) = n_x, \tag{11}$$

where the matrix

$$\mathcal{C} \equiv [\,B \quad AB \quad (A)^2B \quad \dots \quad (A)^{n_x-1}B\,] \tag{12}$$

is called the *controllability matrix*. Condition (11) was introduced into the physics literature from linear systems theory by Romeiras et al. [2] as a simple, but practical test of the controllability.

In order to better understand the restrictions imposed on the control scheme by symmetries, it is beneficial to look at the controllability condition from the geometrical point of view, assuming $n_u = 1$ and, consequently, $B = \mathbf{b}$. The controllability in this context is equivalent to the vectors $\mathbf{h}^1, \mathbf{h}^2, \dots, \mathbf{h}^{n_x}$ spanning the tangent space $\mathcal{T}$. Generically, the matrix A is nondegenerate (has a nondegenerate spectrum), so one can always find a vector $\mathbf{b}$, such that the resulting set (9) forms a basis. However, if A is degenerate, which is a usual consequence of symmetry, there will exist an eigenspace of the Jacobian, $L^r \subset \mathcal{T}$, such that $\mathbf{x}^\dagger A = \lambda_r\mathbf{x}^\dagger\ \forall\mathbf{x} \in L^r$ with the dimension $d_r = \dim(L^r) > 1$, where † denotes the (complex conjugate) transpose of a matrix or vector. The dynamics of the system in such an eigenspace cannot be controlled with just one control parameter (see [2] for an example of such a situation), because the vectors $\mathbf{h}^k$ only span a one-dimensional subspace of L^r. Indeed, since $d_r > 1$ there will exist $d_r - 1$ adjoint eigenvectors $\mathbf{f}_j \in L^r$ orthogonal to $\mathbf{b}$ and each other. Then

$$(\mathbf{f}_j \cdot \mathbf{h}^k) = \mathbf{f}_j^\dagger(A)^{n_x-k}\mathbf{b} = \lambda_r^{n_x-k}\mathbf{f}_j^\dagger\mathbf{b}\lambda_r^{n_x-k}(\mathbf{f}_j \cdot \mathbf{b}) = 0, \tag{13}$$

so every basis vector $\mathbf{h}^k$ is orthogonal to every eigenvector $\mathbf{f}_j$, $j = 1, \dots, d_r - 1$.

It is often convenient to define the notion of controllability for individual eigenvectors. We will say that the adjoint eigenvector $\mathbf{f}$ of the Jacobian A is controllable, if there exists m, $1 \le m \le n_u$, such that $(\mathbf{f} \cdot \mathbf{b}_m) \neq 0$.

[1] Here and in the text we use the notation $(A)^n$ to indicate that A is taken to the power of n to differentiate it from the notation A^t, where index t defines the time dependence.

Respectively, the eigenvector that is orthogonal to every column of the control matrix B is called uncontrollable. Using these definitions we can, therefore, conclude that the controllability of the linearized system is equivalent to the controllability of each and every adjoint eigenvector of the Jacobian matrix (also see [17]). Similarly, the stabilizability is equivalent to the controllability of each and every *unstable* adjoint eigenvector. Identifying the adjoint eigenvectors with normal modes of the system, we can say that the control does not affect the amplitude of the uncontrollable normal modes, so they cannot be suppressed.

If the system dynamics in L^r happens to be stable (e.g., when the system is stabilizable, but uncontrollable), the system can still be stabilized similarly to the nondegenerate case, but we have to ensure the controllability in case the dynamics in this eigenspace is unstable. This can be achieved by increasing the number of control parameters n_u, which extends the set (9), until it spans every eigenspace of $\mathcal{T}$. This would lead one to assume that the minimal value of n_u should be defined by the highest degeneracy of the Jacobian matrix A. We will see, however, that various kinds of degeneracy have a somewhat different effect on the controllability of the system.

2.2. Symmetries of the system

Symmetries usually significantly simplify the analysis of system dynamics, and the control problem is no exception. In particular, even when the exact form of the Jacobian matrix is unknown, the structure of the symmetry group describing the symmetries of the system allows one to reduce the controllability condition (11) to a set of much simpler conditions, which provide a number of system-independent results. The discussion below is based on bifurcation theory [19] and closely parallels the treatment of degeneracy in quantum mechanics and spontaneous symmetry breaking in quantum field theory and phase transitions.

In general we call the system symmetric, if the nonlinear evolution equation preserves its form under a set of linear transformations $g : \mathbf{x} \to \mathbf{x}' = g(\mathbf{x})$ of the phase space. More formally, we say that the evolution equation (2) possesses a *structural* symmetry described by a symmetry group $\mathcal{G}$, if the map $\mathbf{F}$ defined in (2) commutes with all group actions

$$\mathbf{F}(g(\mathbf{x}), \mathbf{u}) = g(\mathbf{F}(\mathbf{x}, \mathbf{u})) \quad \forall g \in \mathcal{G},\ \mathbf{x} \in \mathcal{R}, \tag{14}$$

or, in other words, if the function $\mathbf{F}(\mathbf{x}, \mathbf{u})$ is $\mathcal{G}$-equivariant with respect to its first argument. The group $\mathcal{G}$ is usually a byproduct of symmetries of the underlying physical space, such as rotational and translational symmetry (domain symmetry), and symmetries of the phase space, such as phase symmetry $\phi \to \phi + 2\pi$ (range symmetry). Since all interesting physical symmetries are unitary (such rare exceptions as the Lorentz group are hardly relevant in the context of control problems), we will assume that $\mathcal{G}$ is a unitary group.

Usually, the symmetry demonstrates itself in more than just one way: often steady (as well as time-periodic) states $\bar{\mathbf{x}}$ of symmetric systems will too be symmetric with respect to transformations $g \in \mathcal{H}_{\bar{\mathbf{x}}}$, where $\mathcal{H}_{\bar{\mathbf{x}}} \subseteq \mathcal{G}$ is called an *isotropy* subgroup of $\bar{\mathbf{x}}$. In general, the target state $\bar{\mathbf{x}}$ might also be symmetric with respect to transformations which do not belong to $\mathcal{G}$. However, considering those does not provide any additional information, so we will assume just that

$$g(\bar{\mathbf{x}}) = \bar{\mathbf{x}} \quad \forall g \in \mathcal{H}_{\bar{\mathbf{x}}}. \tag{15}$$

The states with high isotropy symmetry are most interesting from the control point of view: such states tend to have a very regular structure making them ideal candidates for a target state.

For the purpose of control it is important to observe that upon linearization about the target state $\bar{\mathbf{x}}$ the structural symmetry of the evolution equation (2) does not disappear, but is replaced with a related *dynamical* symmetry

$$g(A\Delta\mathbf{x}) = Ag(\Delta\mathbf{x}) \quad \forall g \in \mathcal{L},\ \forall \Delta\mathbf{x} \in \mathcal{T}. \tag{16}$$

Using the definitions (14) and (15) and the fact that symmetry transformations are linear, one concludes that the group $\mathcal{L}$ describing the dynamical symmetry of the system in the vicinity of the target state $\bar{\mathbf{x}}$ includes all transformations $g \in \mathcal{H}_{\bar{\mathbf{x}}}$, and therefore

$$\mathcal{H}_{\bar{\mathbf{x}}} \subseteq \mathcal{L}. \tag{17}$$

One can speculate that typically $\mathcal{L}$ will coincide with $\mathcal{H}_{\bar{\mathbf{x}}}$. As a consequence, if the target state $\bar{\mathbf{x}}$ has low symmetry, the symmetry of the evolution equation will be reduced upon linearization to a subgroup of $\mathcal{G}$. However, as we will see in Section 3, $\mathcal{L}$ might be equal to $\mathcal{G}$, or even include $\mathcal{G}$ as a subgroup for highly symmetric target states, with the apparent symmetry increased by linearization.

2.3. Conditions for control

As we have argued in the previous sections, the increase in the number of control parameters is due to the degeneracies of the Jacobian A, which in turn are due to dynamical symmetries. The information that is most easily accessible concerns the isotropy symmetry of the target state. In practice, however, it is usually not known whether the isotropy group $\mathcal{H}_{\bar{\mathbf{x}}}$ exhausts the dynamical symmetries of the system or the group $\mathcal{L}$ contains some hidden symmetries as well. It is, therefore, important to show that a number of restrictions on the set of control parameters can be obtained using an arbitrary unitary subgroup $\mathcal{L}'$ of $\mathcal{L}$. It is entirely possible that the spectrum of the Jacobian will contain degeneracies resulting from such hidden symmetries, in addition to those produced by $\mathcal{L}'$. We will call these degeneracies accidental with respect to the group $\mathcal{L}'$.

Let us consider the matrix representation T generated in the tangent space $\mathcal{T}$ by the action of transformations g from the group $\mathcal{L}'$

$$(g(\mathbf{x}))_i = (T(g)\mathbf{x})_i = \sum_{j=1}^{n_x} T_{ij}(g) x_j \quad \forall \mathbf{x} \in \mathcal{T}, \tag{18}$$

where, according to (16), all matrices $T(g)$ commute with the Jacobian

$$T(g)A = AT(g) \quad \forall g \in \mathcal{L}' \subseteq \mathcal{L}. \tag{19}$$

The knowledge of the representation T is sufficient to derive a very simple criterion for the admissibility of the control matrix. Observe that, if $T(g)B = B$ for an arbitrary transformation $g \in \mathcal{L}'$, then

$$\mathcal{C} = [T(g)B \quad AT(g)B \quad \cdots \quad (A)^{n_x-1}T(g)B] = [T(g)B \quad T(g)AB \quad \cdots \quad T(g)(A)^{n_x-1}B] = T(g)\mathcal{C}. \tag{20}$$

As a result, since $T_{ij}(g) \neq \delta_{i,j}$ for any $g \neq e$ (where we defined e as the identity transformation: $e(\mathbf{x}) = \mathbf{x}$), the rows $\tilde{\mathbf{c}}_j$ of the controllability matrix become linearly dependent,

$$\sum_{j=1}^{n_x} (T_{ij}(g) - \delta_{i,j})\tilde{\mathbf{c}}_j = \mathbf{0}, \tag{21}$$

and the controllability condition (11) is violated. Therefore, we obtain a necessary condition on the control matrix

$$T(g)B \neq B \quad \forall g \in \mathcal{L}' \setminus \{e\}. \tag{22}$$

In other words, the control parameters should be chosen such that the symmetry of the linearized evolution equation (4) is completely broken for (almost all) nonzero control perturbations $\Delta\mathbf{u} \neq \mathbf{0}$.

Though simple and general, criterion (22) is not very helpful for finding the minimal set of control parameters satisfying the controllability condition. A more general and practically useful criterion can be derived using group representation theory.

Decomposing the representation T into a sum of irreducible representations T^r of the group $\mathcal{L}'$ with respective dimensionalities d_r, we obtain

$$T = p_1 T^1 \oplus p_2 T^2 \oplus \cdots \oplus p_q T^q, \tag{23}$$

with

$$n_x = p_1 d_1 + p_2 d_2 + \cdots + p_q d_q, \tag{24}$$

where p_r denotes the number of equivalent representations T^r present in the decomposition (23), and q is the total number of nonequivalent irreducible representations. Since $\mathcal{L}'$ is unitary, all irreducible representations T^r in (23) can be chosen as unitary [20].

The tangent space $\mathcal{T}$ is similarly decomposed into a sum of invariant subspaces $L^{r\alpha}_{\mathcal{L}'}$ such that $T(g)\mathbf{x} \in L^{r\alpha}_{\mathcal{L}'}$ $\forall \mathbf{x} \in L^{r\alpha}_{\mathcal{L}'}$ and $\forall g \in \mathcal{L}'$

$$\mathcal{T} = L^1_{\mathcal{L}'} \oplus L^2_{\mathcal{L}'} \oplus \cdots \oplus L^q_{\mathcal{L}'}, \tag{25}$$

where

$$L^r_{\mathcal{L}'} = L^{r1}_{\mathcal{L}'} \oplus L^{r2}_{\mathcal{L}'} \oplus \cdots \oplus L^{rp_r}_{\mathcal{L}'}, \tag{26}$$

and $\alpha = 1, \ldots, p_r$ indexes different invariant subspaces, which correspond to the same group of equivalent irreducible representations T^r. It should be noted that even though the decomposition (25) is unique, the decomposition (26) is not, unless $p_r = 1$.

It is useful to define the projection operator $\hat{P}^r$ onto the invariant subspace $L^r_{\mathcal{L}'} \subset \mathcal{T}$. This operator can be obtained directly from the matrix representation T for most symmetry groups of interest. For finite discrete groups it is given by

$$\hat{P}^r = \frac{d_r}{n_g} \sum_{g \in \mathcal{L}'} \chi^r(g) T(g), \tag{27}$$

where n_g is the number of elements of the group $\mathcal{L}'$ and $\chi^r(g)$ is the character of the group element g in the representation T^r. Similarly, for compact continuous groups we have

$$\hat{P}^r = d_r \int_{\mathcal{L}'} \chi^r(g) T(g)\, \mathrm{d}\mu(g), \tag{28}$$

where $\mathrm{d}\mu(g)$ is the group measure [20].

Now we are finally ready to formulate the restrictions imposed by symmetries on the controllability condition. (We will not present all details of the analysis here — interested readers can find them in [18].) First of all one concludes that as long as the decomposition (23) contains nontrivial irreducible representations, there exists a lower bound on the minimal number of control parameters necessary to satisfy the controllability condition

$$\bar{n}_u \geq \max\nolimits_{r=1,\ldots,q} d_r. \tag{29}$$

In addition, symmetry imposes a number of restrictions on the control matrix B,

$$\mathrm{rank}(\hat{P}^r B) \geq d_r, \quad r = 1, \ldots, q, \tag{30}$$

which can be interpreted as the requirement of the mutual independence of control parameters. In other words, an arbitrary (unitary) subgroup $\mathcal{L}'$ of the full dynamical symmetry group $\mathcal{L}$ does not completely define the minimal set of control parameters. It does, however, define a set of necessary conditions required for controllability. In general, the knowledge of all dynamical symmetries, both unitary and nonunitary, described by the group $\mathcal{L}$ is required in order to completely resolve the structure of the Jacobian matrix and obtain the necessary and sufficient condition for controllability.

Nevertheless, even without knowing the full symmetry group $\mathcal{L}$ one can obtain the necessary and sufficient conditions by assuming that there are no accidental degeneracies. It is usually safe to do so if, e.g., $\mathcal{L}'$ is taken to coincide with $\mathcal{H}_{\bar{\mathbf{x}}}$: we ensure that all physical symmetries are taken into account, and accidental degeneracies should only appear for certain special values of system parameters. Then inequalities (29) and (30) are replaced by the equalities

$$\bar{n}_u = \max_{r=1,\dots,q} d_r. \tag{31}$$

$$\mathrm{rank}(\hat{P}^r B) = d_r, \quad r = 1, \dots, q, \tag{32}$$

respectively.

It should be noted that (30) trivially reduces to (32), if no irreducible representation T^r of $\mathcal{L}'$ enters the decomposition (23) more than once, such that $p_r = 1$ for all r. Although situations in which this is not true are not uncommon (inverted pendulum discussed in Section 3.1 is a typical example), this condition can be easily verified, while it might be impossible to prove that there are no hidden symmetries resulting in accidental degeneracies.

Summing up, we conclude that in the absence of accidental degeneracies the system is controllable, if and only if the two conditions are met. The first one requires the number n_u of control parameters to be greater or equal to the dimensionality d_r of the largest irreducible representation T^r present in the decomposition of the matrix representation T of the subgroup $\mathcal{L}' \subseteq \mathcal{L}$ in the tangent space $\mathcal{T}$. The second one requires the control parameters to be independent: the columns $\mathbf{b}_m$ of the control matrix B have to be chosen such that d_r of the projections $\hat{P}^r\mathbf{b}_m$, $m = 1, \dots, n_u$ are linearly independent (and, therefore, span the eigenspace $L^r_{\mathcal{L}'}$) for every $r = 1, \dots, q$. The last condition represents the restrictions imposed on the admissible form of the linear response of the system to perturbations of control parameters and might require additional information about the specific system for verification (Section 3 contains a more detailed discussion).

Of course, symmetry does not always make the Jacobian degenerate, and the nondegenerate case can be handled in the same way as the one with no symmetries. Neither does the degeneracy by itself imply that multi-parameter control is required: if the degeneracy is produced by a nonunitary symmetry (in which case the Jacobian matrix is not diagonalizable), one control parameter is sufficient to ensure the controllability [18]. In both cases, however, the dynamical symmetry should be rather low. Specifically, the decomposition (23) of the matrix representation T should not contain any multi-dimensional irreducible representations.

In principle, all of these results could have been derived by directly linearizing the continuous-time evolution equation (1). Indeed, the definitions of symmetries, respective symmetry groups, and the notions of stabilizability and controllability in the *continuous-time* case are completely analogous to the ones given in Section 2.1 for the *discrete-time* case (see, e.g., [17]). As a consequence, all steps in the above analysis of time-invariant target states are equally applicable to continuous-time systems. This is a rather valuable asset of the developed theory, since continuous-time control is, in general, a much more flexible and powerful technique than discrete-time control. In the presence of an adequate mathematical model continuous-time control can often achieve far superior results. It is, however, a much more complicated technique as well, so we will not discuss it in any detail.

The reason for choosing the discrete-time approach in the present paper is that, on the one hand it is more transparent, and on the other hand it affords a natural generalization to the *time-periodic* case. In particular, if the control matrix does not depend on time, $B^t = B$ (otherwise the results become too complicated to be useful, and as such will not be cited here), one obtains the following independence condition:

$$\mathrm{rank}(\hat{P}^r B) \geq \left\lceil \max\left(\frac{d_r}{p_r}, \frac{d_r}{\tau}\right) \right\rceil, \quad r = 1, \ldots, q. \tag{33}$$

Here $\lceil d \rceil$ denotes the smallest integer larger than or equal to d, and $\tau \geq 1$ is the period of the trajectory. Instead of (29) one respectively obtains the restriction on the minimal number of control parameters

$$\bar{n}_u \geq \left\lceil \max_{r=1,\ldots,q} \max\left(\frac{d_r}{p_r}, \frac{d_r}{\tau}\right) \right\rceil. \tag{34}$$

It is interesting to note that a periodic trajectory can be made controllable using the number of control parameters n_u that could be smaller than the number required for a steady state with the same symmetry.

2.4. Symmetry violation

In reality symmetries of physical systems displaying dynamical instabilities are almost never exact. Indeed, the cylinders in a Taylor–Couette experiment are never perfectly cylindrical, the temperature inside a chemical reactor is never absolutely uniform, neither are the rotor blades of a turbocompressor exactly identical. The above analysis, on the other hand, has been conducted in the assumption of exact symmetry. Therefore, it is essential to understand how the obtained results change, if the symmetry is not exact or, in other words, what the effect of a weak symmetry violation is. Such an analysis is also crucial in the vicinity of points in the parameter space where symmetry increasing bifurcations or accidental degeneracies occur.

For simplicity let us again consider the time-invariant case. The Jacobian A of a weakly perturbed symmetric system takes the form [2]

$$A = A_0 + \epsilon A_1 + \mathrm{O}(\epsilon^2), \tag{35}$$

where ϵ denotes the magnitude of the perturbation and the unperturbed Jacobian A_0 is exactly symmetric with respect to all transformations g of the group $\mathcal{L}$. For the group representation T we thus have

$$T(g)A_0 - A_0 T(g) = 0 \quad \forall g \in \mathcal{L}. \tag{36}$$

In general, the perturbation ϵA_1 will not be symmetric with respect to any element of the group $\mathcal{L}$, except the identity transformation e

$$T(g)A_1 - A_1 T(g) \neq 0 \quad \forall g \in \mathcal{L} \setminus \{e\}. \tag{37}$$

Therefore, since (up to the second-order in ϵ)

$$T(g)A - AT(g) = \epsilon(T(g)A_1 - A_1 T(g)), \tag{38}$$

the perturbation (35) completely destroys the symmetry of the linearized evolution equation (4) for any $\epsilon \neq 0$. As a result, the perturbed system can be made controllable using a single control parameter, irrespectively of the

[2] As pointed out by one of the referees, it is known that there are special cases in which weakly breaking the symmetry of a system can introduce nonperturbative, i.e., O(1) terms into the equations of motion. This occurs, e.g., if one considers an unbounded fluid undergoing an oscillatory bifurcation and the translational invariance of the system is weakly broken by introducing distant endwalls into the problem. We are, however, interested in the generic case and will not discuss such rather pathological examples here.

properties of the original symmetry group $\mathcal{L}$. For instance, calculating the controllability matrix of the perturbed system with $n_u = 1$ and $B = \mathbf{b}$ one obtains

$$\mathcal{C} = \mathcal{C}_0 + \epsilon\mathcal{C}_1 + \mathrm{O}(\epsilon^2), \tag{39}$$

where we defined

$$\mathcal{C}_0 = [\,\mathbf{b} \quad A_0\mathbf{b} \quad \cdots \quad (A_0)^{n_x-1}\mathbf{b}\,], \quad \mathcal{C}_1 = [\,0 \quad A_1\mathbf{b} \quad \cdots \quad \textstyle\sum_{n=0}^{n_x-2}(A_0)^{n_x-2-n}(A_1)^n\mathbf{b}\,]. \tag{40}$$

$\mathcal{C}_0$ is clearly the controllability matrix of the unperturbed system with full symmetry, which does not have a full rank, if the decomposition (23) contains at least one irreducible representation T^r with the dimensionality $d_r > 1$. Indeed, in the absence of accidental degeneracies that would mean

$$n_0 \equiv \mathrm{rank}(\mathcal{C}_0) \le \sum_{r=1}^{q} p_r < n_x. \tag{41}$$

The controllability matrix $\mathcal{C}$ of the perturbed system, on the other hand, has full rank for any $\epsilon \neq 0$ because the symmetry is completely destroyed by the perturbation. Therefore, the perturbed *linear* system becomes controllable (in the sense of the definition given in Section 2.1) even though the unperturbed system is not, for *arbitrarily small* perturbations.

The controllability ensures that for any initial and final states of the linear system (4) a sequence of control perturbations $\Delta\mathbf{U}^t \equiv [\Delta u^{t+n_x-1}, \ldots, \Delta u^t]$ can be found mapping the initial state to the final state in n_x iterations. This sequence can be obtained explicitly from (7)

$$\Delta\mathbf{U}^t = (\mathcal{C})^{-1}(\Delta\mathbf{x}^{t+n_x} - (A)^{n_x}\Delta\mathbf{x}^t). \tag{42}$$

Formally, if the system is controllable, the controllability matrix is invertible, and the solution (42) is well defined for any $\Delta\mathbf{x}^t$ and $\Delta\mathbf{x}^{t+n_x}$. However, when the matrix $\mathcal{C}$ is close to being singular its inverse is not well defined. It is convenient to use the singular value decomposition of the controllability matrix

$$\mathcal{C} = Q\Sigma R^\dagger, \tag{43}$$

where $Q = [\,\mathbf{q}_1 \quad \cdots \quad \mathbf{q}_{n_x}\,]$ and $R = [\,\mathbf{r}_1 \quad \cdots \quad \mathbf{r}_{n_x}\,]$ are some orthogonal $n_x \times n_x$ matrices, and

$$\Sigma = \begin{bmatrix} \sigma_1(\epsilon) & & \\ & \ddots & \\ & & \sigma_{n_x}(\epsilon) \end{bmatrix}. \tag{44}$$

The singular values are ordered such that $\sigma_1(\epsilon) \ge \sigma_2(\epsilon) \ge \cdots \ge \sigma_{n_x}(\epsilon)\ \forall\epsilon$. Additionally, Eq. (41) requires

$$\lim_{\epsilon\to 0}\sigma_i(\epsilon) = 0, \quad i = n_0 + 1, \ldots, n_x. \tag{45}$$

In terms of the matrices Q, Σ, and R we can write the inverse of $\mathcal{C}$ as

$$(\mathcal{C})^{-1} = R(\Sigma)^{-1}Q^\dagger = \sum_{i=1}^{n_x}\sigma_i^{-1}(\epsilon)\mathbf{r}_i\mathbf{q}_i^\dagger, \tag{46}$$

and therefore, for small ϵ Eq. (42) gives

$$\Delta\mathbf{U}^t \approx \sum_{i=n_0+1}^{n_x}\frac{(\mathbf{q}_i\cdot\Delta\mathbf{x}^{t+n_x}) - (\mathbf{q}_i\cdot(A)^{n_x}\Delta\mathbf{x}^t)}{\sigma_i(\epsilon)}\mathbf{r}_i. \tag{47}$$

As a consequence, we obtain

$$\lim_{\epsilon\to 0}|\Delta \mathbf{U}^t| = \infty. \tag{48}$$

This result means that at least one control perturbation of the feedback sequence $\Delta u^t, \dots, \Delta u^{t+n_x-1}$ diverges as the symmetry breaking perturbation ϵA_1 of the Jacobian vanishes. Since no specific relation between the initial and the final state of the system was implied, the obtained result is general, and does not depend on the control method used to calculate the feedback.

In fact, a more general statement holds. Suppose the symmetry is violated only partially, such that the perturbed Jacobian (35) remains exactly symmetric with respect to a subgroup $\mathcal{L}'$ of the full symmetry group $\mathcal{L}$. Denote $\bar{n}_u$ and $\bar{n}'_u$ the minimal number of control parameters required (assuming exact symmetry) by the groups $\mathcal{L}$ and $\mathcal{L}'$, respectively. Then it can be shown that, similarly to the single-parameter case, at least one control perturbation of the feedback sequence $\Delta\mathbf{u}^t, \dots, \Delta\mathbf{u}^{t+n_x-1}$ diverges as ϵ goes to 0 whenever $\bar{n}'_u \le n_u < \bar{n}_u$. The same result is obtained if the independent with respect to the group $\mathcal{L}'$ control parameters become dependent with respect to the group $\mathcal{L}$, as indicated by the violation of the general independence condition (30). The time-periodic generalization is also straightforward. We will call this situation *parametric deficiency*.

In other words, although it might be possible to control a *linear* system with approximate symmetry using a number of control parameters which is smaller than that required in the assumption of exact symmetry, the stabilization requires feedback of very large magnitude. Such systems are called *weakly controllable* in the language of control theory. However, the linear system is only an abstraction. The linear approximation (3) of the evolution equation (2) is only valid for small perturbations $\Delta\mathbf{u}^t$ of the control parameters and small deviations $\Delta\mathbf{x}^t$ from the target trajectory. Besides, additional restrictions on the magnitude of the feedback are usually imposed by practical limitations, size and energy constraints, etc., at the implementation stage. One can, therefore, conclude that, since the magnitude of feedback scales linearly with the deviation from the target trajectory, a nonlinear system with parametric deficiency can be stabilized using linear control only in an asymptotically contracting neighborhood of the target trajectory.

As an example, consider the vicinity of the point $\bar{\mathbf{u}}_0$ in the parameter space $\mathbb{R}^{n_u}$ at which two eigenvalues belonging to different invariant subspaces cross (i.e., accidental degeneracy occurs). Then the full dynamical symmetry is described by the group $\mathcal{L}'$ for $\bar{\mathbf{u}} \ne \bar{\mathbf{u}}_0$ and is increased to $\mathcal{L}$ (of which $\mathcal{L}'$ is a subgroup) for $\bar{\mathbf{u}} = \bar{\mathbf{u}}_0$. In this case $\mathcal{L}$ can be considered approximate symmetry in the vicinity of $\bar{\mathbf{u}}_0$, and the distance to that point plays the role of the parameter ϵ. Suppose the control scheme is such that there is a parametric deficiency. Then the system will remain controllable for $\bar{\mathbf{u}} \ne \bar{\mathbf{u}}_0$. However, the strength of feedback required to control the system will diverge as $\bar{\mathbf{u}}$ approaches $\bar{\mathbf{u}}_0$, at which point the system will become uncontrollable.

3. Applications

3.1. Inverted pendulum in three dimensions

Now that the formal theory is constructed, we can illustrate it by applying to a few simple symmetric systems. As our first example we consider the problem of balancing an inverted pendulum (such as a rigid rod) about its unstable equilibrium by moving the support point in the horizontal plane. Although the two-dimensional version of the problem is literally a textbook example, the three-dimensional version is much richer and less trivial. The question we would like to answer is whether the pendulum can be stabilized by moving the support point in just one direction, or using the whole plane is necessary. Intuitively one would assume that the whole plane is necessary regardless of the symmetry. Symmetry however plays a crucial role in the problem, so the answer should depend on whether the pendulum is symmetric or not. A related, although as we shall see below not equivalent, question is what the minimal number of control parameters is for this system. This problem is of particular interest to us because it not only serves as a nontrivial example of the relation between the structural symmetry group $\mathcal{G}$ and the dynamical symmetry group $\mathcal{L}$, but also highlights the role played by the structure of the tangent space $\mathcal{T}$. Besides, the equations of motion of many systems of practical interest, such as a charged particle in an electromagnetic trap, a space booster on takeoff, or a satellite in orbit, can be cast in a similar form.

At first let us assume that the pendulum is axially symmetric, so that the dynamical equations are invariant with respect to rotation about the vertical axis (which we denote z). The equilibrium state is also rotationally invariant, so that $\mathcal{H}_0 = \mathcal{G} = \mathrm{O}(2)$. For a thin cylindrical rod of mass m and length l the principal moments of inertia are $J_x = J_y = ml^2/3$ and $J_z = 0$, so that in polar coordinates the Lagrangian of the system is given by

$$L = \frac{ml^2}{6}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - \frac{mgl}{2}\cos\theta. \tag{49}$$

The equations of motion in Cartesian coordinates are found by expanding the Lagrangian near the equilibrium state $\theta = \dot{\theta} = 0$ and substituting $x = u_x + l\sin\theta\cos\phi$, $y = u_y + l\sin\theta\sin\phi$. After performing the algebra one obtains (up to a constant)

$$L = \frac{m}{6}(\dot{x}^2 + \dot{y}^2) + \frac{mg}{4l}((x-u_x)^2 + (y-u_y)^2), \tag{50}$$

where u_x and u_y are the coordinates of the support point, playing the role of control parameters. From (50) we immediately obtain a system of equations

$$\frac{m}{3}\ddot{x} = \frac{mg}{2l}(x-u_x), \qquad \frac{m}{3}\ddot{y} = \frac{mg}{2l}(y-u_y), \tag{51}$$

which can be trivially reduced to a system of first-order differential equations by introducing velocities v_x and v_y

$$\dot{x} = v_x, \quad \dot{y} = v_y, \quad \dot{v}_x = \omega_0^2(x-u_x), \quad \dot{v}_y = \omega_0^2(y-u_y), \tag{52}$$

where we denoted $\omega_0^2 = 3g/2l$. The respective Jacobian and control matrix are given by

$$A_c = \begin{bmatrix} O & I \\ \omega_0^2 I & O \end{bmatrix}, \qquad B_c = -\omega_0^2\begin{bmatrix} O \\ I \end{bmatrix}, \tag{53}$$

where O and I are 2×2 zero and unit blocks, respectively. Since the eigenvalues $\lambda = \pm\omega_0$ of the Jacobian are doubly degenerate, we immediately conclude that both control parameters are required to make the system controllable. In fact, it can be easily verified that the pair (A_c, B_c) is controllable.

We could arrive at the same conclusion without calculating the Jacobian. Let us take $\mathcal{L}' = \mathcal{H}_0 = \mathrm{O}(2)$. Consider the representation $T(\psi)$ of the rotation by an angle ψ about the z-axis in the tangent space $\mathcal{T} = \mathbb{R}^4$

$$T(\psi) = \begin{bmatrix} R(\psi) & O \\ O & R(\psi) \end{bmatrix}, \qquad R(\psi) = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\phi & \cos\psi \end{bmatrix}. \tag{54}$$

From the form of the matrix $T(\psi)$ it is clear that the representation of the rotation group in the four-dimensional tangent space can be decomposed into the sum of two equivalent two-dimensional irreducible representations (vector representations) of O(2)

$$T = 2T^1, \qquad d_1 = 2. \tag{55}$$

According to the results of Section 2, this indicates that in order to control the unstable steady state $\bar{\mathbf{r}} = \bar{\mathbf{v}} = 0$ one needs at least two independent control parameters, $\bar{n}_u = 2$. Furthermore, it is trivial to check that the action of u_x and u_y is indeed independent.

To see that it is sufficient to have two control parameters, notice that the dynamical symmetry group of the linear system of Eq. (52) is $\mathcal{L} = \mathrm{GL}(2)$, so that $\mathcal{L}' \subset \mathcal{L}$. This means that $\mathcal{G} \subset \mathcal{L}$, i.e., the symmetry of the linearized equations is higher than the symmetry of the original nonlinear evolution equations. However, the representation of the group $\mathrm{GL}(2)$ is decomposed into a sum of two equivalent two-dimensional irreducible representations, identically to the group O(2), so this higher symmetry does not lead to the increase in the minimal number of control parameters. Neither is the independence condition affected.

Now let us repeat the analysis using the polar coordinates. The equation of motion obtained from (49) reads

$$\ddot{\theta} = \left(\omega_\phi^2 + \frac{3g}{2l}\right)(\theta - u_\theta), \tag{56}$$

where u_θ denotes the change in the angle θ resulting from moving the support point by $\mathbf{u} = l \sin u_\theta (\cos\phi\, \mathbf{e}_x + \sin\phi\, \mathbf{e}_y)$, and $\omega_\phi = \dot{\phi} = \text{const}$. Denoting $\omega_1^2 = \omega_\phi^2 + \omega_0^2$ (56) can be written as a first-order system

$$\dot{\theta} = \omega_\theta, \qquad \dot{\omega}_\theta = \omega_1^2(\theta - u_\theta). \tag{57}$$

If we consider u_θ as our new control parameter, the Jacobian and control matrix can be written in the form

$$A_p = \begin{bmatrix} 0 & 1 \\ \omega_1^2 & 0 \end{bmatrix}, \qquad B_p = \omega_1^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{58}$$

The eigenvalues of the Jacobian are distinct, $\lambda = \pm\omega_1$, so a single control parameter should be sufficient. The pair (A_p, B_p) is controllable, so u_θ is definitely suitable for this role. This is not surprising. Indeed, the degrees of freedom affected by the rotational symmetry (i.e., ϕ and ω_ϕ) have been factored out, so the dynamical equations for the remaining degrees of freedom are not affected by the symmetry and can be considered generic in the sense that the evolution operators are nondegenerate.

This result seems to contradict the one obtained previously for the Cartesian space. In fact, there is no contradiction: since the feedback is applied quite differently in the two cases, it is not unreasonable to expect the minimal number of control parameters to be different as well. In the case of the Cartesian coordinates the displacement of the support point, $\mathbf{u} = u_x\mathbf{e}_x + u_y\mathbf{e}_y$, uniquely determines the magnitude and the direction of the restoring force, so that the latter is always proportional to $-\mathbf{u}$. In the case of the polar coordinates u_θ determines only the magnitude of the restoring force, while the direction of that force depends on the deviation $\mathbf{r} = l \sin\theta (\cos\phi\, \mathbf{e}_x + \sin\phi\, \mathbf{e}_y)$ of the rod from the equilibrium point. As a consequence, the restoring force is proportional to $-u_\theta\mathbf{r}$. This difference shows that the minimal number $\bar{n}_u$ of control parameters depends not only on the symmetry properties of the system, but also on the choice of variables used to describe the dynamics, in other words on the structure of the tangent space $\mathcal{T}$. In particular, since $\mathcal{T}$ is a linear space, $\bar{n}_u$ is invariant with respect to any linear transformation of variables, but there is no reason for it to be invariant with respect to a nonlinear transformation of variables.

In contrast, we can conclude that independently of a particular description it is not sufficient to move the support point in any one direction to keep the symmetric rod from falling. Both directions should be used for control, which seems to be an intuitive result. Indeed, we know that using both u_x and u_y corresponds to moving the support point in both the x-and y-direction. Similarly, in order to apply the feedback by changing the angle θ in the polar coordinate formulation one should be able to move the support point in the direction of the deviation from equilibrium, defined by the angle ϕ, which is arbitrary.

We conclude this section with probably the least intuitive result that follows from the developed theory: if the axial symmetry of the rod is broken, it can be kept in the unstable equilibrium by moving the support point in just one direction, which of course implies that a single control parameter is sufficient. To see this let us again assume that the rod is thin, such that the unstable equilibrium is achieved at the point $\mathbf{r}=0$, but allow the principal moments of inertia to be different, $J_x \neq J_y \neq J_z = 0$. The Lagrangian (50) then has to be changed to

$$L = \frac{1}{2l^2}\left(J_y\dot{x}^2 + J_x\dot{y}^2\right) + \frac{mg}{4l}\left((x-u_x)^2 + (y-u_y)^2\right). \tag{59}$$

Respectively, (52) has to be replaced with

$$\dot{x} = v_x, \quad \dot{y} = v_y, \quad \dot{v}_x = \omega_x^2(x-u_x), \quad \dot{v}_y = \omega_y^2(y-u_y), \tag{60}$$

where now $\omega_x \neq \omega_y$. The corresponding Jacobian

$$A'_c = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_x^2 & 0 & 0 & 0 \\ 0 & \omega_y^2 & 0 & 0 \end{bmatrix} \tag{61}$$

is no longer degenerate, so there is no need to use two control parameters. Let us choose as our new control parameter an arbitrary linear combination u of u_x and u_y, such that $u_x = u\cos\psi$ and $u_y = u\sin\psi$, which corresponds to moving the support point along the line which makes an angle ψ with the x-axis. It is easy to check that the Jacobian A'_c and the respective control matrix $B'_c = [\,0 \quad 0 \quad \omega_x^2\cos\psi \quad \omega_y^2\sin\psi\,]^{\mathrm{T}}$ form a controllable pair for any $\psi \neq n\pi/2$, with n-integer.

3.2. Coupled map lattice in one dimension

As our next example we consider a coupled map lattice (CML). On the one hand CMLs are used quite successfully to model the dynamics of many spatially discrete as well as spatially continuous extended systems describing such spatiotemporally chaotic phenomena as surface growth, population dynamics, and turbulence. It is thus reasonable to expect that the results obtained for lattice systems should be representative of a much wider class of systems. On the other hand CMLs are simple enough, so that many interesting results can be easily obtained analytically in an arbitrary number of dimensions. As we shall see, the symmetry of the lattice plays a very important role in the control problem. In fact, the geometry of control turns out to be closely related to the geometry of the system itself.

The one-dimensional deterministic coupled map lattice with nearest neighbor diffusive coupling [21] is described by the following evolution equation:

$$x_i^{t+1} = \epsilon f(x_{i-1}^t, a) + (1-2\epsilon) f(x_i^t, a) + \epsilon f(x_{i+1}^t, a). \tag{62}$$

Here the index $i = 1, 2, \ldots, n_x$ labels the lattice sites, and the periodic boundary condition is imposed. The choice of the map function $f(x,a)$ is usually motivated by the local dynamics of the physical system under consideration. In principle $f(x,a)$ can be chosen as an arbitrary (nonlinear) function with parameter a, which typically represents the process of generation and growth of local fluctuations, while diffusive coupling typically plays the opposite role of dissipating these fluctuations. Therefore, the parameters a and ϵ specify the degree of instability and the strength of dissipation in the system, respectively. For the purpose of control, however, details of the local map are not important. The only aspect of the control problem affected by any particular choice is the set of existing unstable periodic trajectories.

The coupled map lattice is by construction highly symmetric. The symmetry is that of the spatial lattice: the evolution equation (62) is invariant with respect to translations by an integer number of lattice sites (periodic boundary condition makes the group finite) and reflections about any site (or midplane between any adjacent sites), which map the lattice back onto itself without destroying the adjacency relationship between neighboring sites. The corresponding point group $C_{n_x v}$ (we assume n_x even) has a total of $n_x/2+3$ nonequivalent irreducible representations. The first four are one-dimensional, $d_1 = d_2 = d_3 = d_4 = 1$, while the rest $n_x/2-1$ are two-dimensional, $d_r = 2$, $r \geq 5$. In comparison, breaking the reflection symmetry reduces the group to C_{n_x}, which only has one-dimensional irreducible representations.

The dynamical symmetry group $\mathcal{L}$ can be trivially obtained using the observation that the Jacobian matrix in the linearization (3) constructed for the CML (62) can always be represented as a product of two matrices, $A^t = MN^t$, where

$$M_{ij} = (1-2\epsilon)\delta_{i,j} + \epsilon(\delta_{i,j-1} + \delta_{i,j+1}) \tag{63}$$

describes diffusive coupling, and

$$N^t_{ij} = \partial_x f(\bar{x}^t_i, a)\delta_{i,j} \tag{64}$$

defines the strength of local instability, with symbols $\delta_{i,j\pm1}$ extended to comply with periodic boundary condition. This partition of the Jacobian explicitly shows how the symmetry group $\mathcal{L}$ depends on the symmetry properties of the nonlinear evolution equation (62) and those of the controlled state $\bar{\mathbf{x}}^t$. The matrix M contains the symmetries imposed by the chosen coupling of the nonlinear model

$$T(g)M = MT(g) \quad \forall g \in \mathcal{G}, \tag{65}$$

while the matrices N^t reflect the symmetry of the target state $\bar{\mathbf{x}}^t$

$$T(g)N^t = N^t T(g) \quad \forall g \in \mathcal{H}_{\bar{\mathbf{x}}}. \tag{66}$$

Since the Jacobian A^t commutes with all matrices that commute with both M and N^t, we conclude that generically $\mathcal{L} = \mathcal{H}_{\bar{\mathbf{x}}} \subseteq \mathcal{G}$, in agreement with our assumption (17).

Let us take $\mathcal{L}' = \mathcal{L}$ and construct its representation T in $\mathcal{T} = \mathbb{R}^{n_x}$. Decomposing T into a sum of the irreducible representations of $C_{n_x v}$ we can easily determine the restrictions imposed by the symmetry on the minimal number of control parameters n_u and the structure of the control matrix B. For instance, a zigzag state gives $\mathcal{L} = C_{nv}$ with $n = n_x/2$ and, according to (31), $\bar{n}_u = 2$; a nonreflection-invariant state with spatial period s corresponds to $\mathcal{L} = C_n$ with $n = n_x/s$ and $\bar{n}_u = 1$, etc.

Let us consider the spatially uniform target state, which has the highest symmetry possible, $\mathcal{L} = C_{n_x v}$, in more detail. The decomposition (23) gives

$$T = T^1 \oplus T^4 \oplus T^5 \oplus \cdots \oplus T^{n_x/2+3}, \tag{67}$$

and the corresponding basis of normal modes which transform according to these irreducible representations is given by the eigenvectors of the operators of translation and reflection, i.e., Fourier modes $\mathbf{h}(k)$

$$h_j(k) = \mathrm{e}^{\mathrm{i}kj}. \tag{68}$$

To satisfy the periodic boundary condition, the wavevector k should belong to the discrete set $\pm 2\pi m/n_x$, $m = 0, \ldots, n_x/2$. Fourier modes with the wavevectors of the same magnitude define invariant subspaces $L^k \subset \mathcal{T}$. The subspaces L^k with $0 < k < \pi$ correspond to the representations T^r with $r \geq 5$, L^0 corresponds to T^1, and L^π to T^4. The eigenvalues of A corresponding to subspaces with $0 < k < \pi$ should be doubly degenerate, which can be easily verified by calculating the spectrum

$$\lambda(k) = \alpha\{1 - 4\epsilon \sin^2(k/2)\}, \tag{69}$$

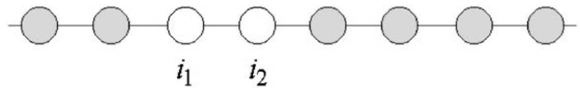

Fig. 1. Minimal local cluster of control sites for a 1D lattice. The feedback is applied at the lattice sites i_1 and i_2 (white) placed next to each other and propagates to the rest of the sites (gray) via the inter-site coupling.

where $\alpha = \partial_x f(\bar{x}, a)$. Since the two-dimensional irreducible representations are present in the decomposition (67), $\bar{n}_u = 2$. Therefore, in order to control an unstable uniform steady state of the coupled map lattice we need at least two control parameters. This is a reflection of the symmetry of coupling in the model (62). Note that, since every two-dimensional irreducible representation occurs in the decomposition (67) once, $p_5 = \cdots = p_{n_x/2+3} = 1$, according to the results of Section 2, the minimal number of control parameters remains the same for a spatially uniform target trajectory of arbitrary time period τ.

Furthermore, since for any length n_x of the lattice the group $\mathcal{G} = C_{n_x v}$ only has one-and two-dimensional irreducible representations and $\mathcal{L}$ is a subgroup of $\mathcal{G}$, it is sufficient to have just two control parameters to make the dynamics of the coupled map lattice controllable in the vicinity of a target state with arbitrary symmetry properties and temporal period. Choosing the minimal number of control parameters, $n_u = 2$, we can determine the conditions making them independent with respect to a particular target state: the linear response of the CML to perturbation of the two parameters, given by the columns of the control matrix $B = [\,\mathbf{b}_1 \quad \mathbf{b}_2\,]$, has to satisfy conditions (22) and (32).

Failure to satisfy the necessary condition (22) rules out the possibility of using global parameters, such as the coupling ϵ or parameter a of the local map $f(x, a)$ for control of symmetric steady states. Taking $\mathbf{u} = (a, \epsilon)$, so that

$$\mathbf{b}_1 = \partial_a \mathbf{F}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = M \begin{bmatrix} \partial_a f(\bar{x}_1, \bar{a}) \\ \vdots \\ \partial_a f(\bar{x}_{n_x}, \bar{a}) \end{bmatrix}, \tag{70}$$

$$\mathbf{b}_2 = \partial_\epsilon \mathbf{F}(\bar{\mathbf{x}}, \bar{\mathbf{u}}) = (\bar{\epsilon})^{-1}(M - I) \begin{bmatrix} f(\bar{x}_1, \bar{a}) \\ \vdots \\ f(\bar{x}_{n_x}, \bar{a}) \end{bmatrix}, \tag{71}$$

we observe that condition (22) is only satisfied, if the group $\mathcal{L}$ is trivial, $\mathcal{L} = \{e\}$. This result holds for time-periodic symmetric target states as well.

Alternatively, one can make the system controllable by directly perturbing the system at the sites i_1 and i_2 (which can be thought of as a simpler version of the method suggested by Hu and Qu [22]). The positions of the control sites cannot be chosen arbitrarily, again due to symmetry. In particular, if the target state is spatially uniform, it is trivial to show that choosing, e.g., $i_2 = i_1 + 1$ satisfies the controllability condition for an arbitrary length n_x of the lattice (see Fig. 1). The control matrix corresponding to this choice of control parameters can be written in the form $B_{ij} = \delta_{j,1}\delta_{i,l} + \delta_{j,2}\delta_{i,l+1}$, where $1 \leq l \leq n_x$.

Such localized control also has its downside. In the weak coupling limit, $\epsilon \to 0$, the coupled map lattice with local feedback becomes a weakly controllable system. The symmetry of the lattice of uncoupled maps is described by the permutation group $\mathcal{G} = \mathrm{S}_{n_x}$, while the linearization about a uniform target state increases the symmetry to $\mathcal{L} = \mathrm{GL}(n_x)$: since the respective Jacobian is a multiple of the unit matrix, $A_{ij} = \alpha\delta_{i,j}$, the linearized system is symmetric with respect to all (complex) nonsingular coordinate transformations. When coupling is restored, $\epsilon > 0$, the symmetry of both the nonlinear evolution equation (62) and its linearization (3) reduces to $\mathcal{G}' = \mathcal{L}' = \mathrm{C}_{n_x v}$.

The matrix representation T of the group $\mathrm{GL}(n_x)$ in $\mathbb{R}^{n_x}$ is already irreducible. Consequently, $n_u = n_x$ independent control parameters are required to control the steady uniform state of the uncoupled lattice. This result is rather intuitive. Obviously, one can no longer control the system applying control perturbations at just two lattice sites, i_1 and i_2. Since the control perturbation does not propagate to adjacent sites of the lattice, feedback has to be applied directly at each site.

If the coupling is nonzero, but very small, the controllability property is restored for $n_u = 2$, but, according to Section 2.4, feedback of very large magnitude is required to control the system due to parametric deficiency. Indeed, in order to affect the dynamics at site i away from i_1 and i_2 the control has to propagate a certain distance decaying by roughly a factor of ϵ per iteration. As a result, the magnitude of the perturbation required to control an arbitrary site of the lattice diverges approximately as $\epsilon^{-n_x/2}$ for $\epsilon \to 0$, resulting in the loss of control [7].

3.3. Coupled map lattice in higher dimensions

It turns out that local dynamics of the coupled maps defined on higher-dimensional lattices can differ quite substantially from the local dynamics of the one-dimensional CMLs. As a consequence, one should not expect that the higher-dimensional control problem could be solved by a simple generalization of the one-dimensional case. In particular, it would be interesting to find out if and when the spatiotemporally chaotic dynamics on a higher-dimensional lattice can be stabilized using a local cluster of control sites, similarly to the one-dimensional lattice. The answer to this question depends on whether there are accidental degeneracies, which can only appear, if the dimension of the lattice is greater than one.

Let us begin with the simplest of the higher-dimensional cases, that of the lattice in $D = 2$ spatial dimensions. The generalization of the one-dimensional CML (62) to two dimensions reads

$$x_{i,j}^{t+1} = f(x_{i,j}^t, a) + \epsilon \Delta f(x_{i,j}^t, a), \tag{72}$$

where we have defined a discrete Laplacian

$$\Delta\chi(x_{i,j}) = \chi(x_{i-1,j}) + \chi(x_{i,j-1}) - 4\chi(x_{i,j}) + \chi(x_{i+1,j}) + \chi(x_{i,j+1}). \tag{73}$$

The double index (i, j) labels the lattice sites and we assume that the lattice is square: $i = 1, 2, \dots, n_x$, $j = 1, 2, \dots, n_x$. We will also assume the periodic boundary conditions.

The symmetry of the CML is again determined by the symmetry of the spatial lattice. It is easy to see that besides retaining the one-dimensional translational and reflectional invariance along each of the two lattice directions, the system (72) is in addition invariant with respect to the reflection along the diagonal direction, i.e., the permutation operation which exchanges the indices i and j. This latter invariance is the product of the discrete rotational symmetry, which arises due to the isotropy of the coupling in the two lattice directions. The corresponding structural symmetry group is then defined as $\mathcal{G} = S_2 \times C_{n_x v} \times C_{n_x v}$. Respectively, for the hypercubic lattice with side n_x in D dimensions one obtains

$$\mathcal{G} = S_D \times \underbrace{C_{n_x v} \times \cdots \times C_{n_x v}}_{D}. \tag{74}$$

The analysis of the controllability condition follows that of the previous section quite closely. In particular, for the uniform target state, $\mathcal{H}_{\bar{\mathbf{x}}} = \mathcal{G}$ and $\mathcal{T} = \mathbb{R}^n$ with $n = n_x^D$. Let us take $\mathcal{L}' = \mathcal{H}_{\bar{\mathbf{x}}}$. The normal modes are just the tensor products of the Fourier modes (68), so for $D = 2$ one has

$$h_{j_1,j_2}(k_1, k_2) = \mathrm{e}^{\mathrm{i}k_1 j_1 + \mathrm{i}k_2 j_2}. \tag{75}$$

The eigenmodes with $k_1 = k_2 = 0, \pi$ transform according to the one-dimensional representations of the group (74) in $\mathcal{T}$, the eigenmodes with $k_1 = 0$, $k_2 = \pi$ — two-dimensional, the eigenmodes with $k_1 = 0, \pi$, $k_2 \neq 0, \pi$ and $k_1 = k_2 \neq 0, \pi$ — four-dimensional, and finally the eigenmodes with $k_1 \neq k_2 \neq 0, \pi$ — eight-dimensional representations. In the absence of accidental degeneracies this leads one to believe that the minimal number of control parameters should be $\bar{n}_u = 8$.

The obtained value, however only gives a lower bound on the minimal number of control parameters, since the actual degeneracy is higher than what is suggested by our previous analysis. This can be shown by examining the respective eigenvalues

$$\lambda(k_1, k_2) = \alpha \left\{ 1 - 4\epsilon \left[\sin^2\left(\frac{k_1}{2}\right) + \sin^2\left(\frac{k_2}{2}\right) \right] \right\}, \tag{76}$$

obtained by linearizing (72). Indeed, note that $\lambda(k, \pi - k) = 1 - 4\epsilon$, irrespectively of the value of k, i.e., there is an accidental degeneracy (for even n_x). The reason is that our choice of $\mathcal{L}'$ does not completely specify the dynamical symmetry group $\mathcal{L}$. (Recall that according to Section 2 the accidental degeneracies are defined with respect to the group $\mathcal{L}'$, not $\mathcal{L}$.) As a result, the subspaces $L_{\mathcal{L}}^{k_1,k_2}$ of $\mathcal{T}$ invariant with respect to $\mathcal{L}$ have dimensions higher than the subspaces $L_{\mathcal{L}'}^{k_1,k_2}$ invariant with respect to $\mathcal{L}'$. Using combinatorial arguments it can be trivially verified that the invariant subspace $L_{\mathcal{L}}^{k_1,k_2}$ corresponding to $k_1 + k_2 = \pi$ has the highest dimensionality, and thus determines the minimal number of control parameters (and hence the number of control sites)

$$\bar{n}_u = \dim\left(L_{\mathcal{L}}^{k,\pi-k}\right) = 0 \times 1 + 1 \times 2 + m_4(n_x) \times 4 + m_8(n_x) \times 8 = 2(n_x - 1). \tag{77}$$

This result is quite interesting. In order to control a dynamical system with symmetric nearest neighbor interactions defined on a square $n_x \times n_x$ lattice one needs a cluster of control sites, the size of which grows linearly with the side of the lattice. To determine whether some particular arrangement of control sites is suitable one has to ensure that the set of independence conditions (32) is satisfied. This can be achieved, for instance, by choosing the arrangement shown in Fig. 2a.

The effect of accidental degeneracy is amplified for higher-dimensional lattices. Combinatorial arguments show that the minimal size $\bar{n}_u$ of the control cluster becomes a function of the lattice size and can grow as fast as $n_x^{\lfloor D/2 \rfloor}$ (for n_x being a multiple of $2D!$), where $\lfloor d \rfloor$ denotes the integer part of d. It should be noted however, that contrary to the symmetry-related degeneracy, the accidental degeneracy depends sensitively on the size of the lattice. For instance, there is no accidental degeneracy for odd n_x, while the symmetry-related degeneracy is always present. Nevertheless, the former does not just disappear for n_x large, but is rather replaced with a near-degeneracy, which, according to Section 2.4 leads to weak controllability, if the size of the control cluster is too small. We therefore expect the scaling of $\bar{n}_u$ obtained above to hold for all sufficiently large lattices.

Fortunately, the evolution equation (72) usually represents nothing more than the leading order approximation of the actual dynamics, where one ignores the interaction with neighbors further away than one lattice spacing. Using this approximation might be advantageous for calculating the values of dynamical averages, but it is undesirable in the control problem. Let us see how the latter is affected by the sub-leading terms in the dynamical equations, adding the next nearest neighbor interaction as an example. Eq. (72) is then replaced with

$$x_{i,j}^{t+1} = f(x_{i,j}^t, a) + \epsilon \Delta f(x_{i,j}^t, a) + \gamma \Delta^2 f(x_{i,j}^t, a). \tag{78}$$

Since we added the next nearest neighbor interaction in a way that does not break the spatial symmetry of the original equation, the eigenvectors and the invariant subspaces do not change either. The eigenvalues however do change and are now given by

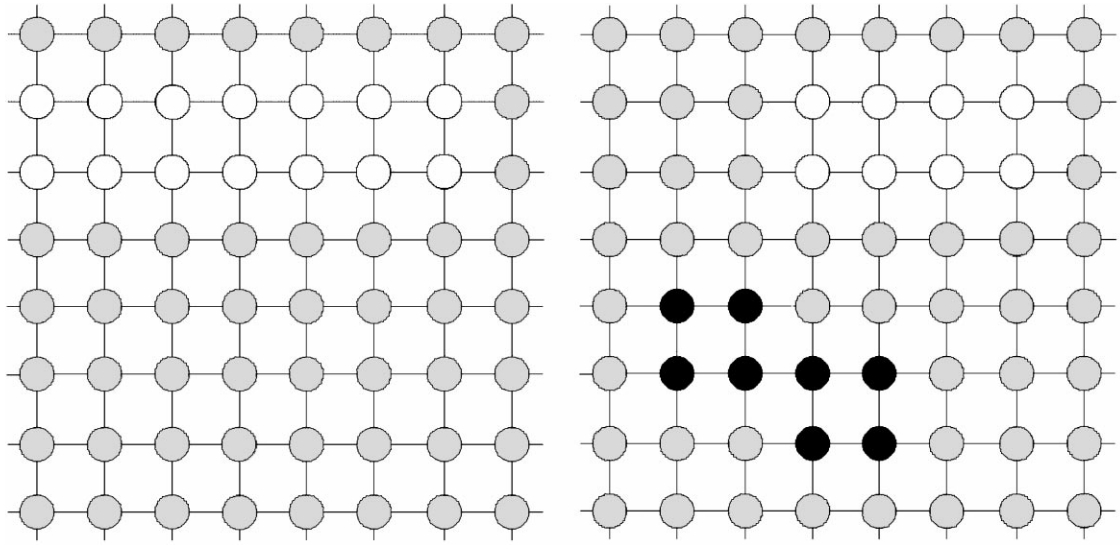

Fig. 2. Clusters of control sites for a 2D lattice. For a lattice with accidental degeneracy (a) the size of the minimal control cluster (white) grows linearly with the size of the lattice. If there is no accidental degeneracy, (b) the lattice can be controlled using a local cluster of control sites. Two examples of allowed minimal local clusters (white and black) are shown.

$$\lambda(k_1, k_2) = \alpha \left\{1 - 4(\epsilon - 4\gamma)\left[\sin^2\left(\frac{k_1}{2}\right) + \sin^2\left(\frac{k_2}{2}\right)\right] - 4\gamma[\sin^2(k_1) + \sin^2(k_2)]\right\}, \tag{79}$$

so the accidental degeneracy disappears, as long as $\gamma \neq 0$. As a consequence, we retrieve the result $\bar{n}_u = 8$ for $D = 2$. Similarly, one concludes that the minimal number of control parameters is given by $\bar{n}_u = D!2^D$ and is thus independent of the size of the lattice in arbitrary dimension D. Moreover, it can be verified, that the coupled map lattice, whose evolution is governed by Eq. (78) can always be controlled with a local cluster of control sites. The choice of the cluster is not unique, and a couple of examples for $D = 2$ are presented in Fig. 2b. Extending the range of coupling further has the same effect, so we expect the obtained results to remain valid for a generic extended system defined on a hypercubic lattice in an arbitrary dimension.

Choosing a different kind of lattice, such as an h.c.p., f.c.c. or b.c.c. lattice, is always possible and will change the symmetry of the system and, with it, the shape and size of the control cluster. However, we again expect that it will be possible to control the dynamics of such lattices using a local cluster of control sites, irrespectively of the size of the system.

4. Conclusions

Summarizing, we have determined that if the system under consideration is symmetric, it cannot be considered generic with respect to conventional chaos control techniques, and its symmetry properties should be understood prior to constructing a control scheme, even if the symmetry is only approximate. The failure to observe the restrictions imposed by the symmetry on the choice of control parameters will result in weak controllability and, as a result, extreme sensitivity to noise, or even worse, complete loss of control.

From the practical point of view, the main result of the symmetry analysis is that the minimal number of independent control parameters required for control can typically be determined without any knowledge of the evolution equations governing the dynamics of the system. One however needs to know the properties, such as spatial symmetry and temporal periodicity, of the target state, and the structural symmetry of the dynamical equations, which in the case of extended chaotic systems is typically defined by the geometry of the underlying physical space. One also needs to determine the structure of the tangent space, or the choice of variables used to describe the state of the system. Of course, the structural symmetry is not always uniquely defined by the geometry. The dynamical equations might, in principle, be symmetric with respect to transformations unrelated to "geometrical" symmetries, such as those describing rotational, reflectional, or translational invariance. Additional "nonphysical" dynamical symmetries can also be introduced as a result of the linearization procedure.

We also showed that the control problem is not fully solved when the sufficient number of control parameters is determined. The action and/or placement of the controllers must also be considered carefully, as certain conditions must be satisfied in order to achieve controllability. In particular, perturbation of the control parameters should completely break the dynamical symmetry. The more restrictive independence condition is specific to each target trajectory and, on the one hand, requires the knowledge of the system's response to variation of different control parameters (which can be obtained experimentally, if necessary), but, on the other hand, allows one to choose the minimal set of control parameters systematically, avoiding trial and error search.

Another area relevant to control, where the above symmetry analysis can be applied equally successfully is phase space reconstruction (or system identification, in engineering terms). Building on the results of [23] one can show [18] that symmetry imposes restrictions, essentially identical to those we obtained for the control parameters, on the number of independent measurements (or number of sensors) one needs to employ to reconstruct the state and the dynamics of the system. If there is a "parametric deficiency" in the measurements, the projected attractors will generically remain "folded" near highly symmetric steady and periodic states, thus preventing the local reconstruction.

A number of more specific conclusions can be made concerning extended chaotic systems. The analysis of the simplified model system containing the defining features of a general spatially extended dynamical system suggests that spatiotemporal chaos generically cannot be controlled using a *single* control parameter, globally or locally (and neither can its state be reconstructed using the measurements from a single sensor). However instabilities can be tamed quite effectively by perturbing the system at a number of distinct spatial locations (control sites). In fact, it can be argued that in experimental setting it is usually much easier to apply feedback locally, which is crucial for practical implementation of control methods based on the presented approach.

We determined that in order to make the target state controllable, the control sites should be arranged properly. Choosing this arrangement in accordance with the underlying symmetries of the system affords a significant reduction of the complexity (smaller density of control sites per unit length, area, or volume of the system) with simultaneous increase in the flexibility of the control algorithm, allowing it to track target trajectories as system parameters change, or switch between different trajectories by changing feedback *without* changing either the density or the location of control sites. Generally speaking, the control sites should be arranged to get rid of uncontrollable normal modes. In particular, in case of systems with both translational and reflectional invariance, the control sites *should not* be arranged in a periodic array.

Perhaps surprisingly, although there is a minimal *number* of control sites (as well as sensors), their minimal *density* is not bounded from below — in the absence of noise an extended system of arbitrary size can, in principle, be controlled using the number of control sites equal to the minimal number of control parameters, which is determined by the symmetry properties alone. (In practice certain restrictions appear due to the fact that the volume of the basin of attraction shrinks exponentially with increasing size of the system.) However, when noise appears, the minimal density of control sites depends on the strength of noise as well as parameters of the system and the choice of feedback gain [7].

Finally, we should comment that the conditions on the set of control parameters derived in Section 2 are imposed by the *controllability* condition and guarantee that control can be achieved. However, in general, only the weaker *stabilizability* condition has to be satisfied, which requires that every *unstable* normal mode of the system is controllable, so that only unstable invariant subspaces have to be considered in the conditions (29)–(32). As a consequence, it might be possible to stabilize highly symmetric states of compact extended systems with strong spatial correlations using a single control parameter — if only a small number of normal modes is excited, there is a chance that all *unstable* modes will correspond to one-dimensional irreducible representations. In strongly chaotic systems a large number of modes will be unstable and many of them will inevitably be degenerate, calling for multi-parameter control. Similar considerations apply to weakly chaotic systems with large spatial extent.

For example, in the liquid bridge convection experiment it was found [9] that the one-sensor, one-heating element arrangement failed to stabilize the unstable axially symmetric state and produced a standing wave with the node at the location of the heating element, which we can immediately identify with an uncontrollable normal mode. The symmetry analysis shows that the dynamical symmetry group is $O(2)$, so that a minimum of two independent control parameters (currents through the two heating elements) are required. The independence condition defines the values of the allowed angular offset ϕ between the heating elements. The controllability condition applicable for both strongly and weakly unstable regimes requires the ratio $2\pi/\phi$ to be irrational. However, since the experimental system was weakly unstable (only the normal modes with $m = 1$ and $m = 2$ were unstable), control could be achieved by satisfying the stabilizability condition, i.e., for all values of ϕ except multiples of $\pi/2$. Similarly, the angle between the two sensors should not be a multiple of $\pi/2$ to allow state reconstruction. Indeed, in the experiment both angles were chosen close to an optimal value of $3\pi/4$.

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